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RANDOM BIPARTITE GRAPHS

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Abstract. In this paper we explore various properties of random bipartite graphs. These structures naturally correspond to independent families, which are very important in various set-theoretic constructions. We investigate their robustness, universality, possibility of factorization and maximality.

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1. Introduction

In the last few decades independent families have frequently been used by set-theorists in various constructions. On the other hand, random graphs (including variations like random digraphs, random tournaments etc.) were investigated in many occasions and, since they lay somewhere on the borderline of several areas of mathematics (graph theory, set theory, model theory), people from all of those areas showed interest in them. Especially well known is the Rado graph, the unique countable random graph. Many useful fact on the Rado graph can be found in [1].

There is a natural way to regard independent families as bipartite graphs (short: bigraphs) that seems to be known, but is rarely mentioned explicitly. In [6] and [4] certain aspects of such graphs were investigated. Here we look into some other properties of these structures, extending them to larger cardinalities when possible. Apart from the benefit of getting results on independent families from the corresponding results on graphs, we find random bigraphs interesting on their own account.

In the next two sections we will go through definitions and some basic properties of these structures, and describe the connection. In section 4 we will examine random bigraphs more closely, proving some universality results and considering the possibility of factorization.

Throughout the text κ, λ, μ and ν will denote infinite cardinals, and α, β, γ and δ ordinals. ω is the set of natural numbers, and $\aleph_0$ its cardinality. Most of our set-theoretic and graph-theoretic notation is standard. With $[X]^\mu$ we denote the set of subsets of X of cardinality μ, and with $[X]^{<\mu}$ the set of subsets of X of cardinality less than μ. $f[X] = \{f(x) : x \in X\}$ is the direct image of a set X under a function f. The denotation of edges in graphs and digraphs will be simplified: we will write xy for an edge $\langle x, y\rangle$ of a digraph (oriented graph).

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2. Definitions and basic properties

Definition 1. *(κ, λ)-bigraph is a structure $G = (X, Y, E)$, where $(X \cup Y, E)$ is a digraph such that $|X| = \kappa$, $|Y| = \lambda$ and $E \subseteq \{xy : x \in X, y \in Y\}$. We call (X, Y) the bipartition of G, X the left side, and Y the right side.*

For disjoint $U, W \in [Y]^{<\mu}$ we denote $\Gamma^G_{U,W} = \{x \in X : \forall u \in U\ xu \in E \wedge \forall w \in W\ xw \notin E\}$. If it is clear from the context which bigraph we are considering, we write only $\Gamma_{U,W}$.

Definition 2. *Let $\mu \leq \lambda$. A (κ, λ)-bigraph (X, Y, E) is (κ, λ, μ)-random if*

$$\forall U, W \in [Y]^{<\mu}\ (U \cap W = \emptyset \Rightarrow \Gamma^G_{U,W} \neq \emptyset). \tag{1}$$

If $\mu \leq \kappa$, a (κ, λ)-bigraph (X, Y, E) is (κ, λ, μ)-dense if

$$\forall U, W \in [X]^{<\mu} U \cap W = \emptyset \Rightarrow \exists y \in Y(\forall u \in U uy \in E \wedge \forall w \in W wy \notin E)). \tag{2}$$

Hence a bigraph is (κ, λ, μ)-dense iff the bigraph obtained by reversing its edges is (κ, λ, μ)-random. If G satisfies both (1) and (2) we will call it (κ, λ, μ)-random dense. A $(\kappa, \lambda, \aleph_0)$-random bigraph is called just (κ, λ)-random. First we have the analogue of a well-known property for random graphs.

Lemma 3. *(a) In a (κ, λ, μ)-random bigraph (X, Y, E) we can find for every disjoint $U, W \in [Y]^{<\mu}$ μ-many vertices $x \in X$ that satisfy $xu \in E$ for all $u \in U$ and $xw \notin E$ for all $w \in W$.*

(b) In a (κ, λ, μ)-dense bigraph (X, Y, E) we can find for every disjoint $U, W \in [X]^{<\mu}$ μ-many vertices $y \in Y$ that satisfy $uy \in E$ for all $u \in U$ and $wy \notin E$ for all $w \in W$.

Proof. We will give only a proof of (a). Suppose the opposite, that $|\Gamma^G_{U,W}| = \nu < \mu$ for some disjoint $U, W \in [Y]^{<\mu}$. Choose an arbitrary set $T \in [Y]^\nu$. Let $\theta : \Gamma^G_{U,W} \to T$ be a bijection. By (1) there is $x \in X$ such that $xu \in E$ for all $u \in U$, $xw \notin E$ for all $w \in W$ and $x\theta(v) \in E \Leftrightarrow v\theta(v) \notin E$ for all $v \in \Gamma^G_{U,W}$. Clearly, $x \in \Gamma^G_{U,W}$ but $x \neq v$ for all $v \in \Gamma^G_{U,W}$, a contradiction. $\square$

It is easy to see that every (κ, λ, μ)-random bigraph is right-extensional, meaning that there are no different vertices $y_1, y_2 \in Y$ such that for all $x \in X$ $xy_1 \in E \Leftrightarrow xy_2 \in E$. Analogously, every (κ, λ, μ)-dense bigraph is left-extensional.

Lemma 4. *Let $G = (X, Y, E)$ be a (κ, λ, μ)-random bigraph. Then the degree of every $y \in Y$ is at least μ. Consequently, $\kappa \geq \mu$.*

Proof. Let $y \in Y$ and let $\{x_\gamma : \gamma < \nu\}$ be the set of all his neighbors. Suppose the opposite, that $\nu < \mu$. Let y_γ, for $\gamma < \nu$, be any elements of Y such that $y_\gamma \neq y$ and $y_\gamma \neq y_\delta$ for $\gamma \neq \delta$. By (1) there is $x \in X$ such that $xy \in E$ and $xy_\gamma \in E \Leftrightarrow x_\gamma y_\gamma \notin E$ for $\gamma < \nu$, so x must be different from all x_γ. This is a contradiction. $\square$

Clearly, the same results hold for vertices in X if G is (κ, λ, μ)-dense. The following two lemmas contain some easy but useful robustness properties of (κ, λ, μ)-random (dense) bigraphs.

Lemma 5. *Every bigraph obtained from a (κ, λ, μ)-random bigraph (X, Y, E) by*
(a) adding $\leq \kappa$ vertices to X (connected with arbitrary vertices from Y)
(b) removing $< \mu$ vertices from X
(c) removing $< \lambda$ vertices from Y
(d) replacing $< \mu$ edges with non-edges and $< \mu$ non-edges with edges
is also a (κ, λ, μ)-random bigraph.

Proof. Adding vertices to X and removing them from Y can not spoil the condition (1). (b) and (d) follow from Lemma 3(a). □

Of course, if we omit the condition "$\leq \kappa$" from (a) we get a (κ', λ, μ)-random bigraph for some $\kappa' \geq \kappa$, and we omit "$< \lambda$" from (c) we get a (κ, λ', μ)-random bigraph for some $\lambda' \leq \lambda$.

Lemma 6. *Let μ be a regular cardinal. Every bigraph obtained from a (κ, λ, μ)-random dense bigraph by deleting $< \mu$ edges from each vertex is also a (κ, λ, μ)-random dense bigraph.*

Proof. We prove for example that (1) still holds after deleting the edges. So let $U, W \in [Y]^{<\mu}$ be disjoint. For every $u \in U$ and every $w \in W$ less than μ edges are deleted, so by the regularity of μ the total number of deleted edges from all of these vertices is also less than μ. But by Lemma 3 there are at least μ vertices $x \in X$ such that $\forall u \in U \ xu \in E \ \wedge \forall w \in W \ xw \notin E$; hence at least one of them remains after deleting the edges. □

3. The connection with independent families

Definition 7. *Let $\mu \leq \lambda$. A family $\mathcal{A} = \{A_\alpha : \alpha < \lambda\}$ of subsets of κ is called (κ, λ, μ)-independent if*

$$\forall U, W \in [\lambda]^{<\mu} (U \cap W = \emptyset \Rightarrow \bigcap_{\alpha \in U} A_\alpha \cap \bigcap_{\alpha \in W} (\kappa \setminus A_\alpha) \neq \emptyset). \tag{3}$$

If $\mu = \aleph_0$, we call $\mathcal{A}$ a (κ, λ)-independent family. Concerning the existence of independent families, we have the following (see [7], Exercise A6 of Chapter VIII and [8]).

Proposition 8. *If $\kappa^{<\mu} = \kappa$ then there is a $(\kappa, 2^\kappa, \mu)$-independent family.*

Definition 9. *Let $\mu \leq \kappa$. A family $\mathcal{A} = \{A_\alpha : \alpha < \lambda\}$ of subsets of κ is called (κ, λ, μ)-dense if*

$$\forall U, W \in [\kappa]^{<\mu} (U \cap W = \emptyset \Rightarrow \exists \alpha \in \lambda \ (U \subseteq A_\alpha \wedge W \cap A_\alpha = \emptyset)). \tag{4}$$

In [3] and [2] the (κ, λ)-independent families such that the cardinality of intersections $\bigcap_{\alpha \in U} A_\alpha \cap \bigcap_{\alpha \in W} (\kappa \setminus A_\alpha)$ is larger than $\aleph_0$ were investigated.

Now we describe how to observe independent families as random bigraphs and vice versa, and list several easy consequences of various known results.

Let $\mathcal{A} = \{A_\alpha : \alpha < \lambda\}$ be a (κ, λ, μ)-independent family. Let X and Y be disjoint sets of cardinalities κ and λ respectively. We enumerate them: $X = \{x_\beta : \beta < \kappa\}$,

$Y = \{y_\alpha : \alpha < \lambda\}$, and define the relation $E \subseteq X \times Y$: let $x_\beta y_\alpha \in E$ iff $\beta \in A_\alpha$. Then $\mathcal{A}^* = (X, Y, E)$ is a (κ, λ, μ)-random bigraph.

On the other hand, let $G = (X, Y, E)$ be a (κ, λ, μ)-random bigraph. We enumerate $X = \{x_\beta : \beta < \kappa\}$ and $Y = \{y_\alpha : \alpha < \lambda\}$ and define, for each $\alpha \in \lambda$, $A_\alpha = \{\beta \in \kappa : x_\beta y_\alpha \in E\}$. Then $G' = \{A_\alpha : \alpha < \lambda\}$ is clearly a (κ, λ, μ)-independent family.

Of course, the described two operations are mutually inverse i.e. $(G')^* = G$ and $(\mathcal{A}^*)' = \mathcal{A}$.

A (κ, λ, μ)-independent family $\mathcal{A}$ is *maximal* if there is no (κ, λ', μ)-independent family $\mathcal{A}'$ for some $\lambda' \geq \lambda$ such that $\mathcal{A} \subset \mathcal{A}'$. Maximal independent families are those that are most frequently used in set-theoretic constructions, so it may be interesting to see how maximality affects the corresponding bigraphs.

Definition 10. *A (κ, λ, μ)-random bigraph $G = (X, Y, E)$ is maximal if for every $Z \subseteq X$*

$$\exists U, W \in [Y]^{<\mu} (U \cap W = \emptyset \wedge (\Gamma^G_{U,W} \subseteq Z \vee \Gamma^G_{U,W} \subseteq X \setminus Z)). \tag{5}$$

Lemma 11. *A (κ, λ, μ)-independent family $\mathcal{A}$ is maximal iff the corresponding bigraph $\mathcal{A}^*$ is maximal.*

Proof. First suppose $\mathcal{A} = \{A_\alpha : \alpha < \lambda\}$ is maximal. Let (X, Y) be the bipartition of $\mathcal{A}^*$ and let $Z \subseteq X$. If $Z \in \mathcal{A}$, there is a vertex $y \in Y$ connected exactly to vertices in Z, so we can clearly take $U = \{y\}$ and $W = \emptyset$. So let $Z \notin \mathcal{A}$. Then $\mathcal{A} \cup \{Z\}$ is not (κ, λ, μ)-independent, so there are $U_1, W_1 \in [\mathcal{A} \cup \{Z\}]^{<\mu}$ such that $\bigcap U_1 \cap (\kappa \setminus \bigcup W_1) = \emptyset$. If $Z \in U_1$, then for the sets $U = U_1 \setminus \{Z\}$ and $W = W_1$ we have $\Gamma^{\mathcal{A}^*}_{U,W} \subseteq X \setminus Z$. Analogously, if $Z \in W_1$, we take $U = U_1$ and $W = W_1 \setminus \{Z\}$ and get $\Gamma^{\mathcal{A}^*}_{U,W} \subseteq Z$.

The other direction is proved in a similar way. □

In [5] Goldstern and Kojman introduced several small cardinals, among them $\mathfrak{r}_\infty$. The following result is a direct consequence of their Theorem 3.1.

Corollary 12. *Let $\lambda < \mathfrak{r}_\infty$ and let G be an $(\aleph_0, \lambda)$-random bigraph. Then there is an $(\aleph_0, \lambda')$-random bigraph (for some $\lambda' \geq \lambda$) containing G as a proper subgraph. Hence if G is a maximal $(\aleph_0, \lambda)$-random bigraph, then $\lambda \geq \mathfrak{r}_\infty$.*

Maximal independent families do not always exist. Kunen in [8] proved the following.

Proposition 13. *Let $\lambda \geq \mu > \aleph_0$. If there is a maximal (κ, λ, μ)-independent family then $2^{<\mu} = \mu$.*

In the same paper Kunen mentions (without proof) a result that we prove in a slightly different form, using bigraphs.

Theorem 14. *If $G = (X, Y, E)$ is a maximal (κ, λ, μ)-random bigraph, then for any $X_1 \supseteq X$ the bigraph $G_1 = (X_1, Y, E)$ is maximal $(|X_1|, \lambda, \mu)$-random. Hence, if $\mathcal{A}$ is a maximal (κ, λ, μ)-independent family then for all $\kappa' > \kappa$ it is also maximal (κ', λ, μ)-independent.*

Proof. By Lemma 5(a) and the remark after it, G_1 is $(|X_1|, \lambda, \mu)$-random. Suppose it is not maximal, so that there is $Z \subseteq X_1$ that doesn't satisfy (5) for G_1. We claim that the set $Z \cap X$ doesn't satisfy (5) for G. Otherwise there would be $U, W \in [Y]^{<\aleph_0}$ such that either $\Gamma^G_{U,W} \subseteq Z \cap X$ or $\Gamma^G_{U,W} \subseteq X \setminus Z$. We can assume without loss of generality that $U \neq \emptyset$; otherwise we add arbitrary $y \notin W$ to U (of course, this does not add new elements to $\Gamma^G_{U,W}$). Then the sets $\Gamma^G_{U,W}$ and $\Gamma^{G_1}_{U,W}$ are same and they are subsets of X (because G_1 doesn't have any "new" edges), so either $\Gamma^{G_1}_{U,W} \subseteq Z$ or $\Gamma^{G_1}_{U,W} \subseteq X_1 \setminus Z$ must be true. This is a contradiction. □

4. More on random bigraphs

The existence od certain random bigraphs can be obtained as a direct consequence of Proposition 8.

Corollary 15. *If $\kappa^{<\mu} = \kappa$ then there is a $(\kappa, 2^\kappa, \mu)$-random bigraph.*

In [4] the homogeneity of bigraphs was investigated, and more results on their existence were obtained. We list some of them in the proposition below.

Definition 16. *Let $G_1 = (X_1, Y_1, E_1)$ and $G_2 = (X_2, Y_2, E_2)$ be bigraphs. A mapping $f : X_1 \cup Y_1 \to X_2 \cup Y_2$ is a homomorphism if for all $x \in X_1$, $y \in Y_1$: $xy \in E_1$ iff $f(x)f(y) \in E_2$.*

Since the edges in bigraphs are oriented, homomorphisms preserve sides ($f[X_1] \subseteq X_2$ and $f[Y_1] \subseteq Y_2$).

Definition 17. *A (κ, λ)-bigraph $G = (X, Y, E)$ is homogeneous if every partial isomorphism $f : U \to X \cup Y$ (where $U \in [X \cup Y]^{<\aleph_0}$) can be extended to an automorphism of G.*

Proposition 18. *(a) There is exactly one (up to isomorphism) $(\aleph_0, \aleph_0)$-random dense bigraph, and it is homogeneous.*

(b) Every homogeneous (κ, λ)-bigraph which is neither empty nor complete is either a perfect matching or its complement or a (κ, λ)-random dense bigraph (of course, when $\kappa \neq \lambda$, only the latter option remains).

(c) There is a $(\kappa, 2^\kappa)$-random dense bigraph for every infinite cardinal κ.

(d) ($\neg$CH$\wedge$MA) For every $\kappa < \mathfrak{c}$ there is a unique $(\aleph_0, \kappa)$-random dense bigraph (up to isomorphism).

(e) ($2^{\kappa^+} > 2^\kappa$) There are 2^{κ^+}-many nonisomorphic (κ, κ^+)-random dense bigraphs.

Unlike (a), $(\aleph_0, \aleph_0)$-random bigraph is not unique up to isomorphism. To see this, we use Lemma 5: if G is an $(\aleph_0, \aleph_0)$-random bigraph without isolated vertices in X, we can add one and get another $(\aleph_0, \aleph_0)$-random bigraph. If G has finitely many isolated vertices, we can exclude one of them, and if it has infinitely many, we can exclude all but finitely many of them. Either way, we get two nonisomorphic $(\aleph_0, \aleph_0)$-random bigraphs.

It is well-known that Rado graph is universal for the class of all finite and countable graphs. For bigraphs we have the following two results.

Theorem 19. *Every (κ_1, λ_1)-bigraph for $\kappa_1 \leq \mu$ and $\lambda_1 < \mu$ can be embedded in any (κ, λ, μ)-random bigraph.*

Proof. Let $G_1 = (X_1, Y_1, E_1)$ be a (κ_1, λ_1)-bigraph. We will define an embedding of G_1 into any (κ, λ, μ)-random bigraph $G = (X, Y, E)$. First, enumerate $X_1 = \{x_\gamma : \gamma < \kappa_1\}$ and choose an arbitrary set $Z \in [Y]^{\lambda_1}$. Let $\varphi : Y_1 \to Z$ be a bijection. Now proceed by recursion on $\gamma < \kappa_1$.

Suppose we defined, for all $\delta < \gamma$, elements $\phi(x_\delta) \in X$. Now let $\phi(x_\gamma) \in X$ be different from all $\phi(x_\delta)$ and such that for all $y = \varphi(y_1) \in Z$: $\phi(x_\gamma)y \in E$ iff $x_\gamma y_1 \in E_1$. (Such an element exists because of Lemma 3(a).) It is now easy to check that $\phi \cup \varphi$ is an embedding of G_1 into G. $\square$

Theorem 20. *Every (κ_1, λ_1)-bigraph for $\kappa_1 \leq \mu$ and $\lambda_1 \leq \mu$ can be embedded in any (κ, λ, μ)-random dense bigraph.*

Proof. First, if $\lambda_1 < \mu$, then the result follows from the previous theorem. If $\kappa_1 < \mu$, then we do the same construction as in the previous theorem, but reversing the roles of the sides of the bigraph.

Now suppose $\kappa_1 = \lambda_1 = \mu$. So let $G_1 = (X_1, Y_1, E_1)$ be a (μ, μ)-bigraph and we will define an embedding of G_1 into any (κ, λ, μ)-random dense bigraph $G = (X, Y, E)$. Enumerate $X_1 = \{x_\gamma : \gamma < \mu\}$ and $Y_1 = \{y_\gamma : \gamma < \mu\}$. Define by recursion one-to-one functions $\varphi : X_1 \to X$ and $\theta : Y_1 \to Y$ as follows. If we already defined $\varphi(x_\delta)$ and $\theta(y_\delta)$ for $\delta < \gamma$, let $\varphi(x_\gamma)$ be different from all $\varphi(x_\delta)$ and such that $\varphi(x_\gamma)\theta(y_\delta) \in E \Leftrightarrow x_\gamma y_\delta \in E_1$ for all $\delta < \gamma$, and let $\theta(y_\gamma)$ be different from all $\theta(y_\delta)$ and such that $\varphi(x_\delta)\theta(y_\gamma) \in E \Leftrightarrow x_\delta y_\gamma \in E_1$ for all $\delta \leq \gamma$.

In the end, $\varphi \cup \theta$ is an embedding of G_1 into G. $\square$

Hence the $(\aleph_0, \aleph_0)$-random dense bigraph is universal for the class of all finite and countable bigraphs.

Theorem 21. *(a) Every (κ, κ, κ)-random dense bigraph has a perfect matching, i.e. a sub-bigraph with the same set of vertices in which every vertex is of degree 1.*

(b) Every (κ, κ, κ)-random dense bigraph has a 1-factorization, i.e. its set of edges can be partitioned into disjoint perfect matchings.

Proof. (a) We use the back-and-forth method. Enumerate both sides of the bipartition: $X = \{x_\alpha : \alpha < \kappa\}$ and $Y = \{y_\alpha : \alpha < \kappa\}$. Now define a sequence $\langle G_\alpha : 0 < \alpha < \kappa \rangle$ of sub-bigraphs of G, each of cardinality less than κ, in the following way by recursion: let $X_0 = Y_0 = E_0 = \emptyset$ and suppose $G_\beta = (X_\beta, Y_\beta, E_\beta)$ are defined for $\beta < \alpha$. If α is limit, let $X_\alpha = \bigcup_{\beta<\alpha} X_\beta$, $Y_\alpha = \bigcup_{\beta<\alpha} Y_\beta$ and $E_\alpha = \bigcup_{\beta<\alpha} E_\beta$. If $\alpha = \beta + 1$, there are several cases.

First let $x_\beta \notin X_\beta$, $y_\beta \notin Y_\beta$ and $x_\beta y_\beta \notin E$. Then by Lemma 3 there is $x' \in X$ such that $x' y_\beta \in E$ and $x' \neq x$ for all $x \in X_\beta$. Analogously we find $y' \in Y \setminus Y_\beta$ such that $x_\beta y' \in E$. Now we put $X_\alpha = X_\beta \cup \{x_\beta, x'\}$, $Y_\alpha = Y_\beta \cup \{y_\beta, y'\}$ and $E_\alpha = E_\beta \cup \{x_\beta y', x' y_\beta\}$.

The other cases ($x_\beta \in X_\beta$, $y_\beta \in Y_\beta$ or $x_\beta y_\beta \in E$) are simpler and they are handled analogously. In the end $(\bigcup_{\alpha<\kappa} X_\alpha, \bigcup_{\alpha<\kappa} Y_\alpha, \bigcup_{\alpha<\kappa} E_\alpha)$ is a sub-bigraph of G that is a perfect matching.

(b) In this part of the proof we will abuse our notation a little, identifying a perfect matching of G with a bijection $f : X \to Y$ such that $xf(x) \in E$ for all $x \in X$.

We enumerate all the edges of G: $E = \{e_\alpha : \alpha < \kappa\}$ (by Lemma 4 there is exactly κ of them). Now we iterate the construction from (a) κ-many times, defining a sequence $\langle S_\alpha : \alpha < \kappa \rangle$ of spanning sub-bigraphs of G by recursion.

Let $E_0 = \emptyset$ and suppose $S_\beta = (X, Y, E_\beta)$ are defined for $\beta < \alpha$ in such way that

(6) the degree of every vertex in S_β is less than κ.

By Lemma 6 this will mean that $(X, Y, E \setminus E_\beta)$ are all (κ, κ, κ)-random dense.

If $\alpha = \beta + 1$ then we repeat the construction from (a) in the graph $(X, Y, E \setminus E_\beta)$ with an additional condition: if $e_\beta = xy \notin E_\beta$, we let $x_0 = x$ and $y_0 = y$. In this way we get a perfect matching f_β including the edge e_β. Now we define $S_\alpha = (X, Y, E_\alpha)$ with $E_\alpha = E_\beta \cup f_\beta$. Clearly, S_α satisfies (6).

If α is a limit ordinal, we put $E_\alpha = \bigcup_{\beta<\alpha} E_\beta$. Again we have (6), since in every of $\alpha < \kappa$ steps we added exactly one edge to each of the vertices.

In the end $E = \bigcup_{\alpha<\kappa} E_\alpha$, so the bigraph G is the disjoint union of matchings f_α for $\alpha < \kappa$. $\square$

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References

[1] Cameron, P., The random graph. In: The Mathematics of Paul Erdös. 2, (J. Nešetřil, R. L. Graham, eds.) Springer, Berlin (1997), 333–351.

[2] Comfort, W. W., Hu, W., Maximal independent families and a topological consequence. Topology Appl. 127 (2003), 343-354.

[3] Eckertson, F. W., Resolvable, not maximally resolvable spaces. Topology Appl. 79 (1997), 1-11.

[4] Goldstern, M., Grossberg, R., Kojman M., Infinite homogeneous bipartite graphs with unequal sides. Discrete Math. 149 (1996) 69-82.

[5] Goldstern, M., Kojman, M., Rules and Reals. Proc. Amer. Math. Soc. 127 (1999), 1517-1524.

[6] Kojman, M., Shelah, S., Homogeneous families and their automorphism groups. J. Lond. Math. Soc. (2) 52 (1995), 303-317.

[7] Kunen, K., Set Theory. An Introduction to Independence Proofs. North-Holland, Amsterdam-London, 1980.

[8] Kunen, K., Maximal σ-independent families. Fund. Math. 117 (1983), 75-80.

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