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ON FULLY ω_1-$p^{\omega+n}$-PROJECTIVE ABELIAN p-GROUPS

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Abstract. We define the class of *fully* ω_1*-*$p^{\omega+n}$*-projective* abelian p-groups and establish its crucial properties. It is shown that this class is situated between the classes of *strongly* ω_1*-*$p^{\omega+n}$*-projective* and ω_1-$p^{\omega+n}$*-projective* abelian p-groups, and it was constructed a fully ω_1-$p^{\omega+n}$-projective group that is not strongly ω_1-$p^{\omega+n}$-projective thus showing that one of the inclusions is proper.

These results strengthen theorems due to Keef in J. Algebra Numb. Theory Acad. (2010) and also continue our own achievements in Hacettepe J. Math. Stat. (2014).

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1. Introduction and terminology

Let all groups discussed here be abelian p-groups, where p is a prime number fixed for the remainder of the paper, and let $n \geq 0$ be an arbitrary integer. Our notions and notations are standard and follow essentially those from [6].

Imitating [9], a group G is said to be $p^{\omega+n}$*-projective* if there exists a p^n-bounded subgroup P such that G/P is Σ-cyclic (i.e., a direct sum of cyclic groups). Equivalently, G is $p^{\omega+n}$-projective exactly when $G \cong S/L$ where S is Σ-cyclic and L is bounded by p^n.

Using this equivalence, the class of so-called ω_1*-*$p^{\omega+n}$*-projective groups* was defined in a parallel way in [7]. Two of their important characterizations are the following:

Theorem 1.1. *The group G is ω_1-$p^{\omega+n}$-projective if exactly one of the following conditions holds:*

(i) There is a countable nice subgroup C such that ($p^{\omega+n}G \subseteq C \subseteq p^{\omega}G$ and) G/C is $p^{\omega+n}$-projective.

(ii) There is a p^n-bounded subgroup P such that G/P is the direct sum of a countable group and a Σ-cyclic group.

The latter point is obviously tantamount to the following one: *There are a countable subgroup K and a p^n-bounded subgroup P such that $G/(K + P)$*

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is Σ*-cyclic.* In fact, using (ii), we write $G/P = (A/P) \oplus (B/P)$ where the first summand is countable while the second one is Σ-cyclic. Hence, $G/A \cong (G/P)/(A/P) \cong B/P$ is Σ-cyclic. But one may write $A = P + K$, where K is countable, which substantiates the claim.

Conversely, $G/(K + P) \cong [G/P]/[(K + P)/P]$ being Σ-cyclic guarantees that with the aid of the well-known Charles' lemma that G/P is the direct sum of a countable group and a Σ-cyclic group, because $(K + P)/P \cong K/(K \cap P)$ is countable, thus obtaining (ii). The claim now sustained.

Note that when P is nice in G, these groups are precisely the *strongly* ω_1*-*$p^{\omega+n}$*-projective groups*, that is, $G/(K + P)$ is Σ-cyclic for some countable subgroup K and some p^n-bounded nice subgroup P – see [2] for more details.

It would be interesting, and perhaps non-trivial, to add in contrast to P the condition of niceness on K. So, we indicate this difference in the following:

Definition 1. A group G is called *fully* ω_1*-*$p^{\omega+n}$*-projective* if there exist a countable nice subgroup K of G and $P \leq G[p^n]$ with Σ-cyclic quotient $G/(K + P)$.

In particular, if K is finite, G is said to be *fully* ω*-*$p^{\omega+n}$*-projective.*

It is self-evident that fully ω_1-$p^{\omega+n}$-projective groups are themselves ω_1-$p^{\omega+n}$-projective. A little surprising fact, proved below in Proposition 3.3, is that strongly ω_1-$p^{\omega+n}$-projective groups are themselves fully ω_1-$p^{\omega+n}$-projective, that is, in a way the niceness of some P yields niceness of some K.

Some elementary properties are the following:

(1) The inclusion $p^\omega G \subseteq K + P = C \oplus L$ holds, where C is countable and L is p^n-bounded. Consequently, $p^\omega G$ is the direct sum of a countable group and a p^n-bounded group or, equivalently, $p^{\omega+n}G$ is countable.

(2) Each $p^{\omega+n}$-projective group is obviously fully ω_1-$p^{\omega+n}$-projective – indeed, just take $K = \{0\}$. Conversely, separable fully ω_1-$p^{\omega+n}$-projective groups are $p^{\omega+n}$-projective. Even something more is true: The direct sum of a countable group and a $p^{\omega+n}$-projective group is fully ω_1-$p^{\omega+n}$-projective being strongly ω_1-$p^{\omega+n}$-projective (see [1] and [2]).

(3) Every fully ω_1-$p^{\omega+n}$-projective group is ω_1-$p^{\omega+n}$-projective – in fact, $G/(K + P) \cong [G/P]/[(K + P)/P]$ is Σ-cyclic and $(K + P)/P \cong K/(K \cap P)$ is countable, whence by the well-known Charles' lemma G/P is the direct sum of a countable group and a Σ-cyclic group, as required in Theorem 1.1 (ii).

Definition 2. A group G is called *separately full* ω_1*-*$p^{\omega+n}$*-projective* if there exist a countable nice subgroup K of G and $P \leq G[p^n]$ with $P \cap K = \{0\}$ such that $G/(K \oplus P)$ is Σ-cyclic.

In particular, if K is finite, G is said to be *separately full* ω*-*$p^{\omega+n}$*-projective.*

2. Three preliminary technicalities

The following three technical claims possess a central position as they subsumed our basic results.

Lemma 2.1. *Let $\alpha \geq \omega$ be an ordinal and let G be a group with a nice subgroup N such that $N \supseteq p^{\alpha+n}G$ for some $n \in \mathbb{N}$, and with a subgroup C such that $C \subseteq p^{\alpha}G$. Then $C + N$ is nice in G.*

Proof. For simpleness denote $M = C + N$. For all limit ordinals γ we can distunguish the following basic cases:

Case 1: $\gamma \leq \alpha$. Thus one observes that $\cap_{\beta<\gamma}(M + p^{\beta}G) = \cap_{\beta<\gamma}(N + C + p^{\beta}G) = \cap_{\beta<\gamma}(N + p^{\beta}G) = N + p^{\gamma}G = M + p^{\gamma}G$, as required.

Case 2: $\gamma > \alpha$. So one sees that $\cap_{\beta<\gamma}(M+p^{\beta}G) = \cap_{\beta<\gamma}(C+N+p^{\beta}G) = [\cap_{\alpha<\beta<\gamma}(C+N+p^{\beta}G)]\cap[\cap_{\beta\leq\alpha<\gamma}(C+N+p^{\beta}G)] = [\cap_{\alpha+n\leq\beta<\gamma}(C+N+p^{\beta}G)] = C + N \subseteq M + p^{\gamma}G$, as required. □

Lemma 2.2. *Suppose that α is an ordinal, and that G and F are groups where F is finite. Then the following formula is fulfilled:*

$$p^{\alpha}(G + F) = p^{\alpha}G + [F \cap p^{\alpha}(G + F)].$$

Proof. We will use a transfinite induction on α. First, if $\alpha - 1$ exists, we have

$$\begin{aligned}
p^{\alpha}(G + F) &= p(p^{\alpha-1}(G + F)) \\
&= p(p^{\alpha-1}G + [F \cap p^{\alpha-1}(G + F)]) \\
&= p(p^{\alpha-1}G) + p(F \cap p^{\alpha-1}(G + F)) \\
&\subseteq p^{\alpha}G + [F \cap p(p^{\alpha-1}(G + F))] = p^{\alpha}G + [F \cap p^{\alpha}(G + F)].
\end{aligned}$$

Since the reverse inclusion "$\supseteq$" is obvious, we obtain the desired equality.

If now $\alpha - 1$ does not exist, we have that $p^{\alpha}(G+F) = \cap_{\beta<\alpha}(p^{\beta}(G+F)) \subseteq \cap_{\beta<\alpha}(p^{\beta}G + F) = \cap_{\beta<\alpha}p^{\beta}G + F = p^{\alpha}G + F$. In fact, the second sign "=" follows like this: Given $x \in \cap_{\beta<\alpha}(p^{\beta}G + F)$, we write that $x = g_{\beta_1} + f_1 = \cdots = g_{\beta_s} + f_s = \cdots$ where $f_1, \cdots, f_s \in F$ are the all elements of F; $g_{\beta_1} \in p^{\beta_1}G, \cdots, g_{\beta_s} \in p^{\beta_s}G$ with $\beta_1 < \cdots < \beta_s < \cdots$.

Since F is finite, while the number of equalities is infinite due to the infinite cardinality of α, we infer that $g_{\beta_s} \in p^{\beta}G$ for any ordinal $\beta < \alpha$ which means that $g_{\beta_s} \in \cap_{\beta<\alpha}p^{\beta}G = p^{\alpha}G$. Thus $x \in \cap_{\beta<\alpha}p^{\beta}G + F = p^{\alpha}G + F$, as claimed. Furthermore, $p^{\alpha}(G + F) \subseteq (p^{\alpha}G + F) \cap p^{\alpha}(G + F) = p^{\alpha}G + [F \cap p^{\alpha}(G + F)]$ which is obviously equivalent to an equality. □

Lemma 2.3. *Let N be a nice subgroup of a group G. Then*

(i) $N + R$ is nice in G for every finite subgroup $R \leq G$;

(ii) N is nice in $G + F$ for each finite group F.

Proof. (i) For any limit ordinal γ, we deduce that $\cap_{\delta<\gamma}(N + R + p^{\delta}G) \subseteq R + \cap_{\delta<\gamma}(N + p^{\delta}G) = R + N + p^{\gamma}G$, as required. Indeed, the relation "$\subseteq$" follows like this: Given $x \in \cap_{\delta<\gamma}(N+R+p^{\delta}G)$, we write $x = a_1+r_1+g_1 = \cdots = a_s + r_s + g_s = \cdots = a_k + r_1 + g_k = \cdots$, where $a_1, \cdots, a_k \in N$; $r_1, \cdots, r_k \in R$; $g_1 \in p^{\delta_1}G, \cdots, g_k \in p^{\delta_k}G$ with $\delta_1 < \cdots < \delta_k$. So $a_1 + g_1 = \cdots = a_k + g_k = \cdots \in \cap_{\delta<\gamma}(N + p^{\delta}G)$ and hence $x \in R + \cap_{\delta<\gamma}(N + p^{\delta}G)$, as requested.

(ii) Since N is nice in G, we may write $\cap_{\delta<\gamma}[N+p^\delta G] = N+p^\gamma G$ for every limit ordinal γ. Furthermore, with Lemma 2.2 at hand, we subsequently deduce that

$$\cap_{\delta<\gamma}[N+p^\delta(G+F)] = \cap_{\delta<\gamma}[N+p^\delta G+(F\cap p^\delta(G+F))] \subseteq$$

$$\cap_{\delta<\gamma}(N+p^\delta G)+[F\cap p^\gamma(G+F)] = N+p^\gamma G+[F\cap p^\gamma(G+F)] = N+p^\gamma(G+F).$$

In fact, the inclusion "$\subseteq$" follows thus: Given $x \in \cap_{\delta<\gamma}[N+p^\delta G+(F\cap p^\delta(G+F))]$, we write $x = a_1+g_1+f_1 = \cdots = a_s+g_s+f_s = \cdots = a_k+g_k+f_1 = \cdots$, where $a_1,\cdots,a_k \in N$; $g_1 \in p^{\delta_1}G,\cdots,g_k \in p^{\delta_k}G$; $f_1 \in F\cap p^{\delta_1}(G+F),\cdots,f_k \in F\cap p^{\delta_k}(G+F)$ with $\delta_1<\cdots<\delta_k$. Hence $a_1+g_1 = \cdots = a_k+g_k = \cdots \in \cap_{\delta<\gamma}(N+p^\delta G)$ and because the number of the f_i's $(1\le i\le k)$ is finite whereas the number of equalities is not, we can deduce that $f_1 \in \cap_{\delta<\gamma}(F\cap p^\delta(G+F)) = F\cap p^\gamma(G+F)$, as needed. □

3. Fully ω_1-$p^{\omega+n}$-projective groups

We start here with a new useful criterion for a group to be ω_1-$p^{\omega+n}$-projective, which cannot be found in [7].

Proposition 3.1. *The group G is ω_1-$p^{\omega+n}$-projective if and only if $G/(K\oplus P)$ is Σ-cyclic for some p^n-bounded subgroup P and some countable subgroup K such that $K\cap P=\{0\}$ and p^nK is nice in G.*

Proof. "$\Rightarrow$". Utilizing Theorem 1.1 (i), let G/C be $p^{\omega+n}$-projective for some countable subgroup satisfying the inequalities $p^{\omega+n}G\subseteq C\subseteq p^\omega G$; note that this automatically forces C to be nice in G. That is why there is $L\le G$ containing C such that $p^nL\subseteq C$ and G/L is Σ-cyclic. Consequently, $L=K\oplus P$ where K is countable and P is p^n-bounded. Thus, $G/(K\oplus P)$ is Σ-cyclic. But $p^\omega G\subseteq L$, whence $p^{\omega+n}G\subseteq p^nL=p^nK\subseteq C\subseteq p^\omega G$. These imply that p^nK is nice in G, as asserted.

"$\Leftarrow$". Observe that $G/(K\oplus P)\cong[G/P]/[(K\oplus P)/P]$ is Σ-cyclic, where $(K\oplus P)/P\cong K$ is countable. Therefore, the well-known Charles' lemma ensures that G/P has to be the direct sum of a countable group and a Σ-cyclic group. Hence Theorem 1.1 (ii) applies to conclude that G is ω_1-$p^{\omega+n}$-projective, as claimed. □

Remark 1. It is noteworthy that if N is a nice subgroup of a group G, then for any natural n the subgroup p^nN need not be nice in G nor in p^nG, and conversely p^nN being nice in G, or in p^nG, for some arbitrary natural n does not yield that N is nice in G. Thereby, at first glance, it seems that the class of ω_1-$p^{\omega+n}$-projective groups properly encompasses the class of fully ω_1-$p^{\omega+n}$-projective groups. However, a concrete example of an ω_1-$p^{\omega+n}$-projective group which is not fully ω_1-$p^{\omega+n}$-projective is not constructed yet.

The next necessary and sufficient condition was originally established in [1], but here we will give a more conceptual proof in terms of strongly n-simply presented groups defined in [8]. Recall that a group G is called *strongly n-simply presented* if there exists $P \leq G[p^n]$ such that P is nice in G and G/P is simply presented.

Theorem 3.2. *The group G is strongly ω_1-$p^{\omega+n}$-projective if and only if $p^{\omega+n}G$ is countable and $G/p^{\omega+n}G$ is $p^{\omega+n}$-projective.*

Proof. **"Necessity".** Referring to [1], the initial definition of a strongly ω_1-$p^{\omega+n}$-projective group G is so that there is a nice subgroup P bounded by p^n such that G/P is the direct sum of a countable group and a Σ-cyclic group. Now appealing to the above statement of strongly n-simply presentness, it is clear that strongly ω_1-$p^{\omega+n}$-projective groups are strongly n-simply presented, so that by virtue of Theorem 3.4 (a) and Proposition 2.5 both from [8] we conclude that $G/p^{\omega+n}G$ must be $p^{\omega+n}$-projective.

That $p^{\omega+n}G$ is countable follows either directly by using the definition, or from [7] because these groups are necessarily ω_1-$p^{\omega+n}$-projective.

"Sufficiency". According to Theorem 3.4 (b) of [8], G must be strongly n-simply presented such that $p^\omega G$ is simply presented. This enables us to utilize Theorem 3.1 from [1] (see also Corollary 3.2 of there) to write that $G \oplus K = T \oplus P$, where K is a unbounded countable Σ-cyclic group, T is simply presented, and P is $p^{\omega+n}$-projective. Since $p^{\omega+n}G = p^{\omega+n}T$ is countable, it follows that T is the direct sum of a countable group and a direct sum of countable groups of length not exceeding $\omega + n$ and so it is the direct sum of a countable group and a $p^{\omega+n}$-projective group. Thus $T \oplus P = C \oplus L$ and hence $G \oplus K = C \oplus L$, where C is countable and L is $p^{\omega+n}$-projective. Let $M \leq L[p^n]$ such that L/M is Σ-cyclic. Therefore, $(G \oplus K)/M = (C \oplus L)/M \cong C \oplus (L/M)$ is the direct sum of a countable group and a Σ-cyclic group. But M is nice in $C \oplus L = G \oplus K$ because M is nice in L. Moreover, writing $M = N \oplus Q$ where $N \leq G$ and $Q \leq K$, we plainly see that N has to be nice in G. We furthermore deduce that $(G/N) \oplus (K/Q) \cong (G \oplus K)/(N \oplus Q)$ is the direct sum of a countable group and a Σ-cyclic group. Since K/Q can easily be embedded in a countable direct summand of such a sum, it follows immediately that G/N is also the direct sum of a countable group and a Σ-cyclic group. But this means that G is strongly ω_1-$p^{\omega+n}$-projective, as expected. □

We are now ready to prove the following:

Proposition 3.3. *Every strongly ω_1-$p^{\omega+n}$-projective group is fully ω_1-$p^{\omega+n}$-projective.*

Proof. In view of Theorem 3.2, a group G being strongly ω_1-$p^{\omega+n}$-projective gives that $p^{\omega+n}G$ is countable and $G/p^{\omega+n}G$ is $p^{\omega+n}$-projective. Thus, one may write that $p^\omega G = C \oplus L$, where C is countable and L is p^n-bounded. Since C is nice in $p^\omega G$, it follows that it is nice in G as well.

On the other hand, $(G/p^{\omega+n}G)/(T/p^{\omega+n}G) \cong G/T$ is Σ-cyclic, and hence $p^\omega G \subseteq T$, where $p^nT \subseteq p^{\omega+n}G$. Thus $T \subseteq p^\omega G + G[p^n]$ and the modular law ensures that $T = T \cap (p^\omega G + G[p^n]) = p^\omega G + (T \cap G[p^n]) = p^\omega G + T[p^n]$. Consequently, $G/(p^\omega G + T[p^n]) = G/(C \oplus L + T[p^n]) = G/(C + P)$ is Σ-cyclic, where $P = L + T[p^n]$ is obviously bounded by p^n, as required. □

As a different and not too direct idea for a proof, it is worth noting that, by property (2) proved above in the introductory section, the quotient $G/p^{\omega+n}G$ being $p^{\omega+n}$-projective is fully ω_1-$p^{\omega+n}$-projective. Hence Corollary 3.12 proved below is applicable to infer that G is fully ω_1-$p^{\omega+n}$-projective, as expected.

So, as a valuable consequence, we derive:

Corollary 3.4. *If G is strongly n-simply presented group whose $p^{\omega+n}G$ is countable, then G is fully ω_1-$p^{\omega+n}$-projective.*

Proof. It was obtained in [1] that such a group G is strongly ω_1-$p^{\omega+n}$-projective, and hence Proposition 3.3 works to get the claim. □

Theorem 3.5. *Suppose that G is a group. Then the following two equivalencies hold:*

(a) G is fully ω_1-$p^{\omega+n}$-projective if and only if G/F is fully ω_1-$p^{\omega+n}$-projective, whenever $F \leq G$ is finite.

(b) If $A \leq G$ such that G/A is finite, then G is fully ω_1-$p^{\omega+n}$-projective if and only if A is fully ω_1-$p^{\omega+n}$-projective.

Proof. (a) "$\Rightarrow$". Assume $G/(K + P)$ is Σ-cyclic for some countable nice subgroup $K \leq G$ and $P \leq G[p^n]$. Since $(K+P+F)/(K+P) \cong F/[F \cap (K+P)]$ is finite, the quotient $[G/(K+P)]/[(K+P+F)/(K+P)] \cong G/(K+P+F) \cong [G/F]/[(K+P+F)/F] = [G/F]/[((K+F)/F) + ((P+F)/F)]$ is Σ-cyclic. But $(K+F)/F$ is countable and nice in G/F in virtue of Lemma 2.3 (i), and moreover it follows at once that $(P+F)/F \cong P/(P \cap F)$ is bounded by p^n, as wanted.

"$\Leftarrow$". Let $[G/F]/[(K/F) + (L/F)] = [G/F]/[(K+L)/F] \cong G/(K+L)$ be Σ-cyclic, where K/F is countable and nice in G/F, whence K is countable and nice in G, and L/F is bounded by p^n, hence $p^n L \subseteq F$. Thus $L = M + P$ where M is finite and $p^n P = \{0\}$. Consequently, $G/(K+L) = G/(K+M+P)$ is Σ-cyclic. However, $K+M$ is countable as well as Lemma 2.3 (i) implies that $K+M$ is nice in G, as desired.

(b) "$\Rightarrow$". Write $G = A + F$ where $F \leq G$ is finite. Observing that $G/F = (A+F)/F \cong A/(A \cap F)$, where $A \cap F$ is obviously finite, we consequently employ point (a) to get that G is fully ω_1-$p^{\omega+n}$-projective uniquely when so is A, as stated.

"$\Leftarrow$". Write $G = A + F$ where $F \leq G$ is finite, and let $A/(K+P)$ be Σ-cyclic where K is a countable nice subgroup of A and $P \leq A[p^n]$. With Lemma 2.3 (ii) at hand, K remains nice in $A + F = G$. We therefore see that

$$G/(K+P) = [A/(K+P)] + [(F+K+P)/(K+P)],$$

where the second term is finite, being isomorphic to $F/(F \cap (K+P))$. Finally, one infers that $G/(K+P)$ is Σ-cyclic, as required. □

As a direct consequence, we obtain:

Corollary 3.6. *Fully ω_1-$p^{\omega+n}$-projective groups are closed under the formation of ω-bijections.*

Another helpful consequence is the following (compare with Corollary 3.11 below):

Corollary 3.7. *Suppose G is a group for which $p^\alpha G$ is finite for some arbitrary ordinal α. Then G is fully ω_1-$p^{\omega+n}$-projective if and only if $G/p^\alpha G$ is fully ω_1-$p^{\omega+n}$-projective.*

Proof. Just put $p^\alpha G = F$, and use Theorem 3.5. □

We recollect that a group G is said to be *n-simply presented* if G/P is simply presented for some $P \leq G[p^n]$ (see [8] for more details). The next assertion demonstrates that under certain circumstances on the first Ulm subgroup the fully ω_1-$p^{\omega+n}$-projective groups can be described in two more equivalent directions:

Proposition 3.8. *Suppose G is a group such that $p^\omega G$ is finite. Then the following three points are equivalent:*
(a) G is n-simply presented;
(b) G is ω_1-$p^{\omega+n}$-projective;
(c) G is fully ω_1-$p^{\omega+n}$-projective.

Proof. The equivalence (a) $\iff$ (b) was proved in [7]. There was actually shown that these clauses are tantamount to $G/p^\omega G$ is $p^{\omega+n}$-projective. Thus appealing to Corollary 3.7 we get the desired claim. □

A rather natural question is whether or not the last statement remains true provided that $p^\omega G$ is infinitely countable, that is, if $G/p^\omega G$ is $p^{\omega+n}$-projective and $p^\omega G$ is countable, is then G fully ω_1-$p^{\omega+n}$-projective? It is noteworthy that if $G/p^\omega G$ is $p^{\omega+n}$-projective and $p^{\omega+n}G$ is countable whereas $p^\omega G$ is uncountable, G need not be ω_1-$p^{\omega+n}$-projective (see [3] too).

We continue in this way by the following:

Proposition 3.9. *Let G be a fully ω_1-$p^{\omega+n}$-projective group. Then*
(i) $p^\alpha G$ is fully ω_1-$p^{\omega+n}$-projective for any ordinal $\alpha \geq \omega$.
(ii) $G/p^\alpha G$ is fully ω_1-$p^{\omega+n}$-projective for any ordinal $\alpha \geq \omega + n$.

Proof. (i) Using the property (1) from Section 1, we have $p^\omega G = C \oplus L$ where C is countable and L is p^n-bounded. So, for each $\alpha \geq \omega$, we deduce $p^\alpha G = X \oplus Y$ where X is countable and Y is p^n-bounded. But such a direct sum is obviously fully ω_1-$p^{\omega+n}$-projective; in fact, $(X \oplus Y)/(X \oplus Y) \cong \{0\}$ is trivially Σ-cyclic, where X is nice in $X \oplus Y$, being a direct summand.

(ii) Assume that $G/(K + P)$ is Σ-cyclic for some countable nice subgroup $K \leq G$ and $P \leq G[p^n]$. Again an appeal to property (1) enables us to conclude that $p^{\omega+n}G \subseteq K$, so that $p^\alpha G \subseteq K$. Observe that

$$G/(K+P) = G/(K+P+p^\alpha G) \cong [G/p^\alpha G]/[(K+P+p^\alpha G)/p^\alpha G]$$

$$= [G/p^\alpha G]/[((K+p^\alpha G)/p^\alpha G) + ((P+p^\alpha G)/p^\alpha G)].$$

Since $(K+p^\alpha G)/p^\alpha G \cong K/(K\cap p^\alpha G)$ is obviously countable and nice in $G/p^\alpha G$ (see, e.g., [6], Lemma 79.3 (1)), as well as $(P+p^\alpha G)/p^\alpha G \cong P/(P\cap p^\alpha G)$ is p^n-bounded, the claim follows. □

The question of validity of Proposition 3.9 when $\alpha \in \mathbb{N}$ in (i) and $\omega \leq \alpha \leq \omega+n-1$ remains unanswered yet.

The next assertion somewhat extends Corollary 3.7 but only for ordinals of the type $\alpha+n$.

Proposition 3.10. *Suppose G is a group for which $p^{\alpha+n}G$ is countable for some arbitrary ordinal α. If $G/p^{\alpha+n}G$ is fully ω_1-$p^{\omega+n}$-projective, then so is G.*

Proof. Since the assertion is trivial when α is a finite non-negative integer, we shall assume that α is infinite, i.e., $\alpha \geq \omega$. By the corresponding definition there is a countable nice subgroup $N/p^{\alpha+n}G \leq G/p^{\alpha+n}G$ with $p^{\alpha+n}G \leq N \leq G$, whence N is a countable nice subgroup of G (cf. [6]). Also, there is $Z/p^{\alpha+n}G \subseteq (G/p^{\alpha+n}G)[p^n]$ with $p^{\alpha+n}G \leq Z \leq G$, and hence $p^nZ \subseteq p^{\alpha+n}G \subseteq Z$. Therefore,

$$[G/p^{\alpha+n}G]/[(N/p^{\alpha+n}G) + (Z/p^{\alpha+n}G)] = [G/p^{\alpha+n}G]/[(N+Z)/p^{\alpha+n}G]$$

$$\cong G/(N+Z)$$

is Σ-cyclic, whence $p^\omega G \subseteq N+Z$ and thus $p^\alpha G \subseteq N+Z$.

Furthermore, let B be a maximal p^n-bounded subgroup of $p^\alpha G$, so that one may decompose $p^\alpha G = B \oplus V$. Let now H be a $p^{\alpha+n}$-high subgroup of G containing B. Since $V[p] = (p^{\alpha+n}G)[p]$, it is self-evident that $V \cap H = \{0\}$. Likewise, $p^\alpha G = B\oplus V \subseteq H[p^n]\oplus V$ and so $p^{\alpha+n}G \subseteq H[p^n]\oplus V$. Moreover, since $p^nV = p^{\alpha+n}G$, it is also clear to see that $(V\oplus H[p^n])/p^{\alpha+n}G \subseteq (G/p^{\alpha+n}G)[p^n]$.

We next claim that $(G/p^{\alpha+n}G)[p^n] = (V\oplus H[p^n])/p^{\alpha+n}G$. To that goal, given $x+p^{\alpha+n}G$ in the left hand-side, we see that $p^nx \in p^{\alpha+n}G = p^nV$, i.e., $p^nx = p^ny$ where $y \in V$. Hence $x \in V + G[p^n]$. But, on the other hand, $G[p] = (p^{\alpha+n}G)[p]\oplus H[p] = V[p]\oplus H[p]$ and since H is pure in G (cf. [6]), it readily follows that $G[p^n] = V[p^n]\oplus H[p^n]$. Finally, $V + G[p^n] = V\oplus H[p^n]$ and thus $x \in V\oplus H[p^n]$, as required.

The last paragraph allows us to infer that $Z \subseteq V\oplus H[p^n]$. Letting

$$E = (Z+V)\cap H[p^n],$$

we detect via the modular law that $Z+V=E+V$ because $Z+V \subseteq V \oplus H[p^n]$ and hence $Z+V=(Z+V)\cap(V\oplus H[p^n])=V+(Z+V)\cap H[p^n]=V+E$. Finally, by adding in both sides of the above equality the member $N+p^\alpha G$, we can conclude that $Z+N+p^\alpha G=E+N+p^\alpha G$, where $E\leq G[p^n]$, because $V\leq p^\alpha G$. This means that $N+Z=N+p^\alpha G+E$. Since $p^{\alpha+n}G$ is countable, one can decompose $p^\alpha G=C\oplus P$, where C is countable and P is p^n-bounded, we infer that $G/(N+Z)=G/(N+p^\alpha G+E)=G/((N+C)+(P+E))=G/(M+L)$ is Σ-cyclic, where in conjunction with Lemma 2.1 we have that $M=N+C$ is countable and nice in G, and $L=P+E$ is bounded by p^n, as required. □

We are not currently able to settle whether or not Corollary 3.7 will be true provided $p^\alpha G$ is countable for some arbitrary ordinal α or even in the case $\alpha=\omega$. We are also wondering whether or not in Proposition 3.10 the ordinal $\alpha+n$ could be replaced by α even if $\alpha=\omega$.

Summarizing the above corresponding assertions, we immediately deduce:

Corollary 3.11. *Suppose that G is a group for which $p^\alpha G$ is countable for some ordinal $\alpha\geq\omega+n$. Then G is fully ω_1-$p^{\omega+n}$-projective if and only if $G/p^\alpha G$ is fully ω_1-$p^{\omega+n}$-projective.*

Proof. As for the "and only if" part, we apply Proposition 3.9 (ii).
Concerning the "if" part, we employ Proposition 3.10. □

For the special case when $\alpha=\omega+n$, we have the following reduction criterion which illustrates that we can investigate only fully ω_1-$p^{\omega+n}$-projective groups of length not exceeding $\omega+n$ (notice that the same follows for ω_1-$p^{\omega+n}$-projective groups – see [7]).

Corollary 3.12. *The group G is fully ω_1-$p^{\omega+n}$-projective if and only if $p^{\omega+n}G$ is countable and $G/p^{\omega+n}G$ is fully ω_1-$p^{\omega+n}$-projective.*

So, we are ready to give an example of a fully ω_1-$p^{\omega+n}$-projective group which need not be a strongly ω_1-$p^{\omega+n}$-projective group (compare with Proposition 3.8).

Example 3.13. *There exists a fully ω_1-$p^{\omega+n}$-projective group with finite first Ulm subgroup that is not strongly ω_1-$p^{\omega+n}$-projective.*

Proof. Suppose that G is a group such that $p^\omega G\cong\mathbb{Z}(p^n)$ and $G/p^\omega G$ is $p^{\omega+n}$-projective. It was shown in [7] that G is ω-$p^{\omega+n}$-projective but not $p^{\omega+n}$-projective. Since $p^{\omega+n}G=\{0\}$, it follows from [1] that G is not strongly ω_1-$p^{\omega+n}$-projective as well. However, it follows from Theorem 3.5 (a) that G is fully ω_1-$p^{\omega+n}$-projective.

We will now demonstrate something more, namely that G is even fully ω-$p^{\omega+n}$-projective. In fact, in virtue of [9], G/L is Σ-cyclic for some $L\leq G$ such that $p^nL\subseteq p^\omega G\subseteq L$. Thus we can write $L=F\oplus M$ where F is finite and M is p^n-bounded. Finally, $G/(F\oplus M)$ is Σ-cyclic with F being nice in G, as required. □

The proof of the last statement suggests the following (compare with Corollary 3.7, too):

Proposition 3.14. *Suppose G is a group whose $p^\alpha G$ is finite for some arbitrary ordinal α. If $G/p^\alpha G$ is fully ω-$p^{\omega+n}$-projective, then G is fully ω-$p^{\omega+n}$-projective, and vice versa.*

Similarly, as in Proposition 3.9, it can be established that if G is fully ω-$p^{\omega+n}$-projective, then so is $p^\alpha G$ for every $\alpha \geq \omega$, as well as that $G/p^\alpha G$ is fully ω-$p^{\omega+n}$-projective for every $\alpha \geq \omega + n$. So, summarizing the corresponding assertions listed above, we directly derive:

Corollary 3.15. *Suppose that G is a group for which $p^\alpha G$ is finite for some ordinal $\alpha \geq \omega + n$. Then G is fully ω-$p^{\omega+n}$-projective if and only if $G/p^\alpha G$ is fully ω-$p^{\omega+n}$-projective.*

The following gives some condition when a direct sum of fully ω_1-$p^{\omega+n}$-projective groups is again a fully ω_1-$p^{\omega+n}$-projective group.

Proposition 3.16. *The countable direct sum of fully ω_1-$p^{\omega+n}$-projective groups is a fully ω_1-$p^{\omega+n}$-projective group.*

Proof. Write $G = \oplus_{\aleph_0} G_i$ where each G_i is fully ω_1-$p^{\omega+n}$-projective. So, let $G_i/(K_i + P_i)$ be Σ-cyclic for some countable nice subgroup K_i and p^n-bounded subgroup P_i. Setting $K = \oplus_{\aleph_0} K_i$ and $P = \oplus_{\aleph_0} P_i$, one easily sees that K remains countable and also nice in G, whereas P remains bounded by p^n. Furthermore, $G/(K+P) = [\oplus_{\aleph_0} G_i]/[(\oplus_{\aleph_0} K_i)+(\oplus_{\aleph_0} P_i)] = [\oplus_{\aleph_0} G_i]/[\oplus_{\aleph_0}(K_i + P_i)] \cong \oplus_{\aleph_0}[G_i/(K_i + P_i)]$ is Σ-cyclic, as required. $\square$

We finish off with a statement which explores when a sum of a countable group and a $p^{\omega+n}$-projective group is fully ω_1-$p^{\omega+n}$-projective.

Proposition 3.17. *Suppose that $G = A + B$ is a group whose subgroup A is countable and B is $p^{\omega+n}$-projective.*

(i) If B is balanced in G, then G is strongly ω_1-$p^{\omega+n}$-projective.

(ii) If A is nice in G and $A \cap B$ is finite, then G is fully ω_1-$p^{\omega+n}$-projective.

Proof. (i) Observing that $G/B = (A + B)/B \cong A/(A \cap B)$ is countable, we apply [6] to write that $G = B \oplus C$ where C is countable. Henceforth, we just employ observation (2) from Section 1.

(ii) We observe that $G = A + B$ implies $G/(A + P) = (A + B)/(A + P) = (A + P + B)/(A + P) \cong B/(B \cap (A + P)) = B/(P + (B \cap A)) \cong [B/P]/[(P + (B \cap A))/P]$, where $P \leq B[p^n]$ such that B/P is Σ-cyclic. But $(P + (B \cap A))/P \cong (B \cap A)/(B \cap A) \cap P) = (B \cap A)/(P \cap A)$ is finite, whence $G/(A + P)$ should be Σ-cyclic, as required. $\square$

4. Some Related Group Classes

We begin here with a few more definitions:

Definition 3. The group G is called *co-strongly ω_1-$p^{\omega+n}$-projective* if $G \cong S/(K+P)$ where S is Σ-cyclic, K is countable, and P is p^n-bounded and nice in G.

In addition, if $K \cap P = \{0\}$, then we will say that G is *co-separately strong ω_1-$p^{\omega+n}$-projective.*

Apparently, $p^{\omega+n}$-projective groups are co-strongly ω_1-$p^{\omega+n}$-projective by taking $K = \{0\}$.

A similar class of groups is one consisting of all groups G such that $G \cong H/T$ where H is a direct sum of a countable group and a Σ-cyclic group, and T is its nice p^n-bounded subgroup.

Some interesting properties of this group class are these:

(*) $p^\omega G$ is countable.

In fact, $p^\omega G \cong p^\omega(H/T) = (p^\omega H + T)/T \cong p^\omega H/(p^\omega H \cap T)$ is countable because so is $p^\omega H$.

(**) $p^\alpha G$ and $G/p^\alpha G$ are again from the same group class for all ordinals α.

In fact, one may observe that

$$p^\alpha G \cong p^\alpha(H/T) = (p^\alpha H + T)/T \cong p^\alpha H/(p^\alpha H \cap T).$$

Since $p^\alpha H$ is again a direct sum of a countable group and a Σ-cyclic group, while $p^\alpha H \cap T$ is p^n-bounded and nice in $p^\alpha H$, the first part follows.

As for the second one, we see that

$$[G/p^\alpha G] \cong [H/T]/[(p^\alpha H+T)/T] \cong H/(p^\alpha H+T) \cong [H/p^\alpha H]/[(p^\alpha H+T)/p^\alpha H],$$

where $H/p^\alpha H$ is obviously again a direct sum of a countable group and a Σ-cyclic group, and $(p^\alpha H + T)/p^\alpha H \cong T/(p^\alpha H \cap T)$ is p^n-bounded and nice in $H/p^\alpha H$.

(***) The direct sum of a countable group and a separable $p^{\omega+n}$-projective group also belongs to that group class.

In fact, write $G = K \oplus L$ where K is countable and L is p^ω-bounded $p^{\omega+n}$-projective. But it is well known that $L \cong S/P$ where S is Σ-cyclic and P is p^n-bounded and nice in S. Therefore, $G \cong K \oplus (S/P) \cong (K \oplus S)/P$. But P is nice in $K \oplus S$ because S is balanced in $K \oplus S$ being its direct summand. This substantiates the claim.

Definition 4. The group G is called *co-fully ω_1-$p^{\omega+n}$-projective* if $G \cong S/(K+P)$ where S is Σ-cyclic, K is countable and nice in G, and P is p^n-bounded.

In addition, if $K \cap P = \{0\}$, then G is said to be *co-separately full ω_1-$p^{\omega+n}$-projective.*

Evidently, $p^{\omega+n}$-projective groups are co-fully ω_1-$p^{\omega+n}$-projective. We shall show now that co-fully ω_1-$p^{\omega+n}$-projective groups are hardly new; actually they are exactly the $p^{\omega+n}$-projective groups and thus, in contrast to Proposition 3.3, co-fully ω_1-$p^{\omega+n}$-projective groups are co-strongly ω_1-$p^{\omega+n}$-projective. Specifically, the following is valid:

Proposition 4.1. *A group is co-fully ω_1-$p^{\omega+n}$-projective if and only if it is $p^{\omega+n}$-projective.*

Proof. Assume that $G \cong S/(K+P)$ where S is Σ-cyclic, K is countable and nice in G, and P is p^n-bounded. Furthermore, one may observe that $G \cong S/(K+P) \cong [S/K]/[(K+P)/K]$, where S/K remains Σ-cyclic, while $(K+P)/K \cong P/(P \cap K)$ is bounded by p^n, as required. □

A useful criterion when a group is co-strongly ω_1-$p^{\omega+n}$-projective is the following:

Proposition 4.2. *The group G is co-strongly ω_1-$p^{\omega+n}$-projective if and only if $G \cong A/C$ where A is separable $p^{\omega+n}$-projective and C is countable.*

Proof. **"Necessity"**. Let $G \cong S/(K+P)$ where S is Σ-cyclic, K is countable and P is p^n-bounded and nice in G. Therefore, $G \cong [S/P]/[(K+P)/P] = A/C$, where we put $A = S/P$ which is separable $p^{\omega+n}$-projective, and $C = (K+P)/P \cong K/(K \cap P)$ is countable.

"Sufficiency". Suppose $A = S/P$ where S is Σ-cyclic and P is p^n-bounded; since A is p^ω-bounded, it follows that P is nice in S. Moreover, write $C = V/P$ is countable, whence $V = P + K$ for some countable $K \leq V$. Finally, $G \cong A/C = [S/P]/[V/P] \cong S/V = S/(P+K)$, as required. □

A valuable consequence is the following:

Corollary 4.3. *Separable co-strongly ω_1-$p^{\omega+n}$-projective groups are themselves $p^{\omega+n}$-projective.*

Proof. Let G be such a group. The application of Proposition 4.2 assures that $G \cong A/C$ where A is separable $p^{\omega+n}$-projective and C is countable. Since G is separable, we next employ Theorem 4.2 of [4] to derive that G is $p^{\omega+n}$-projective. □

Remark 2. Note that another approach could be the usage of [7], because Proposition 4.2 implies that co-strongly ω_1-$p^{\omega+n}$-projective groups are ω_1-$p^{\omega+n}$-projective.

5. Concluding Discussion

As we have previously obtained in the introductory section (compare with Proposition 3.1 too), the group G is said to be *ω_1-$p^{\omega+n}$-projective* if there exist a countable subgroup $K \leq G$ and a p^n-bounded subgroup $P \leq G$ such that $G/(K+P)$ is Σ-cyclic. When P is nice in G, these groups are called *strongly* ω_1-$p^{\omega+n}$-projective, whereas if K is nice in G they are called *fully* ω_1-$p^{\omega+n}$-projective. Likewise, the following inclusions of group classes are fulfilled:

$$\{p^{\omega+n}\text{-projective}\} \subseteq \{\text{strongly } \omega_1\text{-}p^{\omega+n}\text{-projective}\} \subseteq$$
$$\subseteq \{\text{fully } \omega_1\text{-}p^{\omega+n}\text{-projective}\} \subseteq \{\omega_1\text{-}p^{\omega+n}\text{-projective}\}.$$

We close with the following intriguing problems:

Since at present we are unable to construct an ω_1-$p^{\omega+n}$-projective group that is not fully ω_1-$p^{\omega+n}$-projective whenever $n > 0$ (for $n = 0$ these classes apparently coincide), the first basic query is the following:

Problem 1. Does there exist an ω_1-$p^{\omega+n}$-projective group which is not fully ω_1-$p^{\omega+n}$-projective? Is any ω_1-$p^{\omega+n}$-projective group with countable first Ulm subgroup fully ω_1-$p^{\omega+n}$-projective?

The concept of nicely ω_1-$p^{\omega+n}$-projective groups can be seen in [2]. The next question arises quite naturally.

Problem 2. Does it follow that nicely ω_1-$p^{\omega+n}$-projective groups are fully ω_1-$p^{\omega+n}$-projective?

As introduced in [5], a group G is $(\omega + n)$*-totally* $p^{\omega+n}$*-projective* if each its $p^{\omega+n}$-bounded subgroup is $p^{\omega+n}$-projective. Since $p^{\omega+n}$-bounded $(\omega + n)$-totally $p^{\omega+n}$-projective groups are necessarily $p^{\omega+n}$-projective, Example 3.13 assures that there is a fully ω-$p^{\omega+n}$-projective group (and hence a fully ω_1-$p^{\omega+n}$-projective group) of length $\omega + n$ which is not $(\omega + n)$-totally $p^{\omega+n}$-projective. However, it was established in [7] that $(\omega + n)$-totally $p^{\omega+n}$-projective groups are ω_1-$p^{\omega+n}$-projective. We suspect that they could be even fully ω_1-$p^{\omega+n}$-projective, so that we pose:

Problem 3. Find the intersection between the classes of $(\omega+n)$-totally $p^{\omega+n}$-projective groups and fully ω_1-$p^{\omega+n}$-projective groups.

Recall that a challenging conjecture is that if G is $(\omega + n)$-totally $p^{\omega+n}$-projective group, then $G/p^{\omega+n}G$ is $p^{\omega+n}$-projective. If yes, then because by [5] we have that $p^{\omega+n}G$ is countable, $(\omega + n)$-totally $p^{\omega+n}$-projective groups should be strongly ω_1-$p^{\omega+n}$-projective in accordance with Theorem 3.2.

We also were unable to decide whether or not fully ω_1-$p^{\omega+n}$-projective groups are closed under taking ω_1-bijections; what we succeed to prove is that they are closed with respect to ω-bijections (see Corollary 3.6). So, we state that in an explicit form.

Problem 4. Is it true that fully ω_1-$p^{\omega+n}$-projective groups are closed under ω_1-bijections?

On the other hand, in the spirit of [3], we state the following two additional queries.

Problem 5. Describe the structure of the class of those groups G for which there exist $P \leq G[p^m]$ and countable nice subgroups $K \leq G$ with the property that $G/(K + P)$ are $p^{\omega+n}$-projective.

- If $m = 0$, that is $P = \{0\}$, then we obtain ω_1-$p^{\omega+n}$-projective groups;
- If $n = 0$, then we obtain fully ω_1-$p^{\omega+m}$-projective groups;
- If $K = \{0\}$, then we obtain $p^{\omega+m+n}$-projective groups.

Problem 6. Describe the structure of the class of those groups G for which there exist $P \leq G[p^m]$ which are nice in G and countable subgroups $K \leq G$ with the property that $G/(K+P)$ are $p^{\omega+n}$-projective.

- If $m = 0$, that is $P = \{0\}$, then we get ω_1-$p^{\omega+n}$-projective groups;
- If $n = 0$, then we get strongly ω_1-$p^{\omega+m}$-projective groups;
- If $K = \{0\}$, then we get $p^{\omega+m+n}$-projective groups.

According to Proposition 3.3 and the above given particular cases, one can expect that the class of groups in Problem 6 is contained in the class of groups in Problem 5.

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References

[1] Danchev, P., On strongly and separably ω_1-$p^{\omega+n}$-projective abelian p-groups. Hacettepe J. Math. Stat. 43 (1) (2014), 51-64.

[2] Danchev, P., On nicely and separately ω_1-$p^{\omega+n}$-projective abelian p-groups. Math. Reports, to appear (2015).

[3] Danchev, P., On m-ω_1-$p^{\omega+n}$-projective abelian p-groups. Demonstrat. Math. 47 (4) (2014), 805-825.

[4] Danchev, P., Keef, P., Generalized Wallace theorems. Math. Scand. 104 (1) (2009), 33-50.

[5] Danchev, P., Keef,P., An application of set theory to $\omega + n$-totally $p^{\omega+n}$-projective primary abelian groups. Mediterr. J. Math. 8 (4) (2011), 525-542.

[6] Fuchs, L., Infinite Abelian Groups. Volumes I and II, Academic Press, New York and London, 1970 and 1973.

[7] Keef, P., On ω_1-$p^{\omega+n}$-projective primary abelian groups. J. Alg. Numb. Th. Acad. 1 (1) (2010), 41-75.

[8] Keef, P., Danchev, P., On n-simply presented primary abelian groups. Houston J. Math. 38 (4) (2012), 1027-1050.

[9] Nunke, R., Purity and subfunctors of the identity. Topics in Abelian Groups, Scott, Foresman and Co., (1963), 121-171.

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