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ON A CURVATURE-TYPE INVARIANT OF A g-HOLOMORPHICALLY SEMI-SYMMETRIC CONNECTION ON A LOCALLY PRODUCT SPACE

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Abstract. We consider an $n-$dimensional locally product space with p and q dimensional components ($p+q=n$). In our previous paper, we have considered two connections, $(F,g)-$holomorphically semi-symmetric (this means that both metric and structure tensor are parallel towards this connection) and $F-$holomorphically semi-symmetric one, both with gradient generators. We have proved that both of these connections have curvature-like invariants which are both equal to product conformal curvature tensor. Here we shall consider the third connection from this family, namely, g-holomorphically semi-symmetric connection and find its curvature-like invariant.

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1. Introduction

The geometrical motivation for such a consideration was the fact found in one of our previous papers ([9]), that $F-$holomorphically semi-symmetric connection and $(F,g)-$holomorphically semi-symmetric connection on a Kähler space with Norden metrics (or anti-Kähler space) have curvature-type invariants which are equal to one of its conformal invariants.

For such a reason, we have considered the same situation on a locally product space and we obtained an analogous result. In [10], we considered the third connection from such a group on anti-Kähler space and found its curvature-type invariant. In [11], we considered situation on locally product spaces, which is analogous to the situation in [9] and got similar results.

The papers [1, 2, 3, 4, 5, 6, 7, 8, 12] also helped us in consideration of this problem.

It is well-known that a locally product space is an $n-$dimensional manifold $\mathcal{M}_n$ with a (positive definite) metric (g_{ij}), which is called a Riemannian space and with structure tensor field $F^i_j \neq \delta^i_j$, satisfying conditions

$$F^i_s F^s_j = \delta^i_j,\ g_{st} F^s_i F^t_j = g_{ij},\ \nabla_k F^i_j = 0,$$

where ∇ denotes the operator of covariant derivative towards to Levi-Civita connection.

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If we set $g_{is}F^s_j = F_{ij}$, then it is clear that the covariant structure tensor is symmetric and parallel towards the Levi-Civita connection. In any neighborhood of any point of a locally product space, its metric tensor can be expressed in the form

$$ds^2 = g_{\alpha\beta}(x^i)dx^\alpha dx^\beta + g_{rs}(x^i)dx^r dx^s, \tag{1.1}$$

where $\alpha, \beta = 1, ..., p; r, s = p+1, ..., p+q = n$ $(n = \dim \mathcal{M}_n)$, or, equivalently,

$$(g_{ij}) = \begin{pmatrix} g_{\alpha\beta} & 0 \\ 0 & g_{rs} \end{pmatrix} \tag{1.2}$$

and then its tangent space is a product of two tangent subspaces of dimensions p and q. Then the structure tensor in such a coordinate system has the form

$$(F^i_j) = \begin{pmatrix} \delta^\alpha_\beta & 0 \\ 0 & -\delta^r_s \end{pmatrix}, \tag{1.3}$$

or, for its covariant form

$$(F_{ij}) = \begin{pmatrix} g_{\alpha\beta} & 0 \\ 0 & -g_{rs} \end{pmatrix}. \tag{1.4}$$

If in the expression (1.1) the conditions $g_{\alpha\beta} = g_{\alpha\beta}(x^\gamma)$ and $g_{rs} = g_{rs}(x^t)$ are satisfied, then the space $\mathcal{M}_n$ is called a locally decomposable space.

There are several papers dedicated to locally product spaces (see, e. g. [1, 2, 3, 4, 5, 6, 7, 8, 12], which were interesting and useful for our consideration), but not so many in last fifteen years. Maybe this angle of consideration is a convenient way to make such kind of spaces rekindle again. It may also be interesting to treat a special case when the considered space is a product of semi-Riemannian or Riemannian spaces of constant curvature ([2, 3]).

The connection with coefficients

$$\Gamma^i_{jk} = \left\{ {i \atop jk} \right\} - p_j\delta^i_k + p^i g_{jk} + q_j F^i_k - q^i F_{jk}, \tag{1.5}$$

where $q_j = p_a F^a_j$, is a $g-$connection; moreover, its torsion tensor is of the form

$$-p_j\delta^i_k + p_k\delta^i_j + q_j F^i_k - q_k F^i_j, \tag{1.6}$$

which is the reason to call it a holomorphically semi-symmetric connection. Besides, holds

$$K_{ijkl} = F^r_i F^s_j K_{rskl} \tag{1.7}$$

and this is a Kähler-type condition for Riemann-Christoffel tensor.

2. The curvature tensor of a $g-$holomorphically semi-symmetric connection and its algebraic properties

Taking into account (1.5) , we can calculate the curvature tensor for such connection. We obtain that, after lowering the upper index, holds,

$$\begin{aligned} R_{ijkl} &= K_{ijkl} + g_{il}p_{kj} - g_{ik}p_{lj} + g_{jk}p_{li} - g_{jl}p_{ki} \\ &\quad -F_{il}q_{kj} + F_{ik}q_{lj} - F_{jk}q_{li} + F_{jl}q_{ki}, \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} p_{kj} &= \nabla_k p_j + p_k p_j - q_k q_j - \frac{1}{2}p_s p^p g_{kj} + \frac{1}{2}p_s q^s F_{kj}, \\ q_{kj} &= \nabla_k q_j + q_k p_j - p_k q_j + \frac{1}{2}p_s p^s F_{kj} - \frac{1}{2}p_s q^s g_{kj}. \end{aligned} \tag{2.2}$$

It is obvious that

$$p_{kj} = 2\nabla_k p_j - q_{ka}F_j^a \tag{2.3}$$

and, consequently

$$q_{kj} = 2\nabla_k q_j - p_{ka}F_j^a. \tag{2.4}$$

Now we want the tensor R_{ijkl} to be an algebraic curvature tensor. Its component is skew-symmetric in last two indices by definition. Also, it is visible from (2.1) that its component is skew-symmetric in first two indices. Its components must also be invariant under changing places of the first and the second pair of indices. Then, we are getting

$$\begin{aligned} 0 &= g_{il}(p_{kj} - p_{jk}) - g_{ik}(p_{lj} - p_{jl}) + g_{jk}(p_{li} - p_{il}) - \\ &\quad g_{jl}(p_{ki} - p_{ik}) + F_{ik}(q_{lj} - q_{jl}) - F_{il}(q_{kj} - q_{jk}) \\ &\quad +F_{jl}(q_{ki} - q_{ik}) - F_{jk}(q_{li} - q_{il}). \end{aligned} \tag{2.5}$$

If we transvect the upper equality by g^{il},we obtain

$$(n-2)(p_{kj} - p_{jk}) + F_k^l(q_{lj} - q_{jl}) - \psi(q_{kj} - q_{jk}) + F_j^i(q_{ki} - q_{ik}) = 0,$$

where ψ stands for $p - q$. If we take into account (2.3), then it holds that

$$\begin{aligned} &(n-3)(p_{kj} - p_{jk}) + 2(\nabla_k p_j - \nabla_j p_k) - \psi(q_{kj} - q_{jk}) \\ =\ &F_j^a q_{ak} - F_k^a q_{aj}. \end{aligned}$$

From (2.2), it holds that $\nabla_k p_j - \nabla_j p_k = p_{kj} - p_{jk}$; so, we obtain

$$(n-1)(p_{kj} - p_{jk}) - \psi(q_{kj} - q_{jk}) = F_j^a q_{ak} - F_k^a q_{aj}. \tag{2.6}$$

If we transvect (2.5) by F^{il} and take into account (2.4), we shall obtain that

$$\begin{aligned} &\psi(p_{kj} - p_{jk}) - (n-2)(q_{kj} - q_{jk}) \\ =\ &F_k^l(p_{lj} - p_{jl}) + F_j^i(p_{ki} - p_{ik}). \end{aligned} \tag{2.7}$$

If we suppose that, like in [9, 10, 11], the generator (p_i) is a gradient, then p_{kj} will be a symmetric tensor. Then q_{kj} is also a symmetric tensor (from (2.7)) and then the equality (2.6) is satisfied automatically. Besides, it holds that

$$\nabla_k q_j = \nabla_j q_k + 2(q_j p_k - q_k p_j). \tag{2.8}$$

If we use the fact that p_{kj} and q_{kj} are symmetric tensors, then it is easy to prove that the tensor (2.1) satisfies the first Bianchi identity. So, we have proved that the following theorem holds.

Theorem 2.1. *If the generator of $g-$holomorphically semi-symmetric connection on an almost product space is a gradient, then the curvature tensor R_{ijkl} (satisfying (2.1)), of such connection is an algebraic curvature tensor.*

In our following considerations, we would suppose that such a condition is satisfied.

3. Some scalar functions and tensors which are connected with a $g-$ holomorphically semi-symmetric connection

If we set

$$S_{kj} = p_k p_j - q_k q_j - \frac{1}{2} p_s p^s g_{kj} + \frac{1}{2} p_s q^s F_{kj}, \tag{3.1}$$

then, from (2.2), we have that

$$p_{kj} = \nabla_k p_j + S_{kj},\ q_{kj} = F_j^a \nabla_k p_a - S_{ka} F_j^a. \tag{3.2}$$

The tensor S_{kj} is a symmetric one; $S_{ka}F_j^a$ is not symmetric. Also, we can notice that

$$F_j^a q_{ak} = p_{kj} - 2S_{kj}.$$

We also can calculate that

$$\begin{aligned} S_s^s &= \frac{1}{2}(\psi p_s q^s - n p_s p^s); \\ S_{ab}F^{ab} &= \frac{1}{2}(n p_s q^s - \psi p_s p^s); \\ S_{ab}F_k^a F_j^b &= q_k q_j - p_k p_j - \frac{1}{2} p_s p^s g_{kj} + \frac{1}{2} p_s q^s F_{kj} \\ &= -S_{kj} - p_s p^s g_{kj} + p_s q^s F_{kj}. \end{aligned} \tag{3.3}$$

As for the curvature tensor of the connection (2.1) there is not satisfied the condition of Kähler type, but it is satisfied for Levi-Civita connection, we obtain that

$$\begin{aligned} R_{ijkl} - F_i^r F_j^s R_{rskl} &= 2(g_{il}\nabla_k p_j - g_{ik}\nabla_l p_j + g_{jk}\nabla_l p_i \\ &\quad - g_{jl}\nabla_k p_i) - 2(F_{il}\nabla_k q_j - F_{ik}\nabla_l q_j + \\ &\quad F_{jk}\nabla_l q_i - F_{jl}\nabla_k q_i). \end{aligned} \tag{3.4}$$

Other eight terms from the expression (2.1) and from $F_i^r F_j^s R_{rskl}$ are cancelling out each other, as they contain S_{kj} and $S_{ka}F_j^a$. In our next considerations, we shall use abbreviations

$$R_{ijkl}F^{il} = \overline{R}_{jk};\ \overline{R}_{jk}g^{jk} = \overline{R};\ \widetilde{\widetilde{R}} = \overline{R}_{jk}F^{jk}$$

and analogous abbreviations for curvature elements for the Levi-Civita connection.

Transvecting (3.4) by g^{il}, we are getting

$$R_{jk} - \overline{R}_{sk}F_j^s = 2(n-1)\nabla_k p_j - 2\psi\nabla_k q_j + 2g_{jk}\nabla_s p^s - 2F_{jk}\nabla_s q^s + 2F_k^a\nabla_a q_j. \tag{3.5}$$

If we transvect (3.5) by g^{jk}, we obtain

$$R - \widetilde{\widetilde{R}} = 4n\nabla_s p^s - 4\psi\nabla_s q^s$$

and, consequently

$$n\nabla_s p^s - \psi\nabla_s q^s = \frac{R - \widetilde{\widetilde{R}}}{4}, \tag{3.6}$$

which is an important relation between these two scalar functions. If we transvect the equality (3.4) by F^{kj}, we shall obtain an identity. We shall transvect the equality (3.4) by F^{il} and obtain that

$$\overline{R}_{jk} - R_{sk}F_j^s = -2(n-1)\nabla_k q_j + 2\psi\nabla_k p_j - 2\nabla_j q_k + 2g_{jk}\nabla_s q^s - 2F_{jk}\nabla_s p^s. \tag{3.7}$$

If we transvect (3.7) by F^{jk}, we obtain the relation (3.6) again; if we transvect it by g^{jk}, we obtain an identity again. If we change places of indices j and k in (3.7), we obtain

$$\overline{R}_{kj} - R_{sj}F_k^s = -2(n-1)\nabla_j q_k + 2\psi\nabla_j p_k - 2\nabla_k q_j + 2g_{kj}\nabla_s q^s - 2F_{kj}\nabla_s p^s. \tag{3.8}$$

Substracting (3.8) from (3.7) and taking into account that the tensor $\overline{R}_{jk}$ is symmetric, we obtain

$$R_{sj}F_k^s - R_{sk}F_j^s = 2(n-2)(\nabla_j q_k - \nabla_k q_j)$$

and, consequently

$$\nabla_j q_k = \nabla_k q_j - \frac{R_{sk}F_j^s - R_{sj}F_k^s}{2(n-2)}. \tag{3.9}$$

Substituting (3.9) into (3.7), we obtain

$$
\begin{aligned}
2n\nabla_k q_j &= \frac{n-1}{n-2}R_{sk}F^s_j - \frac{1}{n-2}R_{sj}F^s_k - \overline{R}_{kj} + 2\psi\nabla_k p_j\\
&\quad +2g_{jk}\nabla_s q^s - 2F_{jk}\nabla_s p^s
\end{aligned}
$$

and, consequently

$$
\begin{aligned}
\nabla_k q_j &= \frac{n-1}{2n(n-2)}R_{sk}F^s_j - \frac{R_{sj}F^s_k}{2n(n-2)} - \frac{\overline{R}_{kj}}{2n}\\
&\quad +\frac{\psi}{n}\nabla_k p_j + \frac{\nabla_s q^s}{n}g_{kj} - \frac{\nabla_s p^s}{n}F_{kj}.
\end{aligned}
\tag{3.10}
$$

Applying the relation $\nabla_k p_j = F^a_j\nabla_k q_a$, we obtain

$$
\begin{aligned}
\nabla_k p_j &= \frac{n-1}{2n(n-2)}R_{kj} - \frac{\overline{R}_{sk}F^s_j}{2n} - \frac{R_{ab}F^b_j F^a_k}{2n(n-2)}\\
&\quad +\frac{\psi}{n}\nabla_k q_j - \frac{\nabla_s p^s}{n}g_{kj} + \frac{\nabla_s q^s}{n}F_{kj}.
\end{aligned}
\tag{3.11}
$$

If we substitute (3.10) into (3.11) and take into account (3.6), we shall obtain

$$
\begin{aligned}
\nabla_k p_j &= \frac{1}{n^2-\psi^2}\Big[\frac{n(n-1)}{2n(n-2)}R_{kj} - \frac{n}{2}\overline{R}_{sk}F^s_j - \frac{nR_{ab}F^b_j F^a_k}{2(n-2)}\\
&\quad +\frac{\psi(n-1)}{2(n-2)}R_{sk}F^s_j - \frac{\psi}{2(n-2)}R_{sj}F^s_k - \frac{\psi}{2}\overline{R}_{kj}\\
&\quad -\frac{R-\tilde{\tilde{R}}}{4}g_{jk} - \frac{\psi}{4n}(R-\tilde{\tilde{R}})F_{jk}\Big] + \lambda F_{jk},
\end{aligned}
\tag{3.12}
$$

$$
\begin{aligned}
\nabla_k q_j &= \frac{1}{n^2-\psi^2}\Big[\frac{n(n-1)}{2n(n-2)}R_{ka}F^a_j - \frac{n}{2}\overline{R}_{jk} - \frac{nR_{aj}F^a_k}{2(n-2)}\\
&\quad +\frac{\psi(n-1)}{2(n-2)}R_{jk} - \frac{\psi}{2(n-2)}R_{ab}F^b_j F^a_k - \frac{\psi}{2}\overline{R}_{ka}F^a_j\\
&\quad -\frac{R-\tilde{\tilde{R}}}{4}F_{jk} - \frac{\psi}{4n}(R-\tilde{\tilde{R}})g_{jk}\Big] + \lambda g_{jk},
\end{aligned}
\tag{3.13}
$$

where in both expressions λ stands for $\frac{\nabla_s q^s}{n}$, which cannot be eliminated.

If we use expressions (2.1) and (3.2), we can state that

$$
\begin{aligned}
&R_{ijkl} - g_{il}\nabla_k p_j + g_{ik}\nabla_l p_j - g_{jk}\nabla_l p_i + g_{jl}\nabla_k p_i - F_{jk}\nabla_l q_j\\
&\quad + F_{il}\nabla_k q_j - F_{jl}\nabla_k q_i + F_{jk}\nabla_l q_i\\
&= K_{ijkl} + g_{il}S_{jk} - g_{ik}S_{lj} + g_{jk}S_{li} - g_{jl}S_{ki} + F_{il}S_{ka}F^a_j\\
&\quad - F_{ik}S_{la}F^a_j + F_{jk}S_{la}F^a_i - F_{jl}S_{ka}F^a_i.
\end{aligned}
\tag{3.14}
$$

Using (3.3), we can obtain

$$S_{aj}F^a_k = -S_{ka}F^a_j - p_s p^s F_{kj} + p_s q^s g_{kj}. \tag{3.15}$$

Using standard method, by transvecting the expression (3.14) for the curvature tensor first by g^{il} and then by g^{jk} and using (3.2), we obtain that

$$\frac{R-K}{2} = n\nabla_s p^s + \frac{n-2}{2}(\psi p_s q^s - n p_s p^s) - \psi\nabla_s q^s + \frac{\psi}{2}(n p_s q^s - \psi p_s p^s).$$

From the upper equality, we obtain that

$$\frac{R-K}{2} = n\nabla_s p^s - \psi\nabla_s q^s + \psi(n-1)p_s q^s - \frac{((n-2)+\psi^2))}{2}p_s p^s. \tag{3.16}$$

But, if we transvect (3.14) first by g^{il} and then by F^{jk}, we obtain that

$$\overline{R} - \overline{K} = 2(n-2)S_{ab}F^{ab} + 2\psi S^s_s,$$

and, consequently, using (3.3),

$$\overline{R} - \overline{K} = (n(n-2)+\psi^2)p_s q^s - 2\psi(n-1)p_s p^s.$$

If we set abbreviations

$$\psi(n-1) = \alpha;\ n(n-2)+\psi^2 = \beta, \tag{3.17}$$

we are getting the relationship between scalar products

$$p_s q^s = \frac{\overline{R} - \overline{K} + 2\alpha p_s p^s}{\beta}. \tag{3.18}$$

If we substitute (3.17) and (3.18) into (3.16), we obtain

$$\nabla_s p^s = \frac{R-K}{2n} - \frac{\alpha(\overline{R}-\overline{K})}{n\beta} + \frac{\psi}{n}\nabla_s q^s + \frac{(\beta+2\alpha)(\beta-2\alpha)}{2n\beta}p_s p^s. \tag{3.19}$$

Now we shall transvect (3.14) first by F^{il} and after that by F^{jk} and then use (3.3) and (3.18); we shall obtain that

$$\nabla_s p^s = \frac{\widetilde{\widetilde{K}} - \widetilde{\widetilde{R}}}{2n} + \frac{\psi}{n}\nabla_s q^s + \frac{\alpha(\overline{R}-\overline{K})}{n\beta} - \frac{(\beta+2\alpha)(\beta-2\alpha)}{2n\beta}p_s p^s. \tag{3.20}$$

Comparing (3.19) and (3.20), we shall obtain that

$$p_s p^s = \frac{\beta}{2(4\alpha^2-\beta^2)}(R - K + \widetilde{\widetilde{R}} - \widetilde{\widetilde{K}}) - \frac{4\alpha(\overline{R}-\overline{K})}{2(4\alpha^2-\beta^2)} \tag{3.21}$$

and, using (3.18)

$$p_sq^s = \frac{\alpha}{4\alpha^2-\beta^2}(R-K+\tilde{\tilde{R}}-\tilde{\tilde{K}}) - \frac{\beta}{4\alpha^2-\beta^2}(\overline{R}-\overline{K}). \tag{3.22}$$

Using (3.3), we get that

$$\begin{aligned} S_s^s &= \frac{2\psi\alpha-n\beta}{4(4\alpha^2-\beta^2)}(R-K+\tilde{\tilde{R}}-\tilde{\tilde{K}}) + \frac{2n\alpha-\psi\beta}{2(4\alpha^2-\beta^2)}(\overline{R}-\overline{K}); \\ S_{ab}F^{ab} &= \frac{2n\alpha-\psi\beta}{4(4\alpha^2-\beta^2)}(R-K+\tilde{\tilde{R}}-\tilde{\tilde{K}}) + \frac{4\psi\alpha-2n\beta}{4(4\alpha^2-\beta^2)}(\overline{R}-\overline{K}). \end{aligned} \tag{3.23}$$

If we set

$$\mu = \frac{2\psi\alpha-n\beta}{4(4\alpha^2-\beta^2)};\ \nu = \frac{2n\alpha-\psi\beta}{4(4\alpha^2-\beta^2)}, \tag{3.24}$$

then we obtain

$$\begin{aligned} S_s^s &= \mu(R-K+\tilde{\tilde{R}}-\tilde{\tilde{K}}) + 2\nu(\overline{R}-\overline{K}); \\ S_{ab}F^{ab} &= \nu(R-K+\tilde{\tilde{R}}-\tilde{\tilde{K}}) + 2\mu(\overline{R}-\overline{K}). \end{aligned} \tag{3.25}$$

4. Calculating tensors S_{kj} and $S_{ka}F_j^a$

We shall denote the tensor on the left-hand side of (3.14) by L_{ijkl}. Using (3.12) and (3.13), we shall calculate it later. It is a curvature-like tensor, but not an algebraic curvature tensor and its final form will be rather long and complicated. So, we would rewrite (3.14) as

$$\begin{aligned} L_{ijkl} = {} & K_{ijkl} + g_{il}S_{jk} - g_{ik}S_{lj} + g_{jk}S_{li} - g_{jl}S_{ki} + F_{il}S_{ka}F_j^a \\ & -F_{ik}S_{la}F_j^a + F_{jk}S_{la}F_i^a - F_{jl}S_{ka}F_i^a. \end{aligned} \tag{4.1}$$

If we transvect (4.1) by g^{il}, we obtain

$$\begin{aligned} (n-4)S_{kj} + \psi S_{ka}F_j^a = {} & L_{jk} - K_{jk} - g_{jk}(S_s^s + p_sp^s) \\ & -F_{jk}(S_{ab}F^{ab} - p_sq^s). \end{aligned} \tag{4.2}$$

If we transvect (4.1) by F^{il} and use (3.15), we obtain

$$\begin{aligned} \psi S_{kj} + (n-2)S_{ka}F_j^a = {} & \overline{L}_{jk} - \overline{K}_{jk} - g_{kj}(S_{ab}F^{ab} - p_sq^s) \\ & -F_{kj}(S_s^s + p_sp^s), \end{aligned} \tag{4.3}$$

where $\overline{L}_{jk}$ stands for $L_{ijkl}F^{il}$. The system of equations (4.2), (4.3) will give us thenecessary tensors. We are going to solve this system, temporarily using abbreviations

$$(4.4)\qquad S^s_s + p_s p^s = \gamma;\ S_{ab}F^{ab} - p_s q^s = \delta.$$

Then the system will take the form

$$(4.5)\qquad \begin{aligned}(n-4)S_{kj} + \psi S_{ka}F^a_j &= L_{jk} - K_{jk} - \gamma g_{jk} - \delta F_{jk},\\ \psi S_{kj} + (n-2)S_{ka}F^a_j &= \overline{L}_{jk} - \overline{K}_{jk} - \delta g_{jk} - \gamma F_{jk}.\end{aligned}$$

Solving this system of linear equations using the method od opposite coefficients, we obtain

$$(4.6)\quad \begin{aligned}S_{kj} &= \frac{1}{(n-2)(n-4)-\psi^2}[(n-2)(L_{jk}-K_{jk}) - \psi(\overline{L}_{jk} - \overline{K}_{jk})\\ &\quad -((n-2)\gamma - \psi\delta)g_{jk} - ((n-2)\delta - \psi\gamma)F_{jk}].\end{aligned}$$

We can see that, in fact, calculating $S_{ka}F^a_j$ from the system of linear equations is unnecessary, because we can calculate it using their simple mutual relationship.

Now we shall calculate these scalar functions which are factors with metrics and structure. Using (4.4), (3.21), (3.22), (3.23) and (3.3), we obtain that

$$(4.7)\qquad \begin{aligned}(n-2)\gamma - \psi\delta &= -\beta\frac{(n-2)^2-\psi^2}{4(4\alpha^2-\beta^2)}(R - K + \widetilde{\widetilde{R}} - \widetilde{\widetilde{K}})\\ &\quad +\alpha\frac{(n-2)^2-\psi^2}{4\alpha^2-\beta^2}(\overline{R} - \overline{K});\\ (n-2)\delta - \psi\gamma &= \alpha\frac{(n-2)^2-\psi^2}{2(4\alpha^2-\beta^2)}(R - K + \widetilde{\widetilde{R}} - \widetilde{\widetilde{K}})\\ &\quad -\beta\frac{(n-2)^2-\psi^2}{2(4\alpha^2-\beta^2)}(\overline{R} - \overline{K}).\end{aligned}$$

Substituting these scalar quantities in the expression (4.6), we obtain that

$$(4.8)\quad \begin{aligned}S_{kj} &= \frac{1}{(n-2)(n-4)-\psi^2}[(n-2)(L_{jk}-K_{jk}) - \psi(\overline{L}_{jk} - \overline{K}_{jk})\\ &\quad +\frac{(n-2)^2-\psi^2}{4(4\alpha^2-\beta^2)}(\beta(R - K + \widetilde{\widetilde{R}} - \widetilde{\widetilde{K}}) - 4\alpha(\overline{R} - \overline{K}))g_{jk}\\ &\quad -\frac{(n-2)^2-\psi^2}{2(4\alpha^2-\beta^2)}(\alpha(R - K + \widetilde{\widetilde{R}} - \widetilde{\widetilde{K}}) - \beta(\overline{R} - \overline{K}))F_{jk}].\end{aligned}$$

Then, it is easy to calculate

$$(4.9)\quad S_{ka}F^a_j = \frac{1}{(n-2)(n-4)-\psi^2}[(n-2)(L_{ak}-K_{ak})F^a_j$$
$$-\psi(\overline{L}_{ak}-\overline{K}_{ak})F^a_j$$
$$+\frac{(n-2)^2-\psi^2}{4(4\alpha^2-\beta^2)}(\beta(R-K+\tilde{\tilde{R}}-\tilde{\tilde{K}})-4\alpha(\overline{R}-\overline{K}))F_{kj}$$
$$-\frac{(n-2)^2-\psi^2}{2(4\alpha^2-\beta^2)}(\alpha(R-K+\tilde{\tilde{R}}-\tilde{\tilde{K}})-\beta(\overline{R}-\overline{K}))g_{kj}].$$

5. Explicite calculating of tensors $L_{ijkl},\ L_{jk},\ L_{sk}F^s_j,\ \overline{L}_{jk},\ \overline{L}_{sk}F^s_j$

The tensor on the left-hand side of (3.14) we have denoted by L_{ijkl}. As we have calculated covariant derivatives of the generator and its image by the structure ((3.12), (3.13)), we can substitute them into this expression. Then, all the members of these expressions which are containing λ are cancelling and we shall obtain

$$L_{ijkl} =$$
$$R_{ijkl}-\frac{1}{n^2-\psi^2}[\frac{n(n-1)}{2(n-2)}(g_{il}R_{kj}-g_{ik}R_{lj}+g_{jk}R_{li}-g_{jl}R_{ki})$$
$$-\frac{n}{2}(g_{il}\overline{R}_{sk}F^s_j-g_{ik}\overline{R}_{sl}F^s_j+g_{jk}\overline{R}_{sl}F^s_i-g_{jl}\overline{R}_{sk}F^s_i)$$
$$-\frac{n}{2(n-2)}(g_{il}R_{ab}F^b_jF^a_k-g_{ik}R_{ab}F^b_jF^a_l+g_{jk}R_{ab}F^b_iF^a_k$$
$$-g_{jl}R_{ab}F^a_kF^b_i)$$
$$+\frac{\psi(n-1)}{2(n-2)}(g_{il}R_{sk}F^s_j-g_{ik}R_{sl}F^s_j+g_{jk}R_{sl}F^s_i-g_{jl}R_{sk}F^s_i)$$
$$-\frac{\psi}{2(n-2)}(g_{il}R_{sj}F^s_k-g_{ik}R_{sj}F^s_l+g_{jk}R_{si}F^s_l-g_{jl}R_{si}F^s_k)$$
$$-\frac{\psi}{2}(g_{il}\overline{R}_{kj}-g_{ik}\overline{R}_{lj}+g_{jk}\overline{R}_{li}-g_{jl}\overline{R}_{ki})-\frac{R-\tilde{\tilde{R}}}{2}(g_{il}g_{jk}-g_{ij}g_{lk})$$
$$-\frac{\psi}{4n}(R-\tilde{\tilde{R}})(g_{il}F_{kj}-g_{ik}F_{lj}+g_{jk}F_{li}-g_{jl}F_{ki})]$$
$$+\frac{1}{n^2-\psi^2}[\frac{n(n-1)}{2(n-2)}(F_{il}R_{ka}F^a_j-F_{ik}R_{la}F^a_j+F_{jk}R_{la}F^a_j-$$
$$-F_{jl}R_{ka}F^a_i)$$
$$-\frac{n}{2}(F_{il}\overline{R}_{kj}-F_{ik}\overline{R}_{lj}+F_{jk}\overline{R}_{li}-F_{jl}\overline{R}_{ki})-\frac{n}{2(n-2)}(F_{il}R_{aj}F^a_k$$
$$-F_{ik}R_{aj}F^a_l+F_{kj}R_{ai}F^a_l-F_{jl}R_{ai}F^a_l)$$
$$+\frac{\psi(n-1)}{2(n-2)}(R_{jk}F_{il}-F_{ik}R_{jl}+F_{jk}R_{il}-F_{jl}R_{ik})$$
$$-\frac{\psi}{2(n-2)}(F_{il}R_{ab}F^b_jF^a_k-F_{ik}R_{ab}F^b_jF^a_l+F_{jk}R_{ab}F^a_lF^b_i$$

$$- F_{jl}R_{ab}F^a_kF^b_i) - \frac{\psi}{2}(F_{il}\overline{R}_{ka}F^a_j - F_{ik}\overline{R}_{la}F^a_j + F_{jk}\overline{R}_{la}F^a_i$$
$$- F_{jl}\overline{R}_{ka}F^a_i) - \frac{R - \widetilde{\widetilde{R}}}{2}(F_{il}F_{kj} - F_{ik}F_{lj})$$
$$- \frac{\psi}{4n}(R - \widetilde{\widetilde{R}})(F_{il}g_{jk} - F_{ik}g_{jl} + F_{jk}g_{il} - F_{jl}g_{ik})].$$

It is not complicated to see that the eighth and the last row are cancelling each other out. If we change the sign of the second group of members, we obtain that

(5.1)
$$\begin{aligned}
L_{ijkl} = {} & \\
& R_{ijkl} - \frac{1}{n^2 - \psi^2}[\frac{n(n-1)}{2(n-2)}(g_{il}R_{kj} - g_{ik}R_{lj} + g_{jk}R_{li} \\
& - g_{jl}R_{ki} + F_{ik}R_{la}F^a_j - F_{il}R_{ka}F^a_j + F_{jl}R_{ka}F^a_i - F_{jk}R_{la}F^a_i) \\
& - \frac{n}{2}(g_{il}\overline{R}_{sk}F^s_j - g_{ik}\overline{R}_{sl}F^s_j + g_{jk}\overline{R}_{sl}F^s_i - g_{jl}\overline{R}_{sk}F^s_i \\
& + F_{ik}\overline{R}_{lj} - F_{il}\overline{R}_{kj} + F_{jl}\overline{R}_{ki} - F_{jk}\overline{R}_{li}) \\
& - \frac{n}{2(n-2)}R_{ab}(g_{il}F^b_jF^a_k - g_{ik}F^b_jF^a_l + g_{jk}F^b_iF^a_l - g_{jl}F^a_kF^b_i) \\
& + \frac{n}{2(n-2)}(F_{il}R_{aj}F^a_k - F_{ik}R_{aj}F^a_l + F_{jk}R_{ai}F^a_l - F_{jl}R_{ai}F^a_k) \\
& + \frac{\psi(n-1)}{2(n-2)}(g_{il}R_{sk}F^s_j - g_{ik}R_{sl}F^s_j + g_{jk}R_{sl}F^s_i - g_{jl}R_{sk}F^s_i \\
& + F_{ik}R_{jl} - F_{il}R_{kj} + F_{jl}R_{ki} - F_{jk}R_{li}) \\
& - \frac{\psi}{2(n-2)}(g_{il}R_{sj}F^s_k - g_{ik}R_{sj}F^s_l + g_{jk}R_{si}F^s_l - g_{jl}R_{si}F^s_k) \\
& + \frac{\psi}{2(n-2)}R_{ab}(F_{il}F^b_jF^a_k - F_{ik}F^b_jF^a_l + F_{jk}F^b_iF^a_l - F_{jl}F^b_iF^a_k) \\
& - \frac{\psi}{2}(g_{il}\overline{R}_{kj} - g_{ik}\overline{R}_{lj} + g_{jk}\overline{R}_{li} - g_{jl}\overline{R}_{ki} + F_{ik}\overline{R}_{la}F^a_j) \\
& - F_{il}\overline{R}_{ka}F^a_j + F_{jl}\overline{R}_{ka}F^a_i - F_{jk}\overline{R}_{la}F^a_i) \\
& - \frac{R - \widetilde{\widetilde{R}}}{2}(g_{il}g_{kj} - g_{ik}g_{lj} + F_{ik}F_{lj} - F_{il}F_{kj})].
\end{aligned}$$

Transvecting the expression (5.1) by g^{il}, we obtain

(5.2)
$$\begin{aligned}
L_{jk} = {} & \\
& R_{jk} - \frac{1}{n^2 - \psi^2}[(\frac{n^2}{2} - \frac{\psi^2(n-1)}{2(n-2)})R_{jk} \\
& - \frac{n\psi}{2(n-2)}R_{ka}F^a_j + \frac{n\psi}{2(n-2)}R_{ja}F^a_k \\
& + \frac{\psi(\psi - n + 2)}{2(n-2)}R_{ab}F^a_jF^b_k + \frac{\psi^2 - n(n-1)}{2}\overline{R}_{ak}F^a_j
\end{aligned}$$

$$-\frac{n}{2}\overline{R}_{aj}F^a_k-\frac{\psi(n-1)}{2}\overline{R}_{kj}+(R-\widetilde{\widetilde{R}})g_{jk}-\psi(R-\widetilde{\widetilde{R}})F_{jk}]$$

and, consequently

(5.3)
$$\begin{aligned}
&L_{sk}F^s_j=\\
&R_{sk}F^s_j-\frac{1}{n^2-\psi^2}[(\frac{n^2}{2}-\frac{\psi^2(n-1)}{2(n-2)})R_{ks}F^s_j\\
&-\frac{n\psi}{2(n-2)}R_{kj}+\frac{n\psi}{2(n-2)}R_{sa}F^a_kF^s_j\\
&+\frac{\psi(\psi-n+2)}{2(n-2)}R_{jb}F^b_k+\frac{\psi^2-n(n-1)}{2}\overline{R}_{jk}\\
&-\frac{n}{2}\overline{R}_{as}F^a_kF^s_j-\frac{\psi(n-1)}{2}\overline{R}_{ks}F^s_j\\
&+(R-\widetilde{\widetilde{R}})F_{kj}-\psi(R-\widetilde{\widetilde{R}})g_{kj}].
\end{aligned}$$

In the same way, we can calculate components of tensors $\overline{L}_{jk}=L_{ijkl}F^{il}$ and $\overline{L}_{sk}F^s_j$, which will also be necessary for the invariant.

6. Calculating the curvature-type invariant of $g-$holomorphically semi-symmetric connection on a locally product space

We shall use (4.1) to calculate the curvature-type invariant. As we already have calculated tensors S_{kj} and $S_{ka}F^a_j$, we shall substitute (4.8) and (4.9) into the right-hand side of (4.1) and obtain

(6.1)
$$\begin{aligned}
&L_{ijkl}=\\
&K_{ijkl}+\frac{1}{(n-2)(n-4)-\psi^2}\{(n-2)[(L_{jk}-K_{jk})g_{il}\\
&-(L_{jl}-K_{jl})g_{ik}+(L_{il}-K_{il})g_{jk}-(L_{ik}-K_{ik})g_{jl}]\\
&-\psi[(\overline{L}_{jk}-\overline{K}_{jk})g_{il}-(\overline{L}_{jl}-\overline{K}_{jl})g_{ik}\\
&+(\overline{L}_{il}-\overline{K}_{il})g_{jk}-(\overline{L}_{ik}-\overline{K}_{ik})g_{jl}]\\
&+\frac{(n-2)^2-\psi^2}{2(4\alpha^2-\beta^2)}[\beta(R-K+\widetilde{\widetilde{R}}-\widetilde{\widetilde{K}})-4\alpha(\overline{R}-\overline{K})](g_{il}g_{jk}-g_{ik}g_{jl})\\
&-\frac{(n-2)^2-\psi^2}{2(4\alpha^2-\beta^2)}[\alpha(R-K+\widetilde{\widetilde{R}}-\widetilde{\widetilde{K}})-\beta(\overline{R}-\overline{K})]\cdot\\
&(g_{il}F_{kj}-g_{ik}F_{lj}+g_{jk}F_{li}-g_{jl}F_{ki})\\
&+(n-2)[(L_{ak}-K_{ak})F^a_jF_{il}-(L_{al}-K_{al})F^a_jF_{ik}\\
&+(L_{al}-K_{al})F^a_iF_{jk}-(L_{ak}-K_{ak})F^a_iF_{jl}]\\
&-\psi[(\overline{L}_{ak}-\overline{K}_{ak})F^a_jF_{il}-(\overline{L}_{al}-\overline{K}_{al})F^a_jF_{ik}\\
&+(\overline{L}_{al}-\overline{K}_{al})F^a_iF_{jk}-(\overline{L}_{ak}-\overline{K}_{ak})F^a_iF_{jl}
\end{aligned}$$

$$+\frac{(n-2)^2-\psi^2}{2(4\alpha^2-\beta^2)}[\beta(R-K+\tilde{\tilde{R}}-\tilde{\tilde{K}})-4\alpha(\overline{R}-\overline{K})](F_{il}F_{jk}-F_{ik}F_{jl})$$
$$-\frac{(n-2)^2-\psi^2}{2(4\alpha^2-\beta^2)}[\alpha(R-K+\tilde{\tilde{R}}-\tilde{\tilde{K}})-\beta(\overline{R}-\overline{K})]\cdot$$
$$\cdot(F_{il}g_{jk}-F_{ik}g_{jl}+F_{jk}g_{il}-F_{jl}g_{ik}).$$

When we put all tensors and scalars which are depending on curvature tensor of $g-$holomorphically semi-symmetric connection on the left-hand side and the same objects which are depending on the curvature tensor of the Levi-Civita connection, we obtain:

(6.2)

$$L_{ijkl}-\frac{1}{(n-2)(n-4)-\psi^2}\{((n-2)L_{jk}-\psi\overline{L}_{jk})g_{il}$$
$$-((n-2)L_{jl}-\psi\overline{L}_{jl})g_{ik}+((n-2)L_{il}-\psi\overline{L}_{il})g_{jk}$$
$$-((n-2)L_{ik}-\psi\overline{L}_{ik})g_{jl}+((n-2)L_{ak}-\psi\overline{L}_{ak})F^a_jF_{il}$$
$$-((n-2)L_{al}-\psi\overline{L}_{al})F^a_jF_{ik}+((n-2)L_{al}-\psi\overline{L}_{al})F^a_iF_{jk}$$
$$-((n-2)L_{ak}-\psi\overline{L}_{ak})F^a_iF_{jl}$$
$$+\frac{(n-2)^2-\varphi^2}{2(4\alpha^2-\beta^2)}[\beta(R+\tilde{\tilde{R}})-4\alpha\overline{R}]\cdot(g_{il}g_{jk}-g_{ik}g_{jl}+F_{il}F_{jk}-F_{ik}F_{lj})$$
$$-\frac{(n-2)^2-\varphi^2}{2(4\alpha^2-\beta^2)}[\alpha(R+\tilde{\tilde{R}})-\beta\overline{R}]\cdot(g_{il}F_{kj}-g_{ik}F_{lj}+g_{jk}F_{li}-g_{jl}F_{ki})$$
$$=K_{ijkl}-\frac{1}{(n-2)(n-4)-\psi^2}\{((n-2)K_{jk}-\psi\overline{K}_{jk})g_{il}$$
$$-((n-2)K_{jl}-\psi\overline{K}_{jl})g_{ik}+((n-2)K_{il}-\psi\overline{K}_{il})g_{jk}$$
$$-((n-2)K_{ik}-\psi\overline{K}_{ik})g_{jl}+((n-2)K_{ak}-\psi\overline{K}_{ak})F^a_jF_{il}$$
$$-((n-2)K_{al}-\psi\overline{K}_{al})F^a_jF_{ik}+((n-2)K_{al}-\psi\overline{K}_{al})F^a_iF_{jk}$$
$$-((n-2)K_{ak}-\psi\overline{K}_{ak})F^a_iF_{jl}$$
$$+\frac{(n-2)^2-\varphi^2}{2(4\alpha^2-\beta^2)}[\beta(K+\tilde{\tilde{K}})-4\alpha\overline{K}]\cdot(g_{il}g_{jk}-g_{ik}g_{jl}+F_{il}F_{jk}-F_{ik}F_{lj})$$
$$-\frac{(n-2)^2-\varphi^2}{2(4\alpha^2-\beta^2)}[\alpha(K+\tilde{\tilde{K}})-\beta\overline{K}]\cdot(g_{il}F_{kj}-g_{ik}F_{lj}+g_{jk}F_{li}-g_{jl}F_{ki}).$$

So, we have proved that the following theorem holds.

Theorem 6.1. *If the curvature tensor of a $g-$holomorphically semi-symmetric connection of a locally product space is an algebraic curvature tensor, then the tensor on the left-hand side of* (6.2) *(tensor quantities appearing in this formula are given by* (3.14) *and* (5.1) *to* (5.3)*) is independent on the choice of its generator.*

The tensor on the left-hand side of (6.2) is said to be a **curvature-type invariant of a $g-$holomorphically semi-symmetric connection.**

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