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GENERALIZED NONLINEAR VARIATIONAL INEQUALITIES

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Abstract. In this paper, we consider a generalized nonlinear variational inequality problem involving single valued and multivalued nonlinear operators. We also study criteria of its solvability. Iterative methods for approximate solution are also proposed and a convergence result is established. Further, we study iterative methods for finding common element of fixed point set of nonexpansive mapping and solution set of the proposed variational inequality problem.

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1. Introduction and preliminaries

Variational inequalities have emerged as a mathematical programming tool for modeling a wide class of problems arising in different branches of pure and applied sciences see [1, 4, 5, 6, 14, 15] and references therein. Verma [18] studied a variational inequality problem involving a single valued and a set-valued operator. Recently Qin and Shang [16] studied an iterative method to approximate common element of fixed points set of nonexpansive mappings and solution set of a variational inequality.

Let $\mathcal{H}$ be a Hilbert space and K be a nonempty closed subset of $\mathcal{H}$. We consider the following variational inequality problem : Find $(x^*, w^*) \in \mathcal{H} \times T(x^*)$ such that $g(x^*) \in K$ and

$$\langle Ax^* + w^*, y^* - g(x^*) \rangle \geq 0 , \quad \forall\, y^* \in K, \tag{1.1}$$

where $A, g : \mathcal{H} \to \mathcal{H}$ and $T : \mathcal{H} \to 2^{\mathcal{H}}$ are nonlinear mappings.

We call inequality (1.1) as *generalized nonlinear variational inequality problem* and denote by $VI(\mathcal{H}, A, T, g)$.

We now recall some definitions:

Definition 1.1. A mapping $T : \mathcal{H} \to \mathcal{H}$ is said to be :

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(i) strongly monotone, if there exists a constant $\nu > 0$ such that, for each $x \in \mathcal{H}$,
$$\langle T(x) - T(y),\ x - y \rangle \geq \nu \|x - y\|^2$$
holds, for all $y \in \mathcal{H}$;

(ii) $\delta-$cocoercive, if there exists a constant $\delta > 0$ such that, for each $x \in \mathcal{H}$,
$$\langle T(x) - T(y),\ x - y \rangle \geq \delta \|T(x) - T(y)\|^2$$
holds, for all $y \in \mathcal{H}$;

(iii) relaxed $\delta-$cocoercive, if there exists a constant $\delta > 0$ such that, for each $x \in \mathcal{H}$,
$$\langle T(x) - T(y),\ x - y \rangle \geq -\delta \|T(x) - T(y)\|^2$$
holds, for all $y \in \mathcal{H}$;

(iv) relaxed $(\delta, \lambda)-$cocoercive or relaxed cocoercive with constant (δ, λ), if there exist constants $\delta > 0$ and $\lambda > 0$ such that, for each $x \in \mathcal{H}$,
$$\langle T(x) - T(y),\ x - y \rangle \geq -\delta \|T(x) - T(y)\|^2 + \lambda \|x - y\|^2$$
holds, for all $y \in \mathcal{H}$;

(v) $\mu-$Lipschitz continuous or Lipschitz with constant μ, if there exists a constant $\mu > 0$ such that, for each $x, y \in \mathcal{H}$,
$$\|T(x) - T(y)\| \leq \mu \|x - y\| \ ,$$

(vi) nonexpansive, if for each $x, y \in \mathcal{H}$,
$$\|T(x) - T(y)\| \leq \|x - y\| \ .$$

Let $CB(\mathcal{H})$ denote the family of all nonempty closed bounded subsets of $\mathcal{H}$. A set valued mapping $T : \mathcal{H} \to CB(\mathcal{H})$ is said to be :

(v) $\zeta - \hat{H}-$Lipschitz continuous if there exists a constant $\zeta > 0$ such that
$$\hat{H}(T(x), T(y)) \leq \zeta \|x - y\| \ , \ \ \forall\, x, y \in \mathcal{H} \,,$$
where $\hat{H}$ is the Hausdorff metric, i.e. for any two nonempty subsets A and B of $CB(\mathcal{H})$,
$$\hat{H}(A, B) = \max \left\{ \sup_{x \in A} d(x, B),\ \sup_{y \in B} d(y, A) \right\} \ .$$

Lemma 1.2. *[10] Let (X, d) be a complete metric space, $T : X \to CB(X)$ be a set-valued mapping. Then for any $\varepsilon > 0$ and $x, y \in X$, $u \in T(x)$, there exists $v \in T(y)$ such that*
$$d(u, v) \leq (1 + \varepsilon)\, \hat{H}(T(x), T(y)) \,.$$

Lemma 1.3. *[10] Let (X, d) be a complete metric space, $T : X \to CB(X)$ be a set-valued mapping satisfying*

$$\hat{H}(T(x), T(y)) \leq k\, d(x, y)\,, \quad \forall x, y \in X\,,$$

where $0 \leq k < 1$ is a constant. Then the mapping T has a fixed point in X.

Let us recall the following result, which is commonly used in the context of solvability of nonlinear variational inequalities :

Lemma 1.4. *[8] For an element $z \in \mathcal{H}$, we have*

$$x = P_K(z)$$

if and only if

$$x \in K \;:\; \langle x - z,\; y - x\rangle \geq 0\,, \quad \forall\, y \in K\,,$$

where P_K is a projection of $\mathcal{H}$ into K.

It is known that P_K is a nonexpansive mapping, i.e. $\|P_K(x) - P_K(y)\| \leq \|x - y\|\;,\;\; \forall x, y \in \mathcal{H}$.

Let K be a closed convex subset of $\mathcal{H}$ and $\rho > 0$ is fixed. Consider the mapping $F : K \to 2^{\mathcal{H}}$ given by

$$F(u) = u - g(u) + P_K\left(g(u) - \rho\,(A(u) - T(u))\right)\,, \tag{1.2}$$

with the convention $x + \emptyset = x$ for every $x \in \mathcal{H}$, and the orthogonal projection of a set $U \subset \mathcal{H}$ on K is defined as $P_K(U) = \{P_K(u) : u \in U\}$.
A point $x \in \mathcal{H}$ is said to a fixed point of F if $x \in F(x)$.

Using Lemma 1.4, we will establish the following important relation:

Lemma 1.5. *$(x^*, w^*) \in \mathcal{H} \times T(x^*)$ is a solution of (1.1) if and only if x^* is a fixed point of the mapping F given by (1.2).*

Proof. Let $x^* \in \mathcal{H}$ be a fixed point of the mapping F, i.e. $x^* \in F(x^*)$. Then there exists $w^* \in T(x^*)$ such that

$$x^* = x^* - g(x^*) + P_K\left(g(x^*) - \rho(A(x^*) + w^*)\right)\,,$$

i.e.,

$$g(x^*) = P_K\left(g(x^*) - \rho(A(x^*) + w^*)\right)\,,$$

implies that

$$\langle g(x^*) - (g(x^*) - \rho(A(x^*) + w^*))\,,\; y^* - g(x^*)\rangle \geq 0 \quad \forall y^* \in K\,.$$

Hence

$$\langle \rho(A(x^*) + w^*),\; y^* - g(x^*)\rangle \geq 0\,,$$

implies that

$$\langle A(x^*) + w^*,\; y^* - g(x^*)\rangle \geq 0 \;\text{ for some } \rho > 0\,.$$

Conversely, let $(x^*, w^*) \in \mathcal{H} \times T(x^*)$ be a solution of (1.1), then $g(x^*) \in K$ and

$$\langle A(x^*) + w^*,\ y^* - g(x^*)\rangle \geq 0\,, \quad \forall\, y^* \in K\,,$$

hence, for some $\rho > 0$, we have

$$\langle \rho(A(x^*) + w^*),\ y^* - g(x^*)\rangle \geq 0\,, \quad \forall\, y^* \in K\,,$$

or

$$\langle g(x^*) - (g(x^*) - \rho(A(x^*) + w^*))\,,\ y^* - g(x^*)\rangle \geq 0\,, \quad \forall\, y^* \in K\,.$$

By Lemma 1.4, we have

$$g(x^*) = P_K\left[g(x^*) - \rho(A(x^*) + w^*)\right]\,,$$

i.e.,

$$\begin{aligned} x^* &= x^* - g(x^*) + P_K\left[g(x^*) - \rho(A(x^*) + w^*)\right] \\ &\in x^* - g(x^*) + P_K\left[g(x^*) - \rho(A(x^*) + T(x^*))\right] \\ \Rightarrow x^* &\in F(x^*)\,. \end{aligned}$$

i.e. x^* is a fixed point of F. □

Lemma 1.5 implies that the problem (1.1) is equivalent to the fixed point problem (1.2). This alternative equivalent formulation provides a natural connection between variational inequality problem (1.1) and the fixed point theory which will be used to prove the existence result.

2. Main results

Theorem 2.1. *Let K be a closed convex subset of a real Hilbert space $\mathcal{H}$. Let $A, g : \mathcal{H} \to \mathcal{H}$ be relaxed cocoercive with constants (δ_A, λ_A), (δ_g, λ_g) and Lipschitz continuous mappings with constants μ_A, μ_g respectively. Let $T : \mathcal{H} \to CB(\mathcal{H})$ be a $\zeta - \hat{H}$-Lipschitz continuous mapping. Assume that the following assumption holds:*

$$(2.1)\qquad \begin{gathered} \left|\rho - \frac{\Theta}{(\mu_A^2 - \zeta^2)}\right| < \frac{\sqrt{\Theta^2 - 4(\mu_A^2 - \zeta^2)\kappa(1-\kappa)}}{(\mu_A^2 - \zeta^2)}\,, \\ |\Theta| > 2\sqrt{(\mu_A^2 - \zeta^2)\kappa(1-\kappa)}\,,\ \ \mu_A^2 - \zeta^2 > 0\,, \end{gathered}$$

where

$$\Theta = \lambda_A - \zeta(1 - 2\kappa) - \delta_A \mu_A^2$$

$$\kappa = \sqrt{1 - 2\lambda_g + \mu_g^2(1 + 2\delta_g)}\,.$$

Then the problem (1.1) has a solution.

Proof. By Lemma 1.5, it is enough to show that the mapping F defined by (1.2) has a fixed point. Let $x, y \in \mathcal{H}$ be given. For any $p \in F(x)$, there exists $w_1 \in T(x)$ such that

$$p = x - g(x) + P_K\left(g(x) - \rho\left(A(x) + w_1\right)\right) .$$

Since $w_1 \in T(x)$, for any $\varepsilon > 0$, it follows from Lemma 1.2 that there exists $w_2 \in T(y)$ such that

$$\|w_1 - w_2\| \leq (1+\varepsilon)\hat{H}(Tx, Ty) .$$

Taking $q = y - g(y) + P_K\left(g(y) - \rho\left(A(y) + w_2\right)\right)$, we have $q \in F(y)$. Hence,

$$\begin{aligned}
&\|p - q\| \\
&\quad \leq \|x - y - (g(x) - g(y))\| \\
&\qquad + \|P_K\left(g(x) - \rho(A(x) + w_1)\right) - P_K\left(g(y) - \rho(A(y) + w_2)\right)\| \\
&\quad \leq \|x - y - (g(x) - g(y))\| \\
&\qquad + \|g(x) - g(y) - \rho\left\{(A(x) + w_1) - (A(y) + w_2)\right\}\| \\
&\quad \leq 2\|x - y - (g(x) - g(y))\| + \|x - y - \rho\left\{(A(x) + w_1) - (A(y) + w_2)\right\}\| \\
&\quad \leq 2\|x - y - (g(x) - g(y))\| + \|x - y - \rho\left\{A(x) - A(y)\right\}\| + \rho\|w_1 - w_2\| \\
&\quad \leq 2\|x - y - (g(x) - g(y))\| + \|x - y - \rho\left\{A(x) - A(y)\right\}\| \\
&\qquad\quad + \rho(1+\varepsilon)\hat{H}(Tx, Ty) \\
&\quad \leq 2\|x - y - (g(x) - g(y))\| + \|x - y - \rho\left\{A(x) - A(y)\right\}\| \\
&\qquad + \rho(1+\varepsilon)\zeta\|x - y\| . \qquad (2.2)
\end{aligned}$$

Since g is relaxed $(\delta_g, \lambda_g)-$cocoercive and μ_g-Lipschitz mapping, we can compute the following:

$$\begin{aligned}
&\|x - y - (g(x) - g(y))\|^2 \\
&= \|x - y\|^2 - 2\langle g(x) - g(y), x - y\rangle + \|g(x) - g(y)\|^2 \\
&\leq (1 + \mu_g^2)\|x - y\|^2 + 2\delta_g\|g(x) - g(y)\|^2 - 2\lambda_g\|x - y\|^2 \\
&\leq \left(1 - 2\lambda_g + \mu_g^2(1 + 2\delta_g)\right)\|x - y\|^2 . \qquad (2.3)
\end{aligned}$$

Also, since A is relaxed $(\delta_A, \lambda_A)-$cocoercive and μ_A-Lipschitz mapping, we get

$$\begin{aligned}
&\|x - y - \rho\left\{A(x) - A(y)\right\}\|^2 \\
&\quad = \|x - y\|^2 - 2\rho\langle A(x) - A(y), x - y\rangle + \rho^2\|A(x) - A(y)\|^2 \\
&\quad \leq \|x - y\|^2 - 2\rho\left\{-\delta_A\|A(x) - A(y)\|^2 + \lambda_A\|x - y\|^2\right\} \\
&\qquad + \rho^2\|A(x) - A(y)\|^2 \\
&\quad \leq \|x - y\|^2 + 2\rho\delta_A\mu_A^2\|x - y\|^2 - 2\rho\lambda_A\|x - y\|^2 \\
&\qquad + \rho^2\mu_A^2\|x - y\|^2 \\
&\quad = \left[1 + 2\rho\left(\delta_A\mu_A^2 - \lambda_A\right) + \rho^2\mu_A^2\right]\|x - y\|^2 \qquad (2.4)
\end{aligned}$$

Substituting (2.3), (2.4) into (2.2), we have

$$\|p-q\| \le \theta(\varepsilon)\,\|x-y\| \tag{2.5}$$

where

$$\theta(\varepsilon) = 2\kappa + f(\rho_\varepsilon)\,,$$

$$\kappa = \sqrt{1-2\lambda_g+\mu_g^2(1+2\delta_g)}\,,$$

$$f(\rho_\varepsilon) = \sqrt{1+2\rho\,(\delta_A\mu_A^2-\lambda_A)+\rho^2\mu_A^2}\;+\rho(1+\varepsilon)\zeta\,.$$

By using (2.5), we get that

$$d(p,F(y)) = \inf_{q\in F(y)} \|p-q\| \le \theta(\varepsilon)\,\|x-y\|\;,$$

since $p \in F(x)$ is arbitrary, we get

$$\sup_{p\in F(x)} d(p,F(y)) \le \theta(\varepsilon)\,\|x-y\|\;. \tag{2.6}$$

Similarly, we get that

$$\sup_{q\in F(y)} d(q,F(x)) \le \theta(\varepsilon)\,\|x-y\|\;. \tag{2.7}$$

From the definition of Hausdorff metric $\hat{H}$, it follows from (2.6) and (2.7) that

$$\hat{H}(F(x),F(y)) \le \theta(\varepsilon)\,\|x-y\|\;,\quad \forall x,y\in\mathcal{H}\,.$$

Letting $\varepsilon \to 0$, we get that

$$\hat{H}(F(x),F(y)) \le \theta\,\|x-y\|\;,\quad \forall x,y\in\mathcal{H}\,,$$

where,

$$\theta = 2\kappa + f(\rho)\,,$$

$$f(\rho) = \sqrt{1+2\rho\,(\delta_A\mu_A^2-\lambda_A)+\rho^2\mu_A^2}\;+\rho\zeta\,.$$

From (2.1), we get that $\theta < 1$, thus F is a set valued contraction mapping, by Lemma 1.3 it has a fixed point in $\mathcal{H}$, i.e. there exist a point $x^* \in \mathcal{H}$ such that $x^* \in F(x^*)$. Lemma 1.5 implies that $(x^*, w^*) \in \mathcal{H}\times T(x^*)$ is a solution of variational inequality problem (1.1). □

2.1. Iterative algorithm and convergence

For a given $x_0 \in \mathcal{H}$, $w_0 \in T(x_0)$, let

$$x_1 = x_0 - g(x_0) + P_K\left(g(x_0) - \rho(A(x_0)+w_0)\right)\,.$$

By Lemma 1.3 there exists $w_1 \in T(x_1)$ such that

$$\|w_0 - w_1\| \le (1+1)\hat{H}(Tx_0, Tx_1)\,.$$

Let $x_2 = x_1 - g(x_1) + P_K\left(g(x_1) - \rho(A(x_1) + w_1)\right)$, then by Lemma 1.3 there exists $w_2 \in T(x_2)$ such that

$$\|w_1 - w_2\| \leq \left(1 + \frac{1}{2}\right) \hat{H}(Tx_1, Tx_2)\,.$$

By induction, we can get an iterative algorithm, as follows :

Algorithm 1. For a given $x_0 \in \mathcal{H}$, $w_0 \in T(x_0)$, define sequences $\{x_n\}$ and $\{w_n\}$ satisfying

$$\left.\begin{aligned} x_{n+1} &= x_n - g(x_n) + P_K\left(g(x_n) - \rho(A(x_n) + w_n)\right)\,, \\ w_n \in T(x_n)\,,\ \|w_n - w_{n+1}\| &\leq \left(1 + \frac{1}{n+1}\right) \hat{H}\left(T(x_n), T(x_{n+1}\right)\,. \end{aligned}\right\} \tag{2.8}$$

Now, we define Ishikawa type [7] iterative algorithm for approximate solvability of variational inequality problem (1.1).

Algorithm 2. For a given $x_0 \in \mathcal{H}$, compute x_{n+1} by the scheme

$$\begin{aligned} y_n &= (1 - \beta_n)x_n + \beta_n\left[x_n - g(x_n) + P_K\left(g(x_n) - \rho(A(x_n) + w_n)\right)\right]\,, \\ x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n\left[x_n - g(x_n) + P_K\left(g(y_n) - \rho(A(y_n) + u_n)\right)\right]\,, \end{aligned} \tag{2.9}$$

where $w_n \in T(x_n)$, $u_n \in T(y_n)$, $n = 0, 1, 2, \ldots$ and $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0, 1]$, satisfying certain conditions.

We need following result to prove the next result :

Lemma 2.2. *[20] Let $\{a_n\}$ be a non negative sequence satisfying*

$$a_{n+1} \leq (1 - c_n)a_n + b_n\,,$$

with $c_n \in [0, 1]$, $\sum_{n=0}^{\infty} c_n = \infty$, $b_n = o(c_n)$. Then $\lim_{n\to\infty} a_n = 0$.

Theorem 2.3. *Let A, T, g satisfy all the assumptions of Theorem 2.1, and let $\{\alpha_n\}$, $\{\beta_n\}$ be sequences in $[0, 1]$ for all $n \geq 0$ such that $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then the approximate sequences $\{x_n\}$, $\{w_n\}$ constructed by the Algorithm 2 converge strongly to a solution $(x^*, w^*) \in \mathcal{H} \times T(x^*)$ of the problem (1.1).*

Proof. Let $(x^*, w^*) \in \mathcal{H} \times T(x^*)$ is a solution of (1.1), by Lemma 1.5, we have

$$x^* = x^* - g(x^*) + P_K\left(g(x^*) - \rho(A(x^*) + w^*)\right)\,.$$

Using (2.9), we have

$$\begin{aligned}
&\|x_{n+1} - x^*\| \\
&= \|(1-\alpha_n)x_n + \alpha_n\left[x_n - g(x_n) + P_K\left(g(y_n) - \rho(A(y_n)+u_n)\right)\right] - x^*\| \\
&\leq (1-\alpha_n)\|x_n - x^*\| \\
&\quad + \alpha_n \|[x_n - g(x_n) + P_K\left(g(y_n) - \rho(A(y_n)+u_n)\right)] - (x^*)\| \\
&\leq (1-\alpha_n)\|x_n - x^*\| \\
&\quad + \alpha_n \|(x_n - g(x_n) + P_K\left[g(y_n) - \rho(A(y_n)+u_n)\right]) - x^*\| \\
&\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n\|x_n - x^* - (g(x_n) - g(x^*))\| \\
&\quad + \alpha_n\|P_K\left[g(y_n) - \rho(A(y_n)+u_n)\right] - P_K\left[g(x^*) - \rho(A(x^*)+w^*)\right]\| \\
&\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n\|x_n - x^* - (g(x_n) - g(x^*))\| \\
&\quad + \alpha_n\|g(y_n) - g(x^*) - \rho\left((A(y_n)+u_n) - (A(x^*)+w^*)\right)\| \\
&\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n\|x_n - x^* - (g(x_n) - g(x^*))\| \\
&\quad + \alpha_n\|y_n - x^* - (g(y_n) - g(x^*))\| \\
&\quad + \alpha_n\|y_n - x^* - \rho\left(A(y_n) - A(x^*)\right)\| + \alpha_n\rho\|u_n - w^*\| \\
&\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n\sqrt{1 - 2\lambda_g + \mu_g^2(1+2\delta_g)}\,\|x_n - x^*\| \\
&\quad + \alpha_n\sqrt{1 - 2\lambda_g + \mu_g^2(1+2\delta_g)}\,\|y_n - x^*\| \\
&\quad + \alpha_n\sqrt{1 + 2\rho\left(\delta_A\mu_A^2 - \lambda_A\right) + \rho^2\mu_A^2}\,\|y_n - x^*\| \\
&\quad + \alpha_n\rho(1+\varepsilon)\zeta\|y_n - x^*\|
\end{aligned}$$

$$= (1-\alpha_n)\|x_n - x^*\| + \alpha_n\kappa\|x_n - x^*\| + \alpha_n\left(\kappa + f(\rho_\varepsilon)\right)\|y_n - x^*\|, \tag{2.10}$$

where κ and $f(\rho_\varepsilon)$ are as in the Theorem 2.1.
Similarly, we have

$$\begin{aligned}
&\|y_n - x^*\| \\
&\leq (1-\beta_n)\|x_n - x^*\| + \beta_n\|x_n - x^* - (g(x_n) - g(x^*))\| \\
&\quad + \beta_n\|P_K\left(g(x_n) - \rho\left(A(x_n)+w_n\right)\right) - P_K\left(g(x^*) - \rho\left(A(x^*) - w^*\right)\right)\| \\
&\leq (1-\beta_n)\|x_n - x^*\| + \beta_n\kappa\|x_n - x^*\| \\
&\quad + \beta_n\|g(x_n) - g(x^*) - \rho\left\{(A(x_n)+w_n) - (A(x^*) - w^*)\right\}\| \\
&\leq (1-\beta_n)\|x_n - x^*\| + \beta_n\kappa\|x_n - x^*\| \\
&\quad + \beta_n\|x_n - x^* - (g(x_n) - g(x^*))\| \\
&\quad + \beta_n\|x_n - x^* - \rho\left(A(x_n) - A(x^*)\right)\| + \beta_n\rho\|w_n - w^*\| \\
&\leq (1-\beta_n)\|x_n - x^*\| + 2\beta_n\kappa\|x_n - x^*\| + \beta_n f(\rho_\varepsilon)\|x_n - x^*\|
\end{aligned}$$

$$= (1-\beta_n)\|x_n - x^*\| + \beta_n\theta(\varepsilon)\|x_n - x^*\| \tag{2.11}$$

Substituting (2.11) into (2.10), yields that

$$\begin{aligned}
\|x_{n+1} - x^*\| &\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n\kappa\|x_n - x^*\| \\
&\quad + \alpha_n(\kappa + f(\rho_\varepsilon))\left\{(1-\beta_n) + \beta_n\theta(\varepsilon)\right\}\|x_n - x^*\|
\end{aligned}$$

letting $\varepsilon \to 0$, we get that

$$\begin{aligned} \|x_{n+1} - x^*\| &\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n\kappa\|x_n - x^*\| \\ &\quad + \alpha_n(\kappa + f(\rho))\{(1-\beta_n) + \beta_n\theta\}\|x_n - x^*\| \\ &\le [1 - \alpha_n\{1-\theta\}]\|x_n - x^*\| . \end{aligned} \tag{2.12}$$

By virtue of Lemma 2.2, we get from (2.12) that, $\lim_{n\to\infty}\|x_{n+1} - x^*\| = 0$, i.e. $x_n \to x^*$, as $n \to \infty$.
Since

$$\|w_n - w^*\| \le (1+\varepsilon)\zeta\|x_n - x^*\|$$

letting $n \to \infty$, we get that $w_n \to w^*$. This completes the proof. □

2.2. Iterative algorithm for common element

If $(x^*, w^*) \in \mathcal{H} \times T(x^*)$ is a solution of (1.1), then by the relation (1.2), we have

$$x^* = x^* - g(x^*) + P_K\left(g(x^*) - \rho(A(x^*) + w^*)\right) . \tag{2.13}$$

Now, if x^* is a common element of the fixed point set $F(S)$ of a mapping S and solution set of $VI(\mathcal{H}, A, T, g)$, we can see from relation (2.13) that

$$x^* = Sx^* = S\left[x^* - g(x^*) + P_K\left(g(x^*) - \rho(A(x^*) + w^*)\right)\right] . \tag{2.14}$$

Using the fixed point formulation (2.14), we now suggest and analyze the following Ishikawa type [7] iterative methods:

Algorithm 3. For a given $x_0 \in \mathcal{H}$, find the approximate solution x_{n+1} by the iterative scheme
(2.15)

$$\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n S\left[x_n - g(x_n) + P_K\left(g(x_n) - \rho(A(x_n) + w_n)\right)\right] , \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n S\left[x_n - g(x_n) + P_K\left(g(y_n) - \rho(A(y_n) + u_n)\right)\right] , \end{aligned}$$

where $w_n \in T(x_n)$, $u_n \in T(y_n)$, $n = 0, 1, 2, \dots$ and $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0,1]$, satisfying certain conditions.

Theorem 2.4. *Let A, T, g satisfy all the assumptions of Theorem 2.1 and let S be a nonexpansive mapping from K into itself such that $F(S) \cap VI(\mathcal{H}, A, T, g) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0,1]$ for all $n \ge 0$ such that $\sum_{n=0}^{\infty}\alpha_n = \infty$. Then the approximate sequence $\{x_n\}$ constructed by the Algorithm 3 converges strongly to a solution $x^* \in F(S) \cap VI(\mathcal{H}, A, T, g)$.*

Proof. Let x^* be an element of $F(S) \cap VI(\mathcal{H}, A, T, g)$, then using (2.15), we have

$$
\begin{aligned}
&\|x_{n+1} - x^*\| \\
&= \|(1-\alpha_n)x_n + \alpha_n S\left[x_n - g(x_n) + P_K\left(g(y_n) - \rho(A(y_n)+u_n)\right)\right] - x^*\| \\
&\le (1-\alpha_n)\|x_n - x^*\| \\
&\quad + \alpha_n \|S\left[x_n - g(x_n) + P_K\left(g(y_n) - \rho(A(y_n)+u_n)\right)\right] - S(x^*)\| \\
&\le (1-\alpha_n)\|x_n - x^*\| \\
&\quad + \alpha_n \|(x_n - g(x_n) + P_K\left(g(y_n) - \rho(A(y_n)+u_n)\right)) - x^*\| \\
&\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n \|x_n - x^* - (g(x_n) - g(x^*))\| \\
&\quad + \alpha_n \|P_K\left(g(y_n) - \rho(A(y_n)+u_n)\right) - P_K\left(g(x^*) - \rho(A(x^*)+w^*)\right)\|
\end{aligned}
$$

By the similar arguments as in the proof of Theorem 2.3, we get that

$$\lim_{n\to\infty} \|x_{n+1} - x^*\| = 0,$$

i.e. $x_n \to x^*$, as $n \to \infty$. This completes the proof. □

We now discuss some special cases of Variational inequality problem (1.1) :

1. If T is single valued, then the problem (1.1) is equivalent to finding $x^* \in \mathcal{H}$ such that $g(x^*) \in K$ and

   $$\langle Ax^* + Tx^*,\ y^* - g(x^*)\rangle \ge 0\,, \quad \forall\, y^* \in K\,. \tag{2.16}$$

   Inequality (2.16) is studied by Noor et al. [13].

2. If g is identity mapping, then the problem (1.1) is equivalent to finding $(x^*, w^*) \in \mathcal{H} \times T(x^*)$ such that

   $$\langle Ax^* + w^*,\ y^* - x^*\rangle \ge 0\,, \quad \forall\, y^* \in K\,. \tag{2.17}$$

   Inequality (2.17) is studied by Verma [18], Qin et al. [16].

3. If T is single valued and g is identity mappings, then the problem (1.1) is equivalent to finding $x^* \in \mathcal{H}$, such that

   $$\langle Ax^* + Tx^*,\ y^* - x^*\rangle \ge 0\,, \quad \forall\, y^* \in K\,. \tag{2.18}$$

   Inequality (2.18) is studied by Noor [11, 12].

4. If $A = 0$, then the problem (1.1) is equivalent to finding $(x^*, w^*) \in \mathcal{H} \times T(x^*)$ such that $g(x^*) \in K$ and

   $$\langle w^*,\ y^* - g(x^*)\rangle \ge 0\,, \quad \forall\, y^* \in K\,. \tag{2.19}$$

   Inequality (2.19) is studied by Verma [19].

5. If $A = 0$ and g is identity mappings, then the problem (1.1) is equivalent to finding $(x^*, w^*) \in \mathcal{H} \times T(x^*)$ such that

   $$\langle w^*,\ y^* - x^* \rangle \geq 0\,, \quad \forall\, y^* \in K\,. \tag{2.20}$$

   Inequality (2.20) is studied by Bruck [2], Fang et.al [3] and Siddiqi et.al [17].

6. If $T = 0$ and g is identity mappings, then the problem (1.1) is equivalent to finding $x^* \in \mathcal{H}$, such that

   $$\langle Ax^*,\ y^* - x^* \rangle \geq 0\,, \quad \forall\, y^* \in K\,. \tag{2.21}$$

   Inequality (2.21) is studied by Lions and Stampacchia [9].

Conclusion

Results presented in the paper are significant improvement and extension of the results obtained previously by many authors. Especially, our Theorem 2.1, extends the existence of solution in the literature to the case of generalized nonlinear variational inequality (1.1). Algorithm 2 is a very general and unified algorithm for finding the approximate solution of the problem (1.1). Theorem 2.4 provides convergence to common point of fixed point set of nonexpansive mapping and the solution set of the generalized nonlinear variational inequality problem (1.1).

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