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ON UNIQUENESS FOR A CHARACTERISTIC CAUCHY PROBLEM

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Abstract. Using our previous studies on the nonlinear characteristic Cauchy problem for the Wave Equation in canonical form, we focus on the uniqueness of the generalized solution. We show how uniqueness may be recovered in the homogeneous case by searching for a solution in the space of new tempered generalized functions $\mathcal{G}_{\mathcal{O}_M}\left(\mathbb{R}^2\right)$ based on the space of slowly increasing smooth functions in which pointwise characterization exists. In the same way as Biagioni, we can study the sections on the closure of open sets like $[0,T]\times[0,\infty[$ or $[0,\infty[$. The uniqueness can be proved in $\mathcal{G}_{\mathcal{O}_M}([0,T]\times[0,\infty[)$ thanks to an extension of pointwise characterization of elements in $\mathcal{G}_{\mathcal{O}_M}([0,\infty[)$.

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1. Introduction

Solving characteristic Cauchy problem for partial differential equations can meet some obstructions. Some existence results have been proved for characteristic linear problems in distributional framework (Y. V. Egorov [25], Y. V. Egorov and M. A. Shubin [26], L. Hörmander [29]), but uniqueness is still an open question.

Here we consider the characteristic Cauchy problem formally written

$$(P_{form}) : \frac{\partial^2 u}{\partial x \partial y} = F(\cdot,\cdot,u),\ u|_{\gamma} = \varphi,\ \left.\frac{\partial u}{\partial y}\right|_{\gamma} = \psi,$$

where the line γ of equation $x=0$ is globally characteristic for the Cauchy problem, u and F are supposed to be smooth, F is a Lipschitz function and φ and ψ are smooth functions defined in a neighbourhood of γ.

We reformulate it in the framework of generalized functions extending the ideas developed in [17, 18, 20, 21, 22, 32]. We complete the study of [3] focusing on uniqueness of the solution. This article extends the studies of [20], [21] and [3]. The reader will find in [3] the notations and the concepts used in this paper.

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A general reference for the $(\mathcal{C},\mathcal{E},\mathcal{P})$-algebras can be found in [30, 31, 32] and [19].

The characteristic problem is approached by a one-parameter family of classical smooth problems (P_ε) by deforming the characteristic curve into a family of non-characteristic ones $y = f_\varepsilon(x)$. Then we get a one-parameter family of classical solutions which we interpret as a generalized function in a convenient $(\mathcal{C},\mathcal{E},\mathcal{P})$-algebra $\mathcal{A}\left(\mathbb{R}^2\right)$ [31]. We formalize this process by associating a well formulated generalized problem, denoted (P_g). But this solution depends, a priori, on the choice of de-characterization, namely the choice of $(f_\varepsilon)_\varepsilon$. By imposing some restrictions on the asymptotical growth of the f_ε, we have proved in [3] that the generalized solution depends solely on the class of $(f_\varepsilon)_\varepsilon$ as a generalized function, not on the particular representative.

However this generalized solution in $\mathcal{A}\left(\mathbb{R}^2\right)$ fails to be, in general, unique. We show how uniqueness may be recovered in the homogeneous case [2] by working in the space of new tempered generalized functions $\mathcal{G}_{\mathcal{O}_M}\left(\mathbb{R}^2\right)$ based on the space of slowly increasing smooth functions [16] in which pointwise characterization exists [36]. But, in that algebra, it is impossible to obtain uniqueness for nonlinear case.

In recent years there have been a number of linear and nonlinear generalizations of the Gronwall inequality [24], [28]. One of them is the Wendroff inequality for two independent variables. In 1943 R. Bellman introduced the fundamental inequality named Gronwall-Bellman's inequality which is an important tool in the study of existence, uniqueness, boundedness, stability of the solutions of some differential equations and some integral equations [8], [1]. Many Wendroff type inequalities plays a vital role in studying some properties of solutions of differential equations [10], [4], [5]. We establish a inequality of Wendroff type for our study of the characteristic Cauchy problem.

We remark that, in the same way as Biagioni [9], we can study the sections under the open sets and even, on the closure of open sets like $[0,T]\times[0,\infty[$ or $[0,\infty[$. If $|F(x,y,u(x,y))| \leq a(x,y)\,u(x,y)$ where $a\in \mathrm{C}^0(\mathbb{R}^2)$, the uniqueness can be proved in the algebra $\mathcal{G}_{\mathcal{O}_M}([0,T]\times[0,\infty[)$ thanks to an extension of pointwise characterization of elements in $\mathcal{G}_{\mathcal{O}_M}([0,\infty[)$ [36] and some estimates obtained by a inequality of Wendroff type.

The outline of this paper is as follows. Section 2 introduces the generalized algebras. In section 3 we define a well-formulated generalized problem associated to the ill-posed one and we recall some properties of the solutions. In Section 4, we use the framework $\mathcal{G}_{\mathcal{O}_M}(\mathbb{R}^2)$ to show the uniqueness in the homogeneous case. Next, in Section 5 we introduce the algebra $\mathcal{G}_{\mathcal{O}_M}([0,T]\times[0,\infty[)$ and we study the uniqueness for the non-homogeneous problem.

2. Algebras of generalized functions

2.1. The $(\mathcal{C},\mathcal{E},\mathcal{P})$-algebras

2.1.1. Definitions

We recall the definition of the $(\mathcal{C},\mathcal{E},\mathcal{P})$-algebras. Take

(1) Λ a set of indices left-filtering for a given partial order relation $\prec$.

(2) A a solid subring of the ring $\mathbb{K}^{\Lambda}$, ($\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$), that is A has the following stability property: whenever $(|s_\lambda|)_\lambda \leq (r_\lambda)_\lambda$ (i.e. for any λ, $|s_\lambda| \leq r_\lambda$) for any pair $((s_\lambda)_\lambda,(r_\lambda)_\lambda) \in \mathbb{K}^{\Lambda} \times |A|$, it follows that $(s_\lambda)_\lambda \in A$, with $|A| = \{(|r_\lambda|)_\lambda : (r_\lambda)_\lambda \in A\}$ and I_A a solid ideal of A with the same property;

(3) $\mathcal{E}$ a sheaf of $\mathbb{K}$-topological algebras on a topological space X, such that for any open set Ω in X, the algebra $\mathcal{E}(\Omega)$ is endowed with a family $\mathcal{P}(\Omega) = (p_i)_{i\in I(\Omega)}$ of seminorms satisfying

$$\forall i \in I(\Omega), \exists (j,k,C) \in (I(\Omega))^2 \times \mathbb{R}_+^*, \forall f,g \in \mathcal{E}(\Omega) : p_i(fg) \leq C p_j(f) p_k(g).$$

Assume that

(4) For any two open subsets Ω_1, Ω_2 of X such that $\Omega_1 \subset \Omega_2$, we have $I(\Omega_1) \subset I(\Omega_2)$ and if ρ_1^2 is the restriction operator $\mathcal{E}(\Omega_2) \to \mathcal{E}(\Omega_1)$, then, for each $p_i \in \mathcal{P}(\Omega_1)$, the seminorm $\widetilde{p}_i = p_i \circ \rho_1^2$ extends p_i to $\mathcal{P}(\Omega_2)$;

(5) For any family $\mathcal{F} = (\Omega_h)_{h\in H}$ of open subsets of X if $\Omega = \cup_{h\in H}\Omega_h$, then, for each $p_i \in \mathcal{P}(\Omega)$, $i \in I(\Omega)$, there exists a finite subfamily $(\Omega_j)_{1\leq j\leq n(i)}$ of $\mathcal{F}$ and corresponding seminorms $p_j \in \mathcal{P}(\Omega_j)$, $1 \leq j \leq n\,(i)$, such that, for each $u \in \mathcal{E}(\Omega)$, $p_i\,(u) \leq \sum\limits_{j=1}^{j=n(i)} p_1(u_{|\Omega_j})$.

Set $\mathcal{C} = A/I_A$ and

$$\mathcal{X}_{(A,\mathcal{E},\mathcal{P})}(\Omega) = \{(u_\lambda)_\lambda \in [\mathcal{E}(\Omega)]^{\Lambda} : \forall i \in I(\Omega),\ ((p_i(u_\lambda))_\lambda \in |A|\},$$
$$\mathcal{N}_{(I_A,\mathcal{E},\mathcal{P})}(\Omega) = \{(u_\lambda)_\lambda \in [\mathcal{E}(\Omega)]^{\Lambda} : \forall i \in I(\Omega),\ (p_i(u_\lambda))_\lambda \in |I_A|\}.$$

One can prove that $\mathcal{X}_{(A,\mathcal{E},\mathcal{P})}$ is a sheaf of subalgebras of the sheaf $\mathcal{E}^{\Lambda}$ and $\mathcal{N}_{(I_A,\mathcal{E},\mathcal{P})}$ is a sheaf of ideals of $\mathcal{X}_{(A,\mathcal{E},\mathcal{P})}$ [32]. Moreover, the constant sheaf $\mathcal{X}_{(A,\mathbb{K},|.|)}/\mathcal{N}_{(I_A,\mathbb{K},|.|)}$ is exactly the sheaf $\mathcal{C} = A/I_A$, and if $\mathbb{K}=\mathbb{R}$, $\mathcal{C}$ will be denoted $\overline{\mathbb{R}}$. We call presheaf of $(\mathcal{C},\mathcal{E},\mathcal{P})$-algebra the factor presheaf of algebras $\mathcal{A} = \mathcal{X}_{(A,\mathcal{E},\mathcal{P})}/\mathcal{N}_{(I_A,\mathcal{E},\mathcal{P})}$ over the ring $\mathcal{C} = A/I_A$. We denote by $[u_\lambda]$ the class in $\mathcal{A}(\Omega)$ defined by the representative $(u_\lambda)_{\lambda\in\Lambda} \in \mathcal{X}_{(A,\mathcal{E},\mathcal{P})}(\Omega)$.

Remark 2.1. (Overgenerated rings) Let $B_p = \left\{(r_{n,\lambda})_\lambda \in (\mathbb{R}_+^*)^{\Lambda} : 1 \leq n \leq p\right\}$ and B be the subset of $(\mathbb{R}_+^*)^{\Lambda}$ obtained as rational functions with coefficients in $\mathbb{R}_+^*$, of elements in B_p as variables. Define

$$A = \left\{(a_\lambda)_\lambda \in \mathbb{K}^{\Lambda} \mid \exists\,(b_\lambda)_\lambda \in B, \exists \lambda_0 \in \Lambda, \forall \lambda \prec \lambda_0 : |a_\lambda| \leq b_\lambda\right\},$$

we say that A is *overgenerated* by B_p (and it is easy to see that A is a solid subring of $\mathbb{K}^{\Lambda}$). If I_A is some solid ideal of A, we also say that $\mathcal{C} = A/I_A$ is *overgenerated* by B_p, [18].

Remark 2.2. (Relationship with distribution theory) Let Ω be an open subset of $\mathbb{R}^n$. The space of distributions $\mathcal{D}'(\Omega)$ can be embedded into $\mathcal{A}(\Omega)$. If $(\varphi_\lambda)_{\lambda\in(0,1]}$ is a family of mollifiers $\varphi_\lambda\,(x) = \lambda^{-n}\varphi\,(x/\lambda)$, $x \in \mathbb{R}^n$, $\int \varphi\,(x)\,dx = 1$ and if $T \in \mathcal{D}'\,(\mathbb{R}^n)$, the convolution product family $(T * \varphi_\lambda)_\lambda$ is a family of smooth functions slowly increasing in $1/\lambda$. So, for $\Lambda = (0,1]$, we shall choose the subring A overgenerated by some B_p of $(\mathbb{R}_+^*)^{\Lambda}$ containing the family $(\lambda)_\lambda$, [13, 33]. We choose a special kind of mollifiers which moments of higher order vanish.

Remark 2.3. (An association process) Let Ω be an open subset of X, E be a given sheaf of topological $\mathbb{K}$-vector spaces containing $\mathcal{E}$ as a subsheaf, a be a given map from Λ to $\mathbb{K}$ such that $(a(\lambda))_\lambda = (a_\lambda)_\lambda$ is an element of A. We also assume that

$$\mathcal{N}_{(I_A,\mathcal{E},\mathcal{P})}(\Omega) \subset \left\{ (u_\lambda)_\lambda \in \mathcal{X}_{(A,\mathcal{E},\mathcal{P})}(\Omega) : \lim_{E(\Omega),\Lambda} u_\lambda = 0 \right\}.$$

We say that $u = [u_\lambda]$ and $v = [v_\lambda] \in \mathcal{E}(\Omega)$ are a-E associated if $\lim\limits_{E(\Omega),\Lambda} a_\lambda(u_\lambda - v_\lambda) = 0$. *That is to say, for each neighborhood V of* 0 *for the E-topology, there exists $\lambda_0 \in \Lambda$ such that $\lambda \prec \lambda_0 \Longrightarrow a_\lambda(u_\lambda - v_\lambda) \in V$.* We write $u \underset{E(\Omega)}{\overset{a}{\sim}} v$. We can also define an association process between $u = [u_\lambda]$ and $T \in E(\Omega)$ by writing simply $u \sim T \Longleftrightarrow \lim\limits_{E(\Omega),\Lambda} u_\lambda = T$. Taking $E = \mathcal{D}'$, $\mathcal{E} = \mathrm{C}^\infty$, $\Lambda = (0,1]$, we recover the association process defined in the literature [11, 12].

2.2. Algebraic framework

Set $\mathcal{E} = \mathrm{C}^\infty$, $X = \mathbb{R}^d$ for $d = 1, 2$, $E = \mathcal{D}'$ and Λ a set of indices, $\lambda \in \Lambda$. For any open set Ω, in $\mathbb{R}^d$, $\mathcal{E}(\Omega)$ is endowed with the $\mathcal{P}(\Omega)$ topology of uniform convergence of all derivatives on compact subsets of Ω. This topology may be defined by the family of the seminorms $P_{K,l}(u_\lambda) = \sup_{|\alpha| \le l} P_{K,\alpha}(u_\lambda)$ with $P_{K,\alpha}(u_\lambda) = \sup_{x \in K} |D^\alpha u_\lambda(x)|$, $K \Subset \Omega$, where the notation $K \Subset \mathbb{R}^2$ means that K is a compact subset of $\mathbb{R}^2$ and $l \in \mathbb{N}$, $\alpha \in \mathbb{N}^d$.

Let A be a subring of the ring $\mathbb{R}^\Lambda$ of family of reals with the usual laws. We consider a solid ideal I_A of A. Then we have

$$\begin{aligned} &\mathcal{X}(\Omega) = \{(u_\lambda)_\lambda \in [\mathrm{C}^\infty(\Omega)]^\Lambda : \forall K \Subset \Omega, \forall l \in \mathbb{N},\ (P_{K,l}(u_\lambda))_\lambda \in |A|\},\\ &\mathcal{N}(\Omega) = \{(u_\lambda)_\lambda \in [\mathrm{C}^\infty(\Omega)]^\Lambda : \forall K \Subset \Omega, \forall l \in \mathbb{N},\ (P_{K,l}(u_\lambda))_\lambda \in |I_A|\},\\ &\mathcal{A}(\Omega) = \mathcal{X}(\Omega)/\mathcal{N}(\Omega). \end{aligned}$$

The generalized derivation $D^\alpha : u(= [u_\varepsilon]) \mapsto D^\alpha u = [D^\alpha u_\varepsilon]$ provides $\mathcal{A}(\Omega)$ with a differential algebraic structure [19].

Example 2.4. Set $\Lambda = (0,1]$. Consider

$$A = \left\{ (m_\lambda)_\lambda \in \mathbb{R}^\Lambda : \exists p \in \mathbb{R}_+^*, \exists C \in \mathbb{R}_+^*, \exists \mu \in (0,1], \forall \lambda \in (0,\mu], |m_\lambda| \le C\lambda^{-p} \right\},$$

$$I_A = \left\{ (m_\lambda)_\lambda \in \mathbb{R}^\Lambda : \forall q \in \mathbb{R}_+^*, \exists D \in \mathbb{R}_+^*, \exists \mu \in (0,1], \forall \lambda \in (0,\mu], |m_\lambda| \le D\lambda^{q} \right\}.$$

In this case we denote $\mathcal{X}^s(\Omega) = \mathcal{X}(\Omega)$ and $\mathcal{N}^s(\Omega) = \mathcal{N}(\Omega)$. The sheaf of factor algebras $\mathcal{G}^s(\cdot) = \mathcal{X}^s(\cdot)/\mathcal{N}^s(\cdot)$ is called the sheaf of simplified Colombeau algebras. $\mathcal{G}^s\left(\mathbb{R}^d\right)$ is the simplified Colombeau algebra of generalized functions [11], [12].

We have the analogue of theorem 1.2.3. of [27] for $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-algebras.

Proposition 2.5. *Let B be the set introduced in 2.1 and assume that there exists $(a_\lambda)_\lambda \in B$ with $\lim\limits_{\lambda \to 0} a_\lambda = 0$. Consider $(u_\lambda)_\lambda \in \mathcal{X}(\mathbb{R}^2)$ such that: $\forall K \Subset \mathbb{R}^2$, $(P_{K,0}(u_\lambda))_\lambda \in |I_A|$. Then $(u_\lambda)_\lambda \in \mathcal{N}(\mathbb{R}^2)$.*

We refer the reader to [18] and [14]. By the way, update ref. [18] if available.

Definition 2.6. Tempered generalized functions, [27], [34], [35]. For $f \in \mathrm{C}^\infty(\mathbb{R}^n)$, $r \in \mathbb{Z}$ and $m \in \mathbb{N}$, we put $\mu_{r,m}(f) = \sup_{x\in\mathbb{R}^n, |\alpha|\leq m}(1+|x|)^r \left|\mathcal{D}^\alpha f(x)\right|$. The space of functions with slow growth is

$$\mathcal{O}_M(\mathbb{R}^n) = \left\{f \in \mathrm{C}^\infty(\mathbb{R}^n) : \forall m \in \mathbb{N}, \exists q \in \mathbb{N}, \mu_{-q,m}(f) < +\infty\right\}.$$

Definition 2.7. We define

$$\begin{aligned}
&\mathcal{X}_\tau\left(\mathbb{R}^n\right) = \{(f_\varepsilon)_\varepsilon \in \mathcal{O}_M(\mathbb{R}^n)^{(0,1]} : \forall m \in \mathbb{N}, \exists q \in \mathbb{N},\\
&\qquad \exists N \in \mathbb{N}, \mu_{-q,m}(f_\varepsilon) = O(\varepsilon^{-N}); \varepsilon \to 0\},\\
&\mathcal{N}_\tau\left(\mathbb{R}^n\right) =\\
&\quad \{(f_\varepsilon)_\varepsilon \in \mathcal{O}_M(\mathbb{R}^n)^{(0,1]} : \forall m \in \mathbb{N}, \exists q \in \mathbb{N}, \forall p \in \mathbb{N}, \mu_{-q,m}(f_\varepsilon) = O(\varepsilon^{p}); \varepsilon \to 0\}.
\end{aligned}$$

$\mathcal{X}_\tau\left(\mathbb{R}^n\right)$ is a subalgebra of $\mathcal{O}_M(\mathbb{R}^n)^{(0,1]}$ and $\mathcal{N}_\tau\left(\mathbb{R}^n\right)$ an ideal of $\mathcal{X}_\tau\left(\mathbb{R}^n\right)$. The algebra $\mathcal{G}_\tau\left(\mathbb{R}^n\right) = \mathcal{X}_\tau\left(\mathbb{R}^n\right)/\mathcal{N}_\tau\left(\mathbb{R}^n\right)$ is called the algebra of tempered generalized functions.

The generalized derivation $\mathcal{D}^\alpha : u = [u_\varepsilon] \mapsto \mathcal{D}^\alpha u = [\mathcal{D}^\alpha u_\varepsilon]$ provides $\mathcal{G}_\tau\left(\mathbb{R}^n\right)$ with a differential algebraic structure.

2.2.1. Generalized operators and general restrictions

Definition 2.8. *Let Ω be an open subset of $\mathbb{R}^2$ and $F \in \mathrm{C}^\infty(\Omega\times\mathbb{R},\mathbb{R})$. We say that the algebra $\mathcal{A}(\Omega)$ is stable under F if for all $(u_\varepsilon)_\varepsilon \in \mathcal{X}(\Omega)$ and $(i_\varepsilon)_\varepsilon \in \mathcal{N}(\Omega)$, we have $(F(\cdot,\cdot,u_\varepsilon))_\varepsilon \in \mathcal{X}(\Omega)$* and $(F(\cdot,\cdot,u_\varepsilon+i_\varepsilon) - F(\cdot,\cdot,u_\varepsilon))_\varepsilon \in \mathcal{N}(\Omega)$. If $\mathcal{A}\left(\mathbb{R}^2\right)$ if stable under F, the operator

$$\mathcal{F} : \mathcal{A}\left(\mathbb{R}^2\right) \to \mathcal{A}\left(\mathbb{R}^2\right), \quad u = [u_\varepsilon] \mapsto [F(.,.,u_\varepsilon)]$$

is called the *generalized operator associated to F*. See [18].

Proposition 2.9. *Let Ω be an open subset of $\mathbb{R}^2$ and $F \in \mathrm{C}^\infty(\Omega\times\mathbb{R},\mathbb{R})$. We say that F is* smoothly tempered *if the following two conditions are satisfied:*
(i) For each $K \Subset \mathbb{R}^2$, $l \in \mathbb{N}$ and $u \in \mathrm{C}^\infty(\Omega,\mathbb{R})$, there is a positive finite sequence $(C_j)_{0\leq i\leq l}$, such that $P_{K,l}(F(\cdot,\cdot,u)) \leq \sum\limits_{i=0}^{l} C_i \left(P_{K,l}(u)\right)^i$,
(ii) For each $K \Subset \mathbb{R}^2$, $l \in \mathbb{N}$, $u, v \in \mathrm{C}^\infty(\Omega,\mathbb{R})$, there is a positive finite sequence $(D_j)_{1\leq j\leq l}$, such that $P_{K,l}(F(\cdot,\cdot,v) - F(\cdot,\cdot,u)) \leq \sum\limits_{j=1}^{l} D_j \left(P_{K,l}(v-u)\right)^j$.
If F is smoothly tempered then $\mathcal{A}\left(\mathbb{R}^2\right)$ if stable under F.

Consider $(f_\varepsilon)_\varepsilon \in \mathrm{C}^\infty\left(\mathbb{R}\right)^\Lambda$. Set

$$R_\varepsilon : \mathrm{C}^\infty\left(\mathbb{R}^2\right) \to \mathrm{C}^\infty\left(\mathbb{R}\right), g \mapsto R_\varepsilon\left(g\right) \text{ with } R_\varepsilon\left(g\right) : \mathbb{R} \to \mathbb{R},\ t \mapsto g(t, f_\varepsilon(t)).$$

The family $(R_\varepsilon)_\varepsilon$ maps $\mathrm{C}^\infty\left(\mathbb{R}^2\right)^\Lambda$ into $\mathrm{C}^\infty\left(\mathbb{R}\right)^\Lambda$.

Definition 2.10. The family of smooth function $(f_\varepsilon)_\varepsilon$ is *compatible with second side restriction* if for all $(u_\varepsilon)_\varepsilon \in \mathcal{X}((\mathbb{R}^2)$ and all $(i_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R}^2)$, $(u_\varepsilon(\cdot, f_\varepsilon(\cdot)))_\varepsilon \in \mathcal{X}(\mathbb{R})$ and $(i_\varepsilon(\cdot, f_\varepsilon(\cdot)))_\varepsilon \in \mathcal{N}(\mathbb{R})$. The map

$$\mathcal{R} : \mathcal{A}\left(\mathbb{R}^2\right) \to \mathcal{A}\left(\mathbb{R}\right), \quad u = [u_\varepsilon] \mapsto [u_\varepsilon(\cdot, f_\varepsilon(\cdot))] = [R_\varepsilon(u_\varepsilon)]$$

is called the *generalized second side restriction mapping* associated to the family $(f_\varepsilon)_\varepsilon$.

Clearly, if $u = [u_\varepsilon] \in \mathcal{A}(\mathbb{R}^2)$ then $[u_\varepsilon(\cdot, f_\varepsilon(\cdot))]$ is a well defined element of $\mathcal{A}(\mathbb{R})$ (i.e. not depending on the representative of u).

Remark 2.11. The previous process generalizes the standard one defining the restriction of the generalized function $u = [u_\varepsilon] \in \mathcal{A}\left(\mathbb{R}^2\right)$ to a net of the manifolds $\{y = f(x)\}$ obtained when taking $f_\varepsilon = f$ for each $\varepsilon \in \Lambda$.

Definition 2.12. [27] Let $(f_\varepsilon)_\varepsilon$ be a family of $C^\infty(\mathbb{R}^n)$ functions. This family is c-bounded if for all compact set $K \subset \mathbb{R}^n$ there exists another compact set $L \subset \mathbb{R}^n$ such that $f_\varepsilon(K) \subset L$ for all ε (L is independent of ε).

Proposition 2.13. *Assume that for each $K \Subset \mathbb{R}$, there exists $K' \Subset \mathbb{R}$ such that, for all $\varepsilon \in \Lambda$, $f_\varepsilon(K) \subset K'$ and that $(f_\varepsilon)_\varepsilon$ belongs to $\mathcal{X}(\mathbb{R})$. Then the family $(f_\varepsilon)_\varepsilon$ is compatible with second side restriction.*

(See [3]).

3. Solutions for a characteristic Cauchy problem in $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-algebras

3.1. Assumptions

We will use the notations found in [20], [3]. Consider a family of smooth functions $(f_\varepsilon)_\varepsilon$ such that, for all ε, $f_\varepsilon \in \mathrm{C}^\infty(\mathbb{R})$, f_ε strictly increasing, $f_\varepsilon(\mathbb{R}) = \mathbb{R}$, for any $x \in \mathbb{R}$, $f'_\varepsilon(x) \neq 0$, $(f_\varepsilon)_\varepsilon, (f_\varepsilon^{-1})_\varepsilon \in \mathcal{X}_\tau(\mathbb{R})$, $(f_\varepsilon)_\varepsilon$ is c-bounded and $\lim\limits_{\varepsilon \underset{\mathcal{D}'(\mathbb{R})}{\to} 0} f_\varepsilon = f$. Consider the family of smooth non-characteristic curves γ_ε whose equation is $y = f_\varepsilon(x)$, such that γ_ε is diffeomorphic to γ. We approach the Cauchy problem (P_{form}) by a family of non-characteristic ones by replacing the characteristic curve γ by the family of smooth non-characteristic curves γ_ε.

Each compact set $K \Subset \mathbb{R}^2$ is contained in some product $[-a, a] \times [-b, b]$. We define

$$\begin{aligned} \beta_{K,\varepsilon} &= \max(a, f_\varepsilon^{-1}(b)),\ \alpha_{K,\varepsilon} = \min(-a, f_\varepsilon^{-1}(-b)), \\ a_{K,\varepsilon} &= 2\max(\beta_{K,\varepsilon}, |\alpha_{K,\varepsilon}|),\ K_\varepsilon = K_{1\varepsilon} \times K_2 \\ \text{with } K_{1\varepsilon} &= [-a_{K,\varepsilon}/2, a_{K,\varepsilon}/2] \text{ and } K_2 = [-b, b] = [-c/2, c/2]\,. \end{aligned}$$

Then, by construction we have $K \subset K_\varepsilon$. Set $F \in \mathrm{C}^\infty(\mathbb{R}^3, \mathbb{R})$. We make the following assumptions to generate a convenient $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-algebra to our problem.

$$(H_1)\begin{cases}\forall\varepsilon,\in(0,1],\forall K\Subset\mathbb{R}^2,\forall l\in\mathbb{N},\exists\mu_{K,l}>0,\exists M_\varepsilon>0,\\ \sup\limits_{(x,y,z)\in K_\varepsilon\times\mathbb{R},\ |\alpha|\le l}|D^\alpha F(x,y,z)|=M_{K,\varepsilon,l}\le\mu_{K,l}M_\varepsilon.\\ \forall\varepsilon\in(0,1],\forall K\Subset\mathbb{R}^2,\exists\nu_K>0,\exists a_\varepsilon>0,a_{K,\varepsilon}\le\nu_K a_\varepsilon.\end{cases}$$

$(H_2):\varphi,\psi\in O_M(\mathbb{R})$.

$$(H_3)\begin{cases}\mathcal{C}=A/I_A\text{ is overgenerated by the following elements of }\mathbb{R}_*^{(0,1]}\\ (\varepsilon)_\varepsilon,(M_\varepsilon)_\varepsilon,(\exp M_\varepsilon a_\varepsilon)_\varepsilon.\end{cases}$$

$$(H_4)\begin{cases}\mathcal{A}\left(\mathbb{R}^2\right)=\mathcal{X}(\mathbb{R}^2)/\mathcal{N}(\mathbb{R}^2)\text{ is built on }\mathcal{C};\\ (\mathcal{E},\mathcal{P})=(\mathrm{C}^\infty(\mathbb{R}^2),(P_{K,l})_{K\Subset\mathbb{R}^2,l\in\mathbb{N}}).\end{cases}$$

From hypothesis (H_1) it follows that $\mathcal{A}\left(\mathbb{R}^2\right)$ is stable under F relatively to $\mathcal{C}$ [21].

Remark 3.1. When $F=0$ and φ and ψ polynomials, hypotheses (H_1), (H_3) are obviously verified and $\mathcal{A}\left(\mathbb{R}^2\right)$ is simply the Colombeau algebra $\mathcal{G}\left(\mathbb{R}^2\right)$. Moreover, in general, the algebra $\mathcal{A}\left(\mathbb{R}^2\right)$ is not the Colombeau algebra even if only one parameter is used to de-characterize the problem. The $(\mathcal{C},\mathcal{E},\mathcal{P})$-algebra $\mathcal{A}\left(\mathbb{R}^2\right)$ can be of Colombeau-type (See [3]).

3.2. Well-posed problem

To the characteristic Cauchy problem formally written as (P_{form}) is associated the well formulated one

$$(P_g):\frac{\partial^2 u}{\partial x\partial y}=\mathcal{F}(u),\mathcal{R}\left(u\right)=\varphi,\mathcal{R}\left(\frac{\partial u}{\partial y}\right)=\psi$$

where u lies in the algebra $\mathcal{A}\left(\mathbb{R}^2\right)$ and $\mathcal{F}$, $\mathcal{R}$ are defined as previously by taking into account the family $(f_\varepsilon)_\varepsilon$ [3]. In terms of representatives solving (P_g) amounts finding a family $(u_\varepsilon)_\varepsilon\in\mathcal{X}(\mathbb{R}^2)$ such that

$$\begin{cases}\dfrac{\partial^2 u_\varepsilon}{\partial x\partial y}(x,y)-F(x,y,u_\varepsilon\left(x,y\right))=i_\varepsilon\left(x,y\right),\\ u_\varepsilon\left(x,f_\varepsilon(x)\right)-\varphi\left(x\right)=j_\varepsilon\left(x\right),\dfrac{\partial u_\varepsilon}{\partial y}\left(x,f_\varepsilon(x)\right)-\psi\left(x\right)=l_\varepsilon\left(x\right),\end{cases}$$

where $(i_\varepsilon)_\varepsilon\in\mathcal{N}\left(\mathbb{R}^2\right)$, $(j_\varepsilon)_\varepsilon$, $(l_\varepsilon)_\varepsilon\in\mathcal{N}\left(\mathbb{R}\right)$.

Suppose we can find $u_\varepsilon\in\mathrm{C}^\infty\left(\mathbb{R}^2\right)$ verifying

$$(P_\varepsilon)\begin{cases}\dfrac{\partial^2 u_\varepsilon}{\partial x\partial y}(x,y)=F(x,y,u_\varepsilon\left(x,y\right)),\\ u_\varepsilon\left(x,f_\varepsilon(x)\right)=\varphi\left(x\right),\dfrac{\partial u_\varepsilon}{\partial y}\left(x,f_\varepsilon(x)\right)=\psi\left(x\right),\end{cases}$$

then if we can prove that $(u_\varepsilon)_\varepsilon\in\mathcal{X}(\mathbb{R}^2)$, $u=[u_\varepsilon]$ is a solution of (P_g).

3.3. Solution to (P_g)

Theorem 3.1. *Set $F \in \mathrm{C}^\infty(\mathbb{R}^3,\mathbb{R})$. Assume that*

$$(H)\qquad \begin{cases} f_\varepsilon \in \mathrm{C}^\infty(\mathbb{R}),\ f_\varepsilon \text{ strictly increasing, } f_\varepsilon(\mathbb{R}) = \mathbb{R}, \\ \forall x \in \mathbb{R}, f'_\varepsilon(x) \neq 0, \\ \forall \varepsilon, \in (0,1], \forall K \Subset \mathbb{R}^2, \quad \sup\limits_{(x,y,z)\in K\times\mathbb{R}} |\partial_z F(x,y,z)| < \infty, \end{cases}$$

then the Problem (P_ε) has a unique solution u_ε in $\mathrm{C}^\infty(\mathbb{R}^2)$ which satisfies the following integral equation

$$(\mathrm{Int})\qquad u_\varepsilon(x,y) = u_{0,\varepsilon}(x,y) - \iint_{D(x,y,f_\varepsilon)} F(\xi,\eta,u_\varepsilon(\xi,\eta))d\xi d\eta,$$

with $u_{0,\varepsilon}(x,y) = \varphi(x) - \chi_\varepsilon\left(f_\varepsilon\left(x\right)\right) + \chi_\varepsilon\left(y\right)$, where χ_ε is a primitive of $\psi \circ f_\varepsilon^{-1}$ and

$$D(x,y,f_\varepsilon) = \begin{cases} \left\{(\xi,\eta)/x \leq \xi \leq f_\varepsilon^{-1}\left(y\right), f_\varepsilon(\xi) \leq \eta \leq y\right\}, \text{ if } y \geq f_\varepsilon\left(x\right), \\ \left\{(\xi,\eta)/f_\varepsilon^{-1}\left(y\right) \leq \xi \leq x, y \leq \eta \leq f_\varepsilon(\xi)\right\}, \text{ if } y \leq f_\varepsilon\left(x\right). \end{cases}$$

See [20] Section 1, for a detailed proof.

Theorem 3.2. *With the notations and the hypotheses of the above subsection, the generalized function $u \in \mathcal{A}(\mathbb{R}^2)$, represented by the family $\left(u_\varepsilon\right)_\varepsilon$ of solutions to Problems (P_ε), is a solution to the Problem (P_g) and does not depend on the choice of the representative $\left(f_\varepsilon\right)_\varepsilon$ of the class $f = [f_\varepsilon] \in \mathcal{G}_\tau\left(\mathbb{R}\right)$.*

See [3] for a detailed proof.

Remark 3.2. However, following a remark of Michael Oberguggenberger (private communication), we not have here uniqueness of the solution.

Example 3.3. Let us now treat the problem (P_{form}) with $F = 0$ and $\gamma = (Ox)$. Consider the non-characteristic problems $(P_\varepsilon)_\varepsilon$ with the line γ_ε of equation $y = f_\varepsilon\left(x\right) = \varepsilon x$. With the previous notations and hypotheses, $\mathcal{C} = A/I_A$ is overgenerated by $(\varepsilon)_\varepsilon$, element of $\mathbb{R}_*^{]0,1]}$. Moreover $\mathcal{A}\left(\mathbb{R}^2\right)$ is simply the Colombeau algebra $\mathcal{G}\left(\mathbb{R}^2\right)$. If u_ε is the solution to the problem (P_ε) then problem (P_g) admits $u = [u_\varepsilon]_{\mathcal{A}(\mathbb{R}^2)}$ as solution. The solution of (P_ε) has a priori the form $u_\varepsilon(x,y) = H(x) + G(y)$, with H and G in $\mathrm{C}^\infty(\mathbb{R})$. We can find them by

$$H(x) + G(\varepsilon x) = \varphi(x),\ G_y(\varepsilon x) = \psi(x).$$

Then

$$u_\varepsilon(x,y) = \varphi(x) - \varepsilon\int_0^x \psi(t)dt + \varepsilon\int_0^{\frac{y}{\varepsilon}} \psi(t)dt = \varphi(x) - \varepsilon\Psi(x) + \varepsilon\Psi\left(\frac{y}{\varepsilon}\right),$$

where we have set $\Psi = (x \mapsto \int_0^x \psi(t)dt)$. So, $u = [u_\varepsilon] = [\varepsilon u_2] + \left[\varepsilon^2 u_{3,\varepsilon}\right]$ with

$$u_1 = 1_y \otimes \varphi \in \mathrm{C}^\infty(\mathbb{R}^2); u_2 = -\left(1_y \otimes \Psi\right) \in \mathrm{C}^\infty(\mathbb{R}^2);$$
$$[u_{3,\varepsilon}] \sim \left(1_x \otimes \delta_y\right) = \delta_\gamma \in \mathcal{D}'\left(\mathbb{R}^2\right)$$

where $\sim$ means association in $\mathcal{D}'$ and δ_γ is the Dirac distribution of the characteristic line $\{y = 0\}$ [3].

4. The framework $\mathcal{G}_{\mathcal{O}_M}(\mathbb{R}^2)$ and uniqueness in the homogeneous case

The natural topology of $\mathcal{O}_M$ allows us to define a new algebra of tempered generalized function, $\mathcal{G}_{\mathcal{O}_M}\left(\mathbb{R}^d\right)$ (see [16]), which differs from $\mathcal{G}_\tau\left(\mathbb{R}^d\right)$ but still admits a point value characterization (see [36]). As $\mathcal{G}_{\mathcal{O}_M}\left(\mathbb{R}^d\right)$ is of $(\mathcal{C},\mathcal{E},\mathcal{P})$-type and endowed with the sharp topology (see [15]), our goal is to recover uniqueness of the solution of (P_g) in this context (see [2]).

4.1. Point values in $\mathcal{G}_{\mathcal{O}_M}\left(\mathbb{R}^d\right)$

We define $\mathcal{G}_{\mathcal{O}_M}\left(\mathbb{R}^d\right)$ as the quotient algebra $\mathcal{M}_{\mathcal{O}_M}\left(\mathbb{R}^d\right)/\mathcal{N}_{\mathcal{O}_M}\left(\mathbb{R}^d\right)$ where

$$\begin{aligned}\mathcal{M}_{\mathcal{O}_M}(\mathbb{R}^d) = \{&(u_\varepsilon)_\varepsilon \in \mathcal{O}_M(\mathbb{R}^d)^{(0,1]} : \forall\varphi\in\mathcal{S}(\mathbb{R}^d),\ \forall\alpha\in\mathbb{N}^d,\\ &\exists M\in\mathbb{N},\ \exists\varepsilon_0,\ \forall\varepsilon<\varepsilon_0,\ \sup_{x\in\mathbb{R}^d}\left|\varphi(x)\,\partial^\alpha u_\varepsilon(x)\right| \le \varepsilon^{-M}\}\ ;\\ \mathcal{N}_{\mathcal{O}_M}(\mathbb{R}^d) = \{&(u_\varepsilon)_\varepsilon \in \mathcal{O}_M(\mathbb{R}^d)^{(0,1]} : \forall\varphi\in\mathcal{S}(\mathbb{R}^d),\ \forall\alpha\in\mathbb{N}^d,\\ &\forall m\in\mathbb{N},\ \exists\varepsilon_0,\ \forall\varepsilon<\varepsilon_0,\ \sup_{x\in\mathbb{R}^d}\left|\varphi(x)\,\partial^\alpha u_\varepsilon(x)\right| \le \varepsilon^{m}\}.\end{aligned}$$

On one hand, we have $\mathcal{M}_{\mathcal{O}_M}\left(\mathbb{R}^d\right) = \mathcal{M}_\tau\left(\mathbb{R}^d\right)$ (see [16, Prop. 3.2]). However, we only have $\mathcal{N}_{\mathcal{O}_M}\left(\mathbb{R}^d\right) \supsetneq \mathcal{N}_\tau\left(\mathbb{R}^d\right)$. Thus $\mathcal{G}_{\mathcal{O}_M}\left(\mathbb{R}^d\right)$ differs from $\mathcal{G}_\tau\left(\mathbb{R}^d\right)$. On the other hand (see [16, Prop. 3.2]), we get

$$\begin{aligned}\mathcal{N}_{\mathcal{O}_M}(\mathbb{R}^d) = \{&(u_\varepsilon)_\varepsilon \in (\mathcal{O}_M(\mathbb{R}^d)^{(0,1]} : \forall\alpha\in\mathbb{N}^d,\ \forall m\in\mathbb{N},\\ &\exists p\in\mathbb{N},\ \exists\varepsilon_0,\ \forall\varepsilon<\varepsilon_0,\ \sup_{x\in\mathbb{R}^d}(1+|x|)^{-p}|\partial^\alpha u_\varepsilon(x)| \le \varepsilon^m\}.\end{aligned}$$

By the same Taylor argument as in (see [27, Thm. 1.2.25]), we find (see [2])

$$\begin{aligned}\mathcal{N}_{\mathcal{O}_M}(\mathbb{R}^d) = \{&(u_\varepsilon)_\varepsilon \in \mathcal{M}_\tau(\mathbb{R}^d) : \forall m\in\mathbb{N},\ \exists p\in\mathbb{N},\\ &\exists\varepsilon_0,\ \forall\varepsilon<\varepsilon_0,\ \sup_{x\in\mathbb{R}^d}(1+|x|)^{-p}|u_\varepsilon(x)| \le \varepsilon^m\}.\end{aligned}$$

We use the properties about generalized points and point values of $\mathcal{G}_{\mathcal{O}_M}\left(\mathbb{R}^d\right)$ presented in [2]. Let us recall that $\widetilde{\mathbb{K}} = \mathcal{M}_\mathbb{K}/\mathcal{N}_\mathbb{K}$ is the ring of Colombeau generalized numbers ($\mathbb{K}=\mathbb{R},\mathbb{C}$) and similarly $\widetilde{\mathbb{K}^d} = \widetilde{\mathbb{K}}^d$ the set of generalized points.

Definition 4.1. An element $\widetilde{x} = [(x_\varepsilon)_\varepsilon] \in \widetilde{\mathbb{R}}^d$ is of *slow scale* if for all $n\in\mathbb{N}$ there exists ε_0 such that, for all $\varepsilon<\varepsilon_0$, we have $|x_\varepsilon| \le \varepsilon^{-1/n}$.

Theorem 4.1. *Take $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}_{\mathcal{O}_M}(\mathbb{R}^d)$. From [2, theorem 5], we get that, for $\tilde{x} = [(x_\varepsilon)_\varepsilon]$ of slow scale, the point value $u(\tilde{x}) := [(u_\varepsilon(x_\varepsilon))_\varepsilon] \in \widetilde{\mathbb{K}}$ is well-defined. Moreover $u=0$ if and only if $u(\tilde{x})=0$ for each slow scale point $\tilde{x}$ [2, theorem 6].*

4.2. Uniqueness in the homogeneous case

Theorem 4.2. *Suppose that $(f_\varepsilon)_\varepsilon$ is taken in the subset $\mathcal{L}_{\mathcal{O}_M}(\mathbb{R})$ in $\mathcal{M}_{\mathcal{O}_M}(\mathbb{R})$ of families $(g_\varepsilon)_\varepsilon$ such that $g'_\varepsilon > 0$, $\left(g_\varepsilon^{-1}\right)_\varepsilon \in \mathcal{M}_{\mathcal{O}_M}(\mathbb{R})$, $(g_\varepsilon)_\varepsilon$ and $\left(g_\varepsilon^{-1}\right)_\varepsilon$ preserves slow scale points, $\lim_{\varepsilon\to 0,\mathcal{D}'(\mathbb{R})} g_\varepsilon = 0$. Then, if $\varphi\in\mathcal{O}_M(\mathbb{R})$, $\psi\in\mathcal{O}_M(\mathbb{R})$ and $F=0$, the solution $u = [u_\varepsilon]_{\mathcal{G}_{\mathcal{O}_M}(\mathbb{R}^2)}$ of (P_g) is unique in $\mathcal{G}_{\mathcal{O}_M}\left(\mathbb{R}^2\right)$ and depends only on $f = [f_\varepsilon]_{\mathcal{G}_{\mathcal{O}_M}(\mathbb{R})}$.*

Proof. We have

$$\frac{\partial^2 u_\varepsilon}{\partial x \partial y}(x,y) = 0, u_\varepsilon\left(x, f_\varepsilon(x)\right) = \varphi(x), \frac{\partial u_\varepsilon}{\partial y}\left(x, f_\varepsilon(x)\right) = \psi(x).$$

Then

$$u_\varepsilon(x,y) = u_{0,\varepsilon}(x,y) = \varphi(x) - \chi_\varepsilon\left(f_\varepsilon\left(x\right)\right) + \chi_\varepsilon\left(y\right),$$

where χ_ε is a primitive of $\psi \circ f_\varepsilon^{-1}$. Let $u = [u_\varepsilon]$ (with $(u_\varepsilon) \in \mathcal{M}_{\mathcal{O}_M}(\mathbb{R}^2)$) be the solution of (P_g) obtained in Theorem 3.2. Let $v = [v_\varepsilon]$ (with $(v_\varepsilon) \in \mathcal{M}_{\mathcal{O}_M}(\mathbb{R}^2)$) be another solution to (P_g). There are $(i_\varepsilon)_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}(\mathbb{R}^2)$, $(\alpha_\varepsilon)_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}(\mathbb{R})$, $(\beta_\varepsilon)_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}(\mathbb{R})$, such that

$$(P_\varepsilon)\begin{cases} \dfrac{\partial^2 v_\varepsilon}{\partial x \partial y}(x,y) = i_\varepsilon(x,y), \\ v_\varepsilon\left(x, f_\varepsilon(x)\right) = \varphi(x) + \alpha_\varepsilon(x), \dfrac{\partial v_\varepsilon}{\partial y}\left(x, f_\varepsilon(x)\right) = \psi(x) + \beta_\varepsilon(x). \end{cases}$$

The uniqueness of the solution to (P_g) will be the consequence of $(v_\varepsilon - u_\varepsilon)_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}(\mathbb{R}^2)$. We have $v_{0,\varepsilon}(x,y) = u_{0,\varepsilon}(x,y) + \theta_\varepsilon(x,y)$, where

$$\theta_\varepsilon(x,y) = \alpha_\varepsilon(x) - B_\varepsilon(f_\varepsilon\left(x\right)) + B_\varepsilon(y)$$

and B_ε is a primitive of $\beta_\varepsilon \circ f_\varepsilon^{-1}$. Furthermore, as $(\beta_\varepsilon)_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}(\mathbb{R})$, $(f_\varepsilon)_\varepsilon$ is taken in $\mathcal{L}_{\mathcal{O}_M}(\mathbb{R})$ (it preserves slow scale points), then $\left(\beta_\varepsilon \circ f_\varepsilon^{-1}\right)_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}(\mathbb{R})$ and $(B_\varepsilon)_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}(\mathbb{R})$. As $(\alpha_\varepsilon)_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}(\mathbb{R})$, so $(\theta_\varepsilon)_\varepsilon$ belongs to $\mathcal{N}_{\mathcal{O}_M}(\mathbb{R}^2)$. We have

$$v_\varepsilon(x,y) = v_{0,\varepsilon}(x,y) + \iint_{D(x,y,f)} i_\varepsilon(\xi,\eta)d\xi d\eta.$$

We set, for all ε,

$$j_\varepsilon\left(x,y\right) = \iint_{D(x,y,f)} i_\varepsilon(\xi,\eta)d\xi d\eta.$$

Now we have to check that $(j_\varepsilon)_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}\left(\mathbb{R}^2\right)$. Let $[(x_\varepsilon,y_\varepsilon)_\varepsilon] \in \widetilde{\mathbb{R}}^2$ be a slow scale point. Then $[(x_\varepsilon)_\varepsilon] \in \widetilde{\mathbb{R}}$, $[(z_\varepsilon)_\varepsilon] = [(f_\varepsilon\left(x_\varepsilon\right))_\varepsilon] \in \widetilde{\mathbb{R}}$ and $\left[\left(f_\varepsilon^{-1}\left(y_\varepsilon\right)\right)_\varepsilon\right] \in \widetilde{\mathbb{R}}$ are also slow scale points.

We define $\mathrm{A}_\varepsilon(x_\varepsilon,y_\varepsilon) =$ area of $D(x_\varepsilon,y_\varepsilon,f_\varepsilon)$. According to the mean value theorem, for each ε, there exists $(c_\varepsilon,d_\varepsilon) \in D(x_\varepsilon,y_\varepsilon,f_\varepsilon)$ such that

$$\left|j_\varepsilon\left(x_\varepsilon,y_\varepsilon\right)\right| = \left|\iint_{D(x_\varepsilon,y_\varepsilon,f_\varepsilon)} i_\varepsilon(\xi,\eta))d\xi d\eta\right| = \left|i_\varepsilon(c_\varepsilon,d_\varepsilon)\right| \mathrm{A}_\varepsilon(x_\varepsilon,y_\varepsilon).$$

Where $i_\varepsilon(c_\varepsilon,d_\varepsilon)$ is the average value of i_ε on $D(x_\varepsilon,y_\varepsilon,f_\varepsilon)$. As

$$\left|c_\varepsilon\right| \leq \max(\left|x_\varepsilon\right|, \left|f_\varepsilon^{-1}\left(y_\varepsilon\right)\right|), \left|d_\varepsilon\right|) \leq \max(\left|f_\varepsilon\left(x_\varepsilon\right)\right|, \left|y_\varepsilon\right|)$$

then $[(c_\varepsilon)_\varepsilon]$ and $[(d_\varepsilon)_\varepsilon]$ are slow scale points. Thus $[(c_\varepsilon,d_\varepsilon)_\varepsilon]$ is a slow scale point of $\widetilde{\mathbb{R}}^2$. As $(i_\varepsilon)_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}\left(\mathbb{R}^2\right)$ we obtain that $(i_\varepsilon(c_\varepsilon,d_\varepsilon))_\varepsilon \in \mathcal{N}_{\mathbb{R}}$ and $(j_\varepsilon)_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}\left(\mathbb{R}^2\right)$. Thus there is $(\sigma_\varepsilon)_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}\left(\mathbb{R}^2\right)$ $(\sigma_\varepsilon = \theta_\varepsilon + j_\varepsilon)$ such that $v_\varepsilon = u_\varepsilon + \sigma_\varepsilon$. $\square$

Remark 4.2. However, if $F \neq 0$, we cannot prove the existence of a solution to (P_g) in $\mathcal{G}_{\mathcal{O}_M}\left(\mathbb{R}^2\right)$ as can be seen by taking $F(x,y,u(x,y)=-4xyu(x,y)$, $\varphi(x)=1$, $\psi(x)=-2x$. Thus we have

$$(P_\varepsilon): \frac{\partial^2 u_\varepsilon}{\partial x \partial y}(t,x) = -4xyu_\varepsilon(x,y), u_\varepsilon(x,\varepsilon x)=1, \frac{\partial u_\varepsilon}{\partial y}(x,\varepsilon x) = -2x.$$

The solution to (P_ε) is given by $u_\varepsilon(x,y) = \exp(-y^2/\varepsilon)\exp(\varepsilon x^2)$ which does not belong to $\mathcal{M}_{\mathcal{O}_M}\left(\mathbb{R}^2\right)$.

This remark shows that in order to prove the existence and the uniqueness to the characteristic Cauchy problem for the wave equation in canonical form we need another strategy. We may look for solution defined for large but finite x.

5. Uniqueness for the non-homogeneous problem

5.1. The framework of $\mathcal{G}_{\mathcal{O}_M}([0,T]\times[0,\infty[)$

5.1.1. The algebra $\mathcal{G}_{\mathcal{O}_M}\left(\overline{\Omega}\right)$

Thanks to the results of H. A. Biagioni [9], J. Aragona [6], [7] and J-A. Marti (private communication: Generalized functions on the closure of an open set) and using the study of [23], we can define Colombeau spaces on the closure $\overline{\Omega}$ of an open set $\Omega \subset \mathbb{R}^n$, such that $O \subset \Omega \subset \overline{O}$, where O is an open subset of $\mathbb{R}^n$ and $\overline{O}$ the closure of O.

We can easily define $\mathrm{C}^\infty(\overline{\Omega})$ as the space of restrictions to $\overline{\Omega}$ of functions in $\mathrm{C}^\infty(O)$ for any open set $O \supset \overline{\Omega}$. C^∞ being a sheaf, the definition is independent of the choice of O. The usual topology of $\mathrm{C}^\infty(\overline{\Omega})$ involve the family of compact set $K \Subset \overline{\Omega}$. Define now

$$\begin{aligned}
\mathcal{O}_M\left(\overline{\Omega}\right) &= \left\{f \in \mathrm{C}^\infty(\overline{\Omega}), \exists g \in \mathcal{O}_M(O), O \supset \overline{\Omega}, f = g\mid_{\overline{\Omega}}\right\} \\
\mathcal{S}\left(\overline{\Omega}\right) &= \left\{f \in \mathrm{C}^\infty(\overline{\Omega}), \exists g \in \mathcal{S}(O), O \supset \overline{\Omega}, f = g\mid_{\overline{\Omega}}\right\}.
\end{aligned}$$

The function $p_{\varphi,\alpha} : \mathcal{O}_M\left(\overline{\Omega}\right) \to \mathbb{R}_+$

$$f \mapsto p_{\varphi,\alpha}(f) = \sup_{x\in\overline{\Omega}} \left|\varphi(x)\,\mathcal{D}^\alpha f(x)\right|,$$

where $\varphi \in \mathcal{S}\left(\overline{\Omega}\right)$ and $\alpha \in \mathbb{N}^n$, is a semi-norm on $\mathcal{O}_M\left(\overline{\Omega}\right)$ and the family $\mathcal{P} = \left(p_{\varphi,\alpha}\right)_{\varphi,\alpha \in \mathcal{S}\left(\overline{\Omega}\right)\times\mathbb{N}^n}$ endows the algebra $\mathcal{O}_M\left(\overline{\Omega}\right)$ with a locally convex topology. Then, we can define $\mathcal{G}_{\mathcal{O}_M}\left(\overline{\Omega}\right)$ as the quotient algebra $\mathcal{M}_{\mathcal{O}_M}\left(\overline{\Omega}\right)/\mathcal{N}_{\mathcal{O}_M}\left(\overline{\Omega}\right)$ where

$$\begin{aligned}
\mathcal{M}_{\mathcal{O}_M}(\overline{\Omega}) = \{&(u_\varepsilon)_\varepsilon \in \mathcal{O}_M(\overline{\Omega})^{(0,1]} : (\forall\varphi \in \mathcal{S}(\overline{\Omega}))\ (\forall\alpha \in \mathbb{N}^n) \\
&(\exists M \in \mathbb{N})\ (\exists\varepsilon_0)\ (\forall\varepsilon < \varepsilon_0)\ (p_{\varphi,\alpha}(u_\varepsilon) \le \varepsilon^{-M})\}\ ; \\
\mathcal{N}_{\mathcal{O}_M}(\overline{\Omega}) = \{&(u_\varepsilon)_\varepsilon \in \mathcal{O}_M(\overline{\Omega})^{(0,1]} : (\forall\varphi \in \mathcal{S}(\overline{\Omega}))\ (\forall\alpha \in \mathbb{N}^n) \\
&(\forall m \in \mathbb{N})\ (\exists\varepsilon_0)\ (\forall\varepsilon < \varepsilon_0)\ (p_{\varphi,\alpha}(u_\varepsilon) \le \varepsilon^{m})\}.
\end{aligned}$$

This definition is consistent. We can involve, for example, the framework of $(\mathcal{C},\mathcal{E},\mathcal{P})$-algebra with $\mathcal{E} = \mathcal{O}_M(\overline{\Omega})$, $\mathcal{P} = (p_{\varphi,\alpha})$ and $\mathcal{C}$ generated by $(\varepsilon)_\varepsilon$.

5.1.2. Point values in $\mathcal{G}_{\mathcal{O}_M}\left(\mathbb{R}_+^d\right)$

On
$$\overline{\Omega}_M = \left\{ (x_\varepsilon)_\varepsilon \in \left(\overline{\Omega}\right)^{(0,1]} (\exists N \in \mathbb{N}) \left(|x_\varepsilon| = O\left(\varepsilon^{-N}\right)\right) (\varepsilon \to 0) \right\}$$
we introduce an equivalence relation by
$$(x_\varepsilon)_\varepsilon \sim (y_\varepsilon)_\varepsilon \Longleftrightarrow (\forall m \in \mathbb{N}) \left(|x_\varepsilon - y_\varepsilon| = O\left(\varepsilon^{-N}\right)\right) (\varepsilon \to 0)$$
and denote by $\widetilde{\overline{\Omega}} := \overline{\Omega}_M / \sim$ the set of generalized points in $\overline{\Omega}$.

We restrict to the case where $\overline{\Omega}$ is a box, i.e. a Cartesian product of closed (bounded or unbounded) intervals.

$$\begin{aligned} &\mathcal{M}_{\mathcal{O}_M}(\overline{\Omega}) = \\ &\quad \{(u_\varepsilon)_\varepsilon \in \mathcal{O}_M(\overline{\Omega})^{(0,1]} \mid (\forall \alpha \in \mathbb{N}^d)\ (\exists m \in \mathbb{N})\ (\exists p \in \mathbb{N}) \\ &\qquad (\exists \varepsilon_0)\ (\forall \varepsilon < \varepsilon_0)\ (\sup\nolimits_{x \in \overline{\Omega}} (1+|x|)^{-p} |\partial^\alpha u_\varepsilon(x)| \le \varepsilon^{-m})\}. \\ &\mathcal{N}_{\mathcal{O}_M}(\overline{\Omega}) = \\ &\quad \{(u_\varepsilon)_\varepsilon \in \mathcal{O}_M(\overline{\Omega})^{(0,1]} \mid (\forall \alpha \in \mathbb{N}^d)\ (\forall m \in \mathbb{N})\ (\exists p \in \mathbb{N}) \\ &\qquad (\exists \varepsilon_0)\ (\forall \varepsilon < \varepsilon_0)\ (\sup\nolimits_{x \in \overline{\Omega}} (1+|x|)^{-p} |\partial^\alpha u_\varepsilon(x)| \le \varepsilon^{m})\}. \end{aligned}$$

Moreover, we have (see [23])
$$\begin{aligned} \mathcal{N}_{\mathcal{O}_M}(\overline{\Omega}) = \{(u_\varepsilon)_\varepsilon \in \mathcal{O}_M(\overline{\Omega})^{(0,1]} &\mid (\forall m \in \mathbb{N})\ (\exists p \in \mathbb{N}) \\ &(\exists \varepsilon_0)\ (\forall \varepsilon < \varepsilon_0)\ (\sup\nolimits_{x \in \overline{\Omega}} (1+|x|)^{-p} |u_\varepsilon(x)| \le \varepsilon^{m})\}. \end{aligned}$$

Thus
$$\begin{aligned} \mathcal{N}_{\mathcal{O}_M}(\mathbb{R}_+^d) = \{(u_\varepsilon)_\varepsilon \in \mathcal{M}_\tau(\mathbb{R}_+^d) &\mid (\forall m \in \mathbb{N})\ (\exists p \in \mathbb{N}) \\ &(\exists \varepsilon_0)\ (\forall \varepsilon < \varepsilon_0)\ (\sup\nolimits_{x \in \mathbb{R}^d} (1+|x|)^{-p} |u_\varepsilon(x)| \le \varepsilon^{m})\}. \end{aligned}$$

Replacing $\mathbb{R}^2$ by $\mathbb{R}_+^2$ we obtain analogous definitions and theorems as in 2.2. Thus $\mathcal{F}$ can be defined as a mapping of $\mathcal{G}_{\mathcal{O}_M}([0,T] \times [0,\infty[)$ into itself and $\mathcal{R}$ as a mapping $\mathcal{G}_{\mathcal{O}_M}([0,T] \times [0,\infty[) \longrightarrow \mathcal{G}_{\mathcal{O}_M}([0,\infty[)$ then problem (P_g) is correctly formulated.

Replacing $\mathbb{R}^d$ by $\mathbb{R}_+^d$ we obtain also analogous definitions and theorems as in 4.1.

We recall two technical lemmas, the proof of the first one being a simple adaptation of [27, Thm 1.2.29].

Lemma 5.1. *Let $(f_\varepsilon), (g_\varepsilon), (\tilde{f}_\varepsilon), (\tilde{g}_\varepsilon) \in \mathcal{M}_{\mathcal{O}_M}(\mathbb{R}_+)$ be such that $[f_\varepsilon] = [\tilde{f}_\varepsilon]$ and $[g_\varepsilon] = [\tilde{g}_\varepsilon]$. We have that $[f_\varepsilon \circ g_\varepsilon] = [f_\varepsilon \circ \tilde{g}_\varepsilon]$. If moreover $[g_\varepsilon]$ preserves slow scale points then $[\tilde{f}_\varepsilon \circ g_\varepsilon] = [f_\varepsilon \circ g_\varepsilon]$.*

Lemma 5.2. *Let $(f_\varepsilon)_\varepsilon, (g_\varepsilon)_\varepsilon \in \mathcal{M}_{\mathcal{O}_M}(\mathbb{R}_+)$ be such that f_ε and g_ε are bijective, $(f_\varepsilon - g_\varepsilon)_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}(\mathbb{R}_+)$ and $(f_\varepsilon^{-1})_\varepsilon, (g_\varepsilon^{-1})_\varepsilon \in \mathcal{M}_{\mathcal{O}_M}(\mathbb{R}_+)$. Suppose moreover that $\left[g_\varepsilon^{-1}\right]$ preserves slow scale points. Then $(f_\varepsilon^{-1} - g_\varepsilon^{-1})_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}(\mathbb{R}_+)$.*

Proof. We have $(f_\varepsilon^{-1} - g_\varepsilon^{-1}) \circ g_\varepsilon = f_\varepsilon^{-1} \circ g_\varepsilon - Id \in \mathcal{N}_{\mathcal{O}_M}(\mathbb{R}_+)$ because $g_\varepsilon - f_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}(\mathbb{R}_+)$ which implies that $[f_\varepsilon^{-1} \circ g_\varepsilon] = [f_\varepsilon^{-1} \circ f_\varepsilon] = [Id]$. But, as $f_\varepsilon^{-1} - g_\varepsilon^{-1} = \left((f_\varepsilon^{-1} - g_\varepsilon^{-1}) \circ g_\varepsilon\right) \circ g_\varepsilon^{-1}$ and $\left[g_\varepsilon^{-1}\right] \in \mathcal{G}_{OM}(\mathbb{R}_+)$ preserves slow scale points, using the preceding lemma we find that $f_\varepsilon^{-1} - g_\varepsilon^{-1} \in \mathcal{N}_{\mathcal{O}_M}(\mathbb{R}_+)$. $\square$

5.2. The non-homogeneous case

Consider the well formulated generalized problem

$$(P_g) : \frac{\partial^2 u}{\partial x \partial y} = \mathcal{F}(u), \mathcal{R}(u) = \varphi, \mathcal{R}\left(\frac{\partial u}{\partial y}\right) = \psi,$$

where $\mathcal{F}$, $\mathcal{R}$ are defined as previously by taking into account a family $(l_\varepsilon)_\varepsilon$, $\varphi \in \mathcal{O}_M(\mathbb{R}_+) \subset \mathcal{G}_{\mathcal{O}_M}(\mathbb{R}_+)$, $\psi \in \mathcal{O}_M(\mathbb{R}_+)$. The solution u is in the algebra $\mathcal{A}(\mathbb{R}^2)$.

Lemma 5.3. *Let u,w and a be non-negative continuous functions defined for $(x,y) \in \mathbb{R}^2_+$ and let w be non-decreasing in each variable x, y. Define, for $(x,y) \in \mathbb{R}^2_+$,*

$$D(x,y,f) = \begin{cases} \left\{(\xi,\eta)/x \le \xi \le f^{-1}(y), f_\varepsilon(\xi) \le \eta \le y\right\}, \text{ if } y \ge f(x), \\ \left\{(\xi,\eta)/f^{-1}(y) \le \xi \le x, y \le \eta \le f_\varepsilon(\xi)\right\}, \text{ if } y \le f(x). \end{cases}$$

$$n(x,y) = \iint\limits_{D(x,y,f)} a(\xi,\eta)u(\xi,\eta)\, d\xi\, d\eta.$$

If, for each $(x,y) \in \mathbb{R}^2_+$,

$$(A1) \qquad u(x,y) \le w(x,y) + n(x,y)$$

then, for each $(x,y) \in \mathbb{R}^2_+$,

$$u(x,y) \le w(x,y) +$$
$$w(x,y)\Big(\iint\limits_{D(x,y,f)} a(s,t) \exp\Big(\iint\limits_{D(x,y,f)} a(\xi,\eta)\, d\xi\, d\eta - \iint\limits_{D(s,t,f)} a(\xi,\eta)\, d\xi\, d\eta\Big) dt ds\Big).$$

Proof. See [1]. We treat the case $y \ge f(x)$, the other can be solved similarly. Then

$$n(x,y) = \iint\limits_{D(x,y,f)} a(\xi,\eta)u(\xi,\eta) d\xi d\eta = \int_x^{f^{-1}(y)} \Big(\int_{f(\xi)}^{y} a(\xi,\eta)u(\xi,\eta) d\eta\Big) d\xi$$
$$= \int_{f(x)}^{y} \Big(\int_x^{f^{-1}(\eta)} a(\xi,\eta)u(\xi,\eta) d\xi\Big) d\eta.$$

We have, for$(x,y) \in \mathbb{R}^2_+$, $u(x,y) \le w(x,y) + n(x,y)$, and

$$n_{xy}(x,y) = -a(x,y)u(x,y),$$

thus we have

$$\begin{aligned} -n_{xy}(x,y) - a(x,y)n(x,y) &= a(x,y)u(x,y) - a(x,y)n(x,y) \\ &= a(x,y)\left[u(x,y) - n(x,y)\right] \\ &\le a(x,y)w(x,y) \end{aligned}$$

Take

$$k(x,y)=\int_x^{f^{-1}(y)}(\int_{f(\xi)}^{y}a(\xi,\eta)d\xi)d\eta=\int_{f(x)}^{y}(\int_x^{f^{-1}(\eta)}a(\xi,\eta)d\xi)d\eta.$$

Then

$$k_y(x,y)=\int_x^{f^{-1}(y)}a(\xi,y)d\xi;\, k_x(x,y)=-\int_{f(x)}^{y}a(x,\eta)d\eta;\, k_{xy}(x,y)=-a(x,y).$$

Thus

$$\begin{aligned}-n_{xy}(x,y)-a(x,y)n(x,y)&\le a(x,y)w(x,y)-n_x(x,y)k_y(x,y)\\&\quad-n_y(x,y)k_x(x,y)+n(x,y)k_y(x,y)k_x(x,y).\end{aligned}$$

That is

$$\begin{aligned}&[-n_{xy}(x,y)-a(x,y)n(x,y)+n_y(x,y)k_x(x,y)]\\&+[n_x(x,y)k_y(x,y)-n(x,y)k_y(x,y)k_x(x,y)]\\&\le a(x,y)w(x,y).\end{aligned}$$

By multiplication with $\exp(-k(x,y))$, we obtain

$$\begin{aligned}&[-n_{xy}(x,y)-a(x,y)n(x,y)+n_y(x,y)k_x(x,y)]\exp(-k(x,y))\\&+[n_x(x,y)k_y(x,y)-n(x,y)k_y(x,y)k_x(x,y)]\exp(-k(x,y))\\&\le a(x,y)w(x,y)\exp(-k(x,y)).\end{aligned}$$

That is

$$\begin{aligned}&\frac{\partial}{\partial y}[(-n_x(x,y)+n(x,y)k_x(x,y))\exp(-k(x,y))]\\&\le a(x,y)w(x,y)\exp(-k(x,y)).\end{aligned}$$

By keeping x fixed in the above inequality, setting $y=t$ and then integrating with respect to t from $f(x)$ to y we obtain

$$\begin{aligned}&(-n_x(x,y)+n(x,y)k_x(x,y))\exp(-k(x,y))\\&\le\int_{f(x)}^{y}a(x,t)w(x,t)\exp(-k(x,t))dt.\end{aligned}$$

because $n_x(x,f(x))=0$. That is

$$\frac{\partial}{\partial x}(-n(x,y)\exp(-k(x,y)))\le\int_{f(x)}^{y}a(x,t)w(x,t)\exp(-k(x,t))dt.$$

Now, by keeping y fixed in the above inequality, setting $x=s$ and then integrating with respect to s from x to $f^{-1}(y)$, we have

$$\begin{aligned}&-n(f^{-1}(y),y)\exp(-k(f^{-1}(y),y))+n(x,y)\exp(-k(x,y))\\&\le\int_x^{f^{-1}(y)}(\int_{f(s)}^{y}a(s,t)w(s,t)\exp(-k(s,t))dt)ds.\end{aligned}$$

Thus

$$n(x,y)\exp(-k(x,y)) \le \int_x^{f^{-1}(y)} (\int_{f(s)}^{y} a(s,t)w(s,t)\exp(-k(s,t))dt)ds.$$

Then

$$\begin{aligned}&\int_x^{f^{-1}(y)} (\int_{f(\xi)}^{y} a(\xi,\eta)u(\xi,\eta)d\eta)\mathrm{d}\xi \\ &\le \exp(k(x,y))\int_x^{f^{-1}(y)} (\int_{f(s)}^{y} a(s,t)w(s,t)\exp(-k(s,t))dt)ds.\end{aligned}$$

We have

$$\begin{aligned}&\exp(k(x,y))\int_x^{f^{-1}(y)} (\int_{f(s)}^{y} a(s,t)w(s,t)\exp(-k(s,t))dt)ds \\ &= \int_x^{f^{-1}(y)} (\int_{f(s)}^{y} a(s,t)w(s,t)\exp(k(x,y))\exp(-k(s,t))dt)ds \\ &= \int_x^{f^{-1}(y)} \int_{f(s)}^{y} a(s,t)w(s,t)\exp(k(x,y)-k(s,t))dtds.\end{aligned}$$

So

$$(A3)\qquad \int_x^{f^{-1}(y)} (\int_{f(\xi)}^{y} a(\xi,\eta)u(\xi,\eta)d\eta)\mathrm{d}\xi$$

$$(5.1)\qquad \le \int_x^{f^{-1}(y)} \int_{f(s)}^{y} a(s,t)w(s,t)\exp(k(x,y)-k(s,t))dtds.$$

From $A1$ and $A3$ we have

$$u(x,y) \le w(x,y) + \int_x^{f^{-1}(y)} \int_{f(s)}^{y} a(s,t)w(s,t)\exp(k(x,y)-k(s,t))dtds.$$

Since $w(x,y)$ is non-negative and non-decreasing in each variable x, y, we have

$$u(x,y) \le w(x,y)(1 + \int_x^{f^{-1}(y)} \int_{f(s)}^{y} a(s,t)\exp(k(x,y)-k(s,t))dtds).$$

Moreover,

$$\exp(k(x,y)-k(s,t)) = \exp(\int_x^{f^{-1}(y)} \int_{f(\xi)}^{y} a(\xi,\eta)d\xi d\eta - \int_s^{f^{-1}(t)} \int_{f(\xi)}^{t} a(\xi,\eta)d\xi d\eta)$$

hence the result

$$\begin{aligned}&u(x,y) \le w(x,y) + \\ &\le w(x,y)(\iint_{D(x,y,f)} a(s,t)\exp(\iint_{D(x,y,f)} a(\xi,\eta)d\xi d\eta - \iint_{D(s,t,f)} a(\xi,\eta)d\xi d\eta)dtds).\end{aligned}$$

$\square$

Corollary 5.4. *With the previous assumptions, for* $(x,y) \in \mathbb{R}_+^2$*, we have*

$$u(x,y) \leq w(x,y) \exp\Big(\iint_{D(x,y,f)} a(s,t) dtds.\Big)$$

Proof. We consider the case $y \geq f(x)$. For $(x,y) \in \mathbb{R}_+^2$, we have

$$u(x,y) \leq w(x,y)\Big(1 + \iint_{D(x,y,f)} a(s,t) \exp(k(x,y) - k(s,t)) dtds\Big).$$

Take

$$M = 1 + \iint_{D(x,y,f)} a(s,t) \exp(k(x,y) - k(s,t)) dtds.$$

then

$$M \leq 1 + \exp(k(x,y)) \iint_{D(x,y,f)} a(s,t) \exp(-k(s,t)) dtds,$$

hence

$$M \leq 1 + \exp(k(x,y)) \iint_{D(x,y,f)} \exp(-k(s,t))\Big(-\frac{d^2}{dsdt} \iint_{D(s,t,f)} a(\xi,\eta) d\xi d\eta\Big) dtds.$$

Thus

$$\begin{aligned} M &\leq 1 + \exp(k(x,y)) \iint_{D(x,y,f)} \exp(-k(s,t))\Big(\frac{d^2}{dsdt} k(s,t)\Big) dtds \\ &\leq 1 + \exp(k(x,y))(1 - \exp(-k(x,y))). \end{aligned}$$

We deduce that

$$\begin{aligned} M &\leq 1 + \exp(k(x,y)) - \exp(k(x,y)) \exp(-k(x,y)) \\ &\leq 1 + \exp(k(x,y)) - 1. \end{aligned}$$

Then we have $M \leq \exp(k(x,y))$, but w is non-negative and non-decreasing in each variable x, y, then we obtain $u(x,y) \leq w(x,y) \exp(k(x,y))$, that is

$$u(x,y) \leq w(x,y) \exp\Big(\iint_{D(x,y,f)} a(s,t) dtds\Big).$$

□

Corollary 5.5. *Let a be non-negative continuous function defined on $\mathbb{R}_+^2$. Assume that, for each $(x,y) \in \mathbb{R}_+^2$, we have $|F(x,y,u(x,y))| \leq a(x,y)\, u(x,y)$. Let $\mathcal{L}_{\mathcal{O}_M}(\mathbb{R}_+)$ be the subset in $\mathcal{M}_{\mathcal{O}_M}(\mathbb{R}_+)$ of families $(g_\varepsilon)_\varepsilon$ such that $g'_\varepsilon > 0$*

and $\left[g_\varepsilon^{-1}\right] \in \mathcal{M}_{\mathcal{O}_M}(\mathbb{R}_+)$ *preserves slow scale points,* $\lim_{\varepsilon\to 0,\mathcal{D}'(\mathbb{R})} g_\varepsilon = 0$. *Assume that* $\Phi \in \mathcal{O}_M(\mathbb{R}_+)$, $\Psi \in \mathcal{O}_M(\mathbb{R}_+)$, $(L_\varepsilon)_\varepsilon \in \mathcal{L}_{\mathcal{O}_M}(\mathbb{R}_+)$, $[U_\varepsilon] \in \mathcal{G}(\mathbb{R}_+^2)$ *is a solution to*

$$\left(P_g^*\right) : \quad \frac{\partial^2 U}{\partial x \partial y} = \mathcal{F}(U) \quad ; \quad \mathcal{R}(U) = \Phi; \quad \mathcal{R}\left(\frac{\partial U}{\partial y}\right) = \psi.$$

and $|U_{0,\varepsilon}|$ *is non-decreasing in each variable* x, y. *Then, for each* $(x,y) \in \mathbb{R}_+^2$, *we have*

$$|U_\varepsilon(x,y)| \le |U_{0,\varepsilon}(x,y)| + \iint\limits_{D(x,y,f_\varepsilon)} a(\xi,\eta)\,|U_\varepsilon(\xi,\eta)|\,d\xi d\eta,$$

thus

$$|U_\varepsilon(x,y)| \le |U_{0,\varepsilon}(x,y)| \exp\Big(\int_x^{f_\varepsilon^{-1}(y)} \int_{f_\varepsilon(s)}^{y} a(s,t)dtds\Big).$$

Definition 5.6. The generalized function $[u_\varepsilon]$ is a solution to Problem $\left(P_g^{**}\right)$ if there are $U = [U_\varepsilon] \in \mathcal{G}(\mathbb{R}_+^2)$, $\Phi, \Psi \in \mathcal{O}_M(\mathbb{R}_+)$, $(L_\varepsilon)_\varepsilon \in \mathcal{L}_{\mathcal{O}_M}(\mathbb{R}_+)$ such that

$$\begin{array}{l} (1) \quad U \text{ is solution to } \left(P_g^*\right); \\ (2) \begin{cases} u_\varepsilon = U_\varepsilon|_{[0,T]\times[0,\infty[}\,; l_\varepsilon = L_\varepsilon|_{[0,T]}\,; \\ u_\varepsilon\left(x, l_\varepsilon(x)\right) = \Phi|_{[0,T]}(x) = \varphi(x); \\ \dfrac{\partial u_\varepsilon}{\partial y}\left(x, l_\varepsilon(x)\right) = \Psi|_{[0,T]}(x) = \psi(x)\,; \end{cases} \\ (3) \,\{[u_\varepsilon] \in \mathcal{G}_{\mathcal{O}_M}([0,T]\times[0,+\infty[). \end{array}$$

Moreover, for $(t,x) \in \mathbb{R}_+^2$, we have

$$U_\varepsilon(t,x) = U_{0,\varepsilon}(x,y) - \iint\limits_{D(x,y,L_\varepsilon)} F(\xi,\eta,U_\varepsilon(\xi,\eta))d\xi d\eta.$$

Theorem 5.1. *Suppose that* $(l_\varepsilon)_\varepsilon$ *is taken in* $\mathcal{L}_{\mathcal{O}_M}([0,T])$. *If* $\varphi \in \mathcal{O}_M([0,T])$, $\psi \in \mathcal{O}_M([0,T])$, *the generalized function* $u = [u_\varepsilon]_{\mathcal{G}_{\mathcal{O}_M}([0,T]\times[0,\infty[)}$, *where* u_ε *is defined in Definition 5.6, depends only on* $l = [l_\varepsilon]_{\mathcal{G}_{\mathcal{O}_M}([0,T])}$. *Let* a *be non-negative continuous function defined on* $\mathbb{R}_+^2$. *Assume that, for each* $(x,y) \in \mathbb{R}_+^2$, $|F(x,y,u(x,y))| \le a(x,y)\,u(x,y)$, $M_1 = \sup\limits_{(x,y)\in(\mathbb{R}_+)^2}(a(x,y)) < +\infty$ *and* $|U_{0,\varepsilon}|$ *is non-decreasing in each variable* x, y. *Then* u *is the unique solution to* $\left(P_g^{**}\right)$ *in* $\mathcal{G}_{\mathcal{O}_M}([0,T]\times[0,\infty[)$.

Proof. The first step is to prove the existence, and it is not possible to do that in $\mathcal{G}_{\mathcal{O}_M}(\mathbb{R}^2)$ if $F \neq 0$ ([2], Remark 3).We have

$$U_{0,\varepsilon}(x,y) = \Xi_\varepsilon(y) - \Xi_\varepsilon(L_\varepsilon(x)) + \varphi_\varepsilon(x)$$

and Ξ_ε denotes a primitive of $\Psi \circ L_\varepsilon^{-1}$. Furthermore, as $\Psi \in \mathcal{O}_M(\mathbb{R}_+)$, L_ε is taken in $\mathcal{L}_{\mathcal{O}_M}(\mathbb{R}_+)$ (it preserve slow scale points), then $\left(\Psi \circ L_\varepsilon^{-1}\right)_\varepsilon \in$

$\mathcal{M}_{\mathcal{O}_M}(\mathbb{R}_+)$ and $(\Xi_\varepsilon)_\varepsilon \in \mathcal{M}_{\mathcal{O}_M}(\mathbb{R}_+)$. As $\Phi \in \mathcal{O}_M(\mathbb{R}_+)$, so $(U_{0,\varepsilon})_\varepsilon$ belongs to $\mathcal{M}_{\mathcal{O}_M}\left((\mathbb{R}_+)^2\right)$.

Thus $\exists m \in \mathbb{N}$, $\exists p \in \mathbb{N}$, $\exists \varepsilon_0$, $\forall \varepsilon < \varepsilon_0$,

$$\sup_{(x,y)\in(\mathbb{R}_+)^2} (1+|x|+|y|)^{-p} U_{0,\varepsilon}(t,x) \leq \varepsilon^{-m}.$$

From the above Corollary 5.5

$$U_\varepsilon(x,y) \leq U_{0,\varepsilon}(x,y) \exp\left(\int_{f_\varepsilon^{-1}(y)}^{x} \int_{f_\varepsilon(s)}^{y} a(s,t)dtds\right).$$

Let $(x,y) \in \mathbb{R}_+^2$, define $\mathrm{A}_\varepsilon(x,y) =$ area of $D(x,y,f_\varepsilon)$. According to the mean value theorem, for each ε, there exists $(c_\varepsilon, d_\varepsilon) \in D(x,y,f_\varepsilon)$ such that

$$\iint\limits_{D(x,y,f_\varepsilon)} a(s,t)dtds = |a(c_\varepsilon,d_\varepsilon)|\, \mathrm{A}_\varepsilon(x,y).$$

Then

$$U_\varepsilon(x,y) \leq U_{0,\varepsilon}(x,y) \exp\left(|a(c_\varepsilon,d_\varepsilon)|\, \mathrm{A}_\varepsilon(x,y)\right).$$

Take $u_\varepsilon = U_\varepsilon|_{[0,T]\times[0,\infty[}$. So, we have

$$\begin{aligned}
&\sup_{(x,y)\in[0,T]\times[0,\infty[} (1+|x|+|y|)^{-p} |u_\varepsilon(x,y)| \\
&\leq \sup_{(x,y)\in[0,T]\times[0,\infty[} (1+|x|+|y|)^{-p} |u_{0,\varepsilon}(x,y)|\, e^{(-|a(c_\varepsilon,d_\varepsilon)|\mathrm{A}_\varepsilon(x,y))} \\
&\leq \sup_{(x,y)\in(\mathbb{R}_+)^2} \left((1+|x|+|y|)^{-p} |U_{0,\varepsilon}(x,y)|\right) \sup_{(x,y)\in[0,T]\times[0,\infty[} e^{M_1\left|x-L_\varepsilon^{-1}(y)\right||y-L_\varepsilon(x)|} \\
&\leq \varepsilon^{-m} \sup_{(x,y)\in[0,T]\times[0,\infty[} e^{-M_1\left|x-L_\varepsilon^{-1}(y)\right||y-L_\varepsilon(x)|}.
\end{aligned}$$

As $(L_\varepsilon)_\varepsilon \in \mathcal{L}_{\mathcal{O}_M}(\mathbb{R}_+)$, we know that $\left(L_\varepsilon^{-1}\right)_\varepsilon \in \mathcal{M}_{\mathcal{O}_M}(\mathbb{R}_+)$.

Take $l_\varepsilon = L_\varepsilon|_{[0,T]}$. Then, for $(l_\varepsilon)_\varepsilon \in \mathcal{L}_{\mathcal{O}_M}([0,T])$, we have $y = l_\varepsilon\left(l_\varepsilon^{-1}(y)\right)$ and

$$\begin{aligned}
\sup_{(x,y)\in[0,T]\times[0,\infty[} e^{-M_1\left|x-L_\varepsilon^{-1}(y)\right||y-L_\varepsilon(x)|} &= \sup_{(x,y)\in[0,T]\times[0,\infty[} e^{-M_1\left|x-l_\varepsilon^{-1}(y)\right||y-l_\varepsilon(x)|} \\
&\leq \sup_{(x,y)\in[0,T]\times[0,\infty[} e^{-M_1 T l_\varepsilon(T)}.
\end{aligned}$$

So $\exists k \in \mathbb{N}$, $\exists p \in \mathbb{N}$, $\exists \varepsilon_1$, $\forall \varepsilon < \varepsilon_1$,

$$\sup_{(t,x)\in[0,T]\times[0,\infty[} (1+|t|+|x|)^{-p} |u_\varepsilon(t,x)| \leq \varepsilon^{-k}.$$

Thus

$$(u_\varepsilon)_\varepsilon \in \mathcal{M}_{\mathcal{O}_M}([0,T]\times[0,\infty[)$$

and the class of $(u_{\varepsilon,})_\varepsilon$ in $\mathcal{G}_{\mathcal{O}_M}([0,T][\times[0,\infty[)$ is a solution to problem $\left(P_g^{**}\right)$ in $\mathcal{G}_{\mathcal{O}_M}([0,T]\times[0,\infty[)$.

Let us show that u is the unique solution to $\left(P_g^{**}\right)$. Let

$$v = [v_\varepsilon] \in \mathcal{G}_{\mathcal{O}_M}\left([0,T] \times [0,\infty[\right)$$

be another solution to $\left(P_g^{**}\right)$. That is to say there are $[V_\varepsilon] \in \mathcal{G}\left(\mathbb{R}_+^2\right)$, $\Phi \in \mathcal{O}_M\left(\mathbb{R}_+\right)$, $\Psi \in \mathcal{O}_M\left(\mathbb{R}_+\right)$, $\left(L_\varepsilon\right)_\varepsilon \in \mathcal{L}_{\mathcal{O}_M}\left(\mathbb{R}_+\right)$, $\left(I_\varepsilon\right)_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}\left(\mathbb{R}_+^2\right)$, $\left(\alpha_\varepsilon\right)_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}\left[0,T]\right)$, $\left(\beta_\varepsilon\right)_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}\left[0,T]\right)$, $(i_\varepsilon)_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}([0,T] \times [0,\infty[)$, such that

$$(1)\begin{cases} \dfrac{\partial^2 V_\varepsilon}{\partial x \partial y}(x,y) = F(t,x,V_\varepsilon(x,y)) + I_\varepsilon(x,y); \\ \qquad V_\varepsilon\left(x, L_\varepsilon(x)\right) = \Phi\left(x\right) + A_\varepsilon\left(x\right); \\ \qquad \dfrac{\partial V_\varepsilon}{\partial y}\left(x, L_\varepsilon(x)\right) = \Psi_\varepsilon(x) + B_\varepsilon(x). \end{cases}$$

$$(2)\begin{cases} v_\varepsilon = \left.V_\varepsilon\right|_{[0,T]\times[0,\infty[}\,; l_\varepsilon = \left.L_\varepsilon\right|_{[0,T]}\,; \varphi = \left.\Phi\right|_{[0,T]}\,; \psi = \left.\Psi\right|_{[0,T]}\,; \\ v_\varepsilon\left(x, l_\varepsilon(x)\right) = \left.\Phi\right|_{[0,T]}(x) + \left.A\right|_{[0,T]}(x) = \varphi(x) + \alpha_\varepsilon\left(x\right); \\ \dfrac{\partial v_\varepsilon}{\partial y}\left(x, L_\varepsilon(x)\right) = \left.\Psi\right|_{[0,T]}(x) + \left.B_\varepsilon\right|_{[0,T]}(x) = \psi(x) + \left.\beta_\varepsilon\right|_{[0,T]}(x). \end{cases}$$

$$(3)\left\{[v_\varepsilon] \in \mathcal{G}_{\mathcal{O}_M}([0,T] \times [0,\infty[).\right.$$

The uniqueness of the solution to (P_g) will be consequence of

$$(v_\varepsilon - u_\varepsilon)_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}([0,T] \times [0,\infty[).$$

We have $V_{0,\varepsilon}(x,y) = U_{0,\varepsilon}(x,y) + \theta_\varepsilon(x,y)$, where

$$\theta_\varepsilon(x,y) = A_\varepsilon(x) - \Pi_\varepsilon(L_\varepsilon\left(x\right)) + \Pi_\varepsilon(y)$$

and Π_ε denotes a primitive of $B_\varepsilon \circ L_\varepsilon^{-1}$. Furthermore, as $\left(B_\varepsilon\right)_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}(\mathbb{R}_+)$, $\left(L_\varepsilon\right)_\varepsilon$ is taken in $\mathcal{L}_{\mathcal{O}_M}\left(\mathbb{R}_+\right)$ (it preserves slow scale points), then $\left(B_\varepsilon \circ L_\varepsilon^{-1}\right)_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}(\mathbb{R}_+)$ and $\left(\Pi_\varepsilon\right)_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}(\mathbb{R}_+)$. As $\left(A_\varepsilon\right)_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}([0,T])$, so $\left(\theta_\varepsilon\right)_\varepsilon$ belongs to $\mathcal{N}_{\mathcal{O}_M}([0,T] \times [0,\infty[)$. We have

$$V_\varepsilon(x,y) = V_{0,\varepsilon}(x,y) + \iint\limits_{D(x,y,L_\varepsilon)} I_\varepsilon(\xi,\eta)d\xi d\eta - \iint\limits_{D(x,y,L_\varepsilon)} F(\xi,\eta,V_\varepsilon\left(\xi,\eta\right))d\xi d\eta.$$

We set, for all ε,

$$J_\varepsilon\left(x,y\right) = \iint\limits_{D(x,y,L_\varepsilon)} I_\varepsilon(\xi,\eta)d\xi d\eta.$$

Now we have to check that

$$\left(J_\varepsilon\right)_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}\left([0,T] \times [0,\infty[\right).$$

Let $(x,y) \in \mathbb{R}_+^2$, recall that $\mathrm{A}_\varepsilon(x,y) =$ area of $D(x,y,f_\varepsilon)$. Let $[(x_\varepsilon,y_\varepsilon)_\varepsilon] \in \widetilde{\mathbb{R}_+}^2$ be a slow scale point. Then $[(x_\varepsilon)_\varepsilon] \in \widetilde{\mathbb{R}_+}$, $[(z_\varepsilon)_\varepsilon] = [(L_\varepsilon\left(x_\varepsilon\right))_\varepsilon] \in \widetilde{\mathbb{R}_+}$ and

$\left[\left(L_{\varepsilon}^{-1}\left(y_{\varepsilon}\right)\right)_{\varepsilon}\right] \in \widetilde{\mathbb{R}_{+}}$ are also slow scale points. According to the mean value theorem, for each ε, there exists $(c_\varepsilon, d_\varepsilon) \in D(x_\varepsilon, y_\varepsilon, L_\varepsilon)$ such that

$$\left|J_{\varepsilon}\left(x_{\varepsilon}, y_{\varepsilon}\right)\right| = \left| \iint\limits_{D(x,y,L_\varepsilon)} I_\varepsilon(\xi,\eta))d\xi d\eta \right| = \left|I_\varepsilon(c_\varepsilon, d_\varepsilon)\right| \mathrm{A}_\varepsilon(x_\varepsilon, y_\varepsilon).$$

Where $I_\varepsilon(c_\varepsilon, d_\varepsilon)$ is the average value of I_ε on $D(x_\varepsilon, y_\varepsilon, f_\varepsilon)$. As

$$|c_\varepsilon| \leq \max(|x_\varepsilon|, \left|L_\varepsilon^{-1}\left(y_\varepsilon\right)\right|), |d_\varepsilon| \leq \max(\left|L_\varepsilon\left(x_\varepsilon\right)\right|, |y_\varepsilon|)$$

then $\left[(c_\varepsilon)_\varepsilon\right]$ and $\left[(d_\varepsilon)_\varepsilon\right]$ are slow scale points. Thus $\left[(c_\varepsilon, d_\varepsilon)_\varepsilon\right]$ is a slow scale point of $\widetilde{\mathbb{R}_{+}}^{2}$. As $\left(I_\varepsilon\right)_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}\left(\mathbb{R}_{+}^2\right)$ we obtain that $\left(I_\varepsilon(c_\varepsilon, d_\varepsilon)\right)_\varepsilon \in \mathcal{N}_{\mathbb{R}_{+}}$ and $\left(J_\varepsilon\right)_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}\left(\mathbb{R}_{+}^2\right)$. Thus there is $\left(\sigma_\varepsilon\right)_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}\left(\mathbb{R}_{+}^2\right)$, $\sigma_\varepsilon = \theta_\varepsilon + J_\varepsilon$, such that

$$v_\varepsilon(x,y) = U_{0,\varepsilon}(x,y) + \sigma_\varepsilon(x,y) - \iint\limits_{D(x,y,L_\varepsilon)} F(\xi,\eta,V_\varepsilon(\xi,\eta))d\xi d\eta.$$

Take $W_\varepsilon = (V_\varepsilon - U_\varepsilon)$ and show that $(w_\varepsilon)_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}\left(\mathbb{R}_{+}^2\right)$. We have

$$W_\varepsilon(x,y) = \iint\limits_{D(x,y,L_\varepsilon)} \left(-F(\xi,\eta,V_\varepsilon(\xi,\eta)) + F(\xi,\eta,U_\varepsilon(\xi,\eta))\right) d\xi d\eta + \sigma_\varepsilon(x,y),$$

but

$$\begin{aligned} &F(\xi,\eta,V_\varepsilon(\xi,\eta)) - F(\xi,\eta,U_\varepsilon(\xi,\eta)) \\ &\qquad = \left(V_\varepsilon(\xi,\eta) - U_\varepsilon(\xi,\eta)\right) \int_0^1 \frac{\partial F}{\partial z}(\xi,\eta,U_\varepsilon(\xi,\eta) + \rho(W_\varepsilon(\xi,\eta)))\,\mathrm{d}\rho\ , \end{aligned}$$

so

$$\begin{aligned} W_\varepsilon(x,y) &= \sigma_\varepsilon(x,y) \\ &\quad - \iint\limits_{D(x,y,L_\varepsilon)} W_\varepsilon(\xi,\eta)(\int_0^1 \frac{\partial F}{\partial z}(\xi,\eta,U_\varepsilon(\xi,\eta) + \rho(W_\varepsilon(\xi,\eta)))d\rho d\xi)d\eta. \end{aligned}$$

But $M_1 = \sup\limits_{(t,x,z)\in\mathbb{R}_{+}^2\times\mathbb{R}} \dfrac{\partial F}{\partial z}(t,x,z) = \sup\limits_{(x,y)\in(\mathbb{R}_{+})^2} (a(x,y))$, thus

$$|W_\varepsilon(x,y)| \leq \sigma_\varepsilon(x,y) + \iint\limits_{D(x,y,L_\varepsilon)} M_1 \left|W_\varepsilon(\xi,\eta)\right| d\xi d\eta.$$

From the above Corollary 5.5, for any $(x,y) \in \mathbb{R}_{+}^2$, we have

$$|W_\varepsilon(t,x)| \leq \sigma_\varepsilon(x,y)\exp(\iint\limits_{D(x,y,L_\varepsilon)} M_1 dtds) \leq \sigma_\varepsilon(x,y)\exp(M_1 \mathrm{A}_\varepsilon(x,y)).$$

But, as $(\sigma_\varepsilon)_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}\left(\mathbb{R}_+^2\right)$, we have $\forall m \in \mathbb{N}$, $\exists p \in \mathbb{N}$, $\exists \varepsilon_0$, $\forall \varepsilon < \varepsilon_0$,

$$\sup_{(x,y)\in[0,T]\times[0,\infty[} (1+|x|+|y|)^{-p}\sigma_\varepsilon(t,x) \leq \sup_{(x,y)\in\mathbb{R}_+^2} (1+|x|+|y|)^{-p}\sigma_\varepsilon(x,y) \leq \varepsilon^m.$$

Take $l_\varepsilon = L_\varepsilon|_{[0,T]}$. Then, for $(l_\varepsilon)_\varepsilon \in \mathcal{L}_{\mathcal{O}_M}([0,T])$, we have

$$\sup_{(t,x)\in[0,T]\times[0,\infty[} \exp(M_1 \mathrm{A}_\varepsilon(x,y)) = \sup_{(t,x)\in[0,T]\times[0,\infty[} \exp(M_1 \mathrm{A}_\varepsilon(x,y)) \leq e^{M_1 T f_\varepsilon(T)}$$

so, $\forall k \in \mathbb{N}$, $\exists p \in \mathbb{N}$, $\exists \varepsilon_1$, $\forall \varepsilon < \varepsilon_1$,

$$\sup_{(t,x)\in[0,T]\times[0,\infty[} (1+|t|+|x|)^{-p}\,|w_\varepsilon(t,x)| \leq \varepsilon^k.$$

We deduce that

$$(w_\varepsilon)_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}([0,T]\times[0,\infty[)\,.$$

Thus the solution is unique in $\mathcal{G}_{\mathcal{O}_M}([0,T]\times[0,\infty[)$. $\square$

References

[1] Abdeldaim, A., Yakout, M., On Wendroff's Inequality and Applications. Int. Journal of Math. Analysis, Vol. 4 No. 13 (2010), 607–616.

[2] Allaud, E., Delcroix, A., Dévoué, V., Marti, J.-A., Vernaeve, H., Paradigmatic well-posedness in some generalized characteristic Cauchy problems. Proceedings of the 8th congress of the ISAAC-2011, 1 (2012), 476–489. Ceregmia working paper 2012-01. Available at: `http://ideas.repec.org/s/crg/wpaper.html`.

[3] Allaud, A., Dévoué, V., Generalized solutions to a characteristic Cauchy problem. J. Appl. Anal., Vol. 19 No 1 (2013), 1–29.

[4] Agarwal, R. P., Kim, Y.-H., Sen, S. K., Multidimensional Gronwall-Bellman-type integral inequalities with applications. Memoirs on Differential Equations and Mathematical Physics 47 (2009), 19–122.

[5] András, S., Mészáros, A., Remarks on some Wendroff type inequalities. Journal of Mathematical Inequalities, Vol. 5 No 3 (2011), 401–411.

[6] Aragona, J., Colombeau generalized functions on quasi-regular sets. Publ. Math. Debrecen 68(3-4) (2006), 371–399.

[7] Aragona, J., Garcia, R. G., Juriaans, S. O., Generalized solutions of a nonlinear parabolic equation with generalized functions as initial data. Nonlinear Analysis 71 (2009), 5187–5207.

[8] Bainov, D. D., Zahariev, A. I., A Note on Bellman-Gronwall's Inequality. Journal of Mathematical Analysis and Applications. 80 (1981), 147–149.

[9] Biagioni, H., A Nonlinear Theory of Generalized Functions. Lecture Notes Math. 1421, Springer-Verlag, 1990.

[10] Bondge, B. K., Pachpatte, B. G., On Nonlinear Integral Inequalities of the Wendroff Type. J. Math. Anal. Appl. 70 (1979), 161–169.

[11] Colombeau, J.-F., New Generalized Functions and Multiplication of Distributions. North-Holland, Amsterdam, Oxford, New-York, 1984.

[12] Colombeau, J.-F., Elementary introduction to new generalized functions, North Holland Math. Studies 113. North-Holland, Amsterdam, 1984.

[13] Delcroix, A., Remarks on the embedding of spaces of distributions into spaces of Colombeau generalized functions, Novi Sad J. math., Vol. 35 No. 2, (2005), 27–40.

[14] Delcroix, A., Some properties of $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-algebras: Overgeneration and 0-order estimates. Available at: `http://hal.archives-ouvertes.fr/hal-00326671/fr/`.

[15] Delcroix, A., Topology and functoriality in $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-algebras. Application to singular differential problems. J. Math. Anal. Appl. 359 (2008), 394–403.

[16] Delcroix, A., A new approach to temperate generalized Colombeau functions. Publ. Inst. Math. (Beograd) (N.S.) 84(98) (2008), 109–121.

[17] Delcroix, A., Dévoué, V., Marti, J.-A., Generalized solutions of singular differential problems. Relationship with classical solutions. J. Math. Anal. Appl. 353(1) (2009), 386–402.

[18] Delcroix, A., Dévoué, V., Marti, J.-A., Well posed problems in algebras of generalized functions. Appl. Anal., Vol. 90 No. 11, (2011), 1747–1761.

[19] Delcroix, A., Scarpalézos, D., Topology on Asymptotic Algebras of Generalized Functions and Applications. Monatsh. Math. 129 (2000), 1–14.

[20] Dévoué, V., On generalized solutions to the wave equation in canonical form. Dissertationes math. 443 (2007), 1–69.

[21] Dévoué, V., Generalized solutions to a non Lipschitz Cauchy problem. J. Appl. Anal., Vol.15, No.1 (2009), 1–32.

[22] Dévoué, V., Generalized solutions to a singular nonlinear Cauchy problem. Novi Sad J. Math., Vol.41, No.1 (2011), 85–121.

[23] Dévoué, V., Marti, J.-A., Vernaeve, H., Vindas, J., Generalized functions on the closure of an open set. Application to uniqueness of some characteristic Cauchy problem. Preprint. Ceregmia working paper 2013-18. Available at: `http://ideas.repec.org/s/crg/wpaper.html`.

[24] Dragomir, S. S., Some Gronwall type inequalities and applications. School of Communications and Informatics. Victoria University of Technology. Available at: `http//rgmia.vu.edu/SSDragomirWeb.html`.

[25] Egorov, Y. V., On the solubility of differential equations with simple characteristics. Russian Math. Surveys 26 (1971), 113-130.

[26] Egorov, Y. V., Shubin, M. A., Partial Differential Equations. Springer Verlag 1993.

[27] Grosser, M., Kunzinger, M., Oberguggenberger, M., Steinbauer, R., Geometric Theory of Generalized Functions with Applications to General Relativity. Dordrecht: Kluwer Academic Publ., 2001.

[28] Hacia, L., On Some Integral Inequalities and Their Applications. J. Math. Anal. Appl. 206 (1997), 611-622. Article AY975258.

[29] Hörmander, L., On the characteristic Cauchy problem. Ann. Math. 88 (1968), 341-370.

[30] Marti, J.-A., Fundamental structures and asymptotic microlocalization in sheaves of generalized functions, Integral Transforms Spec. Funct. 6(1- 4), (1998), 223–228.

[31] Marti, J.-A., $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-Sheaf structure and applications. In: Nonlinear theory of generalized functions. pp.175–186. (eds. M. Grosser and al.), Research Notes in Mathematics. Chapman & Hall/CRC, 1999.

[32] Marti, J.-A., Non linear Algebraic analysis of delta shock wave to Burgers' equation. Pacific J. Math. 210(1) (2003), 165–187.

[33] Oberguggenberger, M., Multiplication of Distributions and Applications to Partial Differential Equations. Pitman Research Notes in Mathematics 259, Longman Scientific & Technical, Harlow 1992.

[34] Scarpalézos, D., Colombeau's generalized functions: Topological structures; Microlocal properties. A simplified point of view. Part I. Bull. Cl. Sci. Math. Nat. Sci. Math. 25 (2000), 89–114.

[35] Scarpalézos, D., Colombeau's generalized functions: Topological structures; Microlocal properties. A simplified point of view. Part II. Publ. Inst. Math. (Beograd) (N.S.) 76(90) (2004), 111–125.

[36] Vernaeve, H., Pointwise Characterizations in generalized function algebras. Monatsh. Math. 158 (2009), 195–213.

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