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STUDY ON SEMI-SYMMETRIC METRIC SPACES

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Abstract. Many differential geometer studied different types of manifolds with a semi-symmetric metric connection. In this paper, we have considered a Riemannian manifold (M^n, g), $(n > 2)$, equipped with a semi-symmetric metric connection and studied the properties of the curvature tensor, the conformal curvature tensor, the Weyl projective curvature tensor and the conharmonic curvature tensor. We have also studied the Einstein spaces and recurrent space with respect to semi-symmetric metric connection and obtained certain results related to them.

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1. Introduction

Let (M^n, g), $(n > 2)$ be an n-dimensional Riemannian manifold with Riemannian metric g. A connection is said to be symmetric if the torsion tensor with respect to that connection is equal to zero otherwise it is called a non-symmetric connection. If covariant derivative of the metric tensor with respect to a given connection is equal to zero, then the connection is called a metric connection otherwise it is called a non-metric connection. The Riemannian manifold equipped with a semi-symmetric metric connection has been studied by O. C. Andonie [1], M. C. Chaki and A. Konar [2], B. B. Chaturvedi and P. N. Pandey [3, 4, 5].

[6] relate the semi-symmetric metric connection ∇ and Riemannian connection D of (M^n, g) by the relation

$$\nabla_X Y = D_X Y + \omega(Y)\, X - g(X, Y)U, \tag{1.1}$$

where X, Y are vector fields and ω is a 1-form, define by $\omega(X) =$ g(X,U).

K. Yano [7] found the relation between the curvature tensor with respect to the connection ∇ and D

$$\begin{aligned} &\overline{R}(X, Y)\, Z \\ (1.2) = \quad & R(X, Y)\, Z - \pi(Y, Z)X + \pi(X, Z)Y - g(Y, Z)AX + g(X, Z)AY, \end{aligned}$$

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where $\overline{R}$ is the curvature tensor with respect to a semi-symmetric metric connection, R is the curvature tensor with respect to Riemannian connection and π is the tensor field of type $(0, 2)$ defined by

$$\pi(X, Y) = (\nabla_X \omega)Y - \omega(X)\omega(Y) + \frac{1}{2}\omega(\rho)g(X, Y), \tag{1.3}$$

also A is the tensor field of type $(1, 1)$ defined by

$$g(AX, Y) = \pi(X, Y), \text{ for all vector fields X and Y.} \tag{1.4}$$

A Friedman and J. A. Schouten [8] considered the semi-symmetric metric connection ∇ and Riemannian manifold D with coefficients $\Gamma^h_{i\,j}$ and $\{^{\,h}_{i\,j}\}$ respectevely. According to them if the torsion tensor T of the connection ∇ on M^n, $(n > 2)$ is

$$T^h_{i\,j} = \delta^h_i \omega_j - \delta^h_j \omega_i, \tag{1.5}$$

then

$$\Gamma^h_{i\,j} = \{^{\,h}_{i\,j}\} + \delta^h_i\ \omega_j - g_{ij}\ \omega^h, \tag{1.6}$$

where $\omega^h = \omega_t g^{t\,h}$, ω^h being the contravariant components of the generating vector w_h and

$$\nabla_j\ \omega_i = D_j\ \omega_i - \omega_i\ \omega_j + g_{i\,j}\ \omega, \text{ where } \omega = \omega^h\ \omega_h. \tag{1.7}$$

A. Friedman and J. A. Schouten [8] obtained the relation between the curvature tensor with respect to a semi-symmetric metric connection and the Riemannian connection i.e.

$$\overline{R}_{i\,j\,k\,h} = R_{i\,j\,k\,h} - g_{i\,h}\ \pi_{j\,k} + g_{j\,h}\ \pi_{i\,k} - g_{j\,k}\ \pi_{i\,h} + g_{i\,k}\ \pi_{j\,h}, \tag{1.8}$$

where

$$\pi_{j\,k} = \nabla_j\ \omega_k - \omega_j\ \omega_k + \frac{1}{2}\ g_{j\,k}\ \omega. \tag{1.9}$$

Transvecting (1.8) by $g^{i\,h}$ and using $\pi = \pi_{i\,h}\ g^{i\,h}$, we get

$$\overline{R}_{j\,k} = R_{j\,k} - n\pi_{j\,k} + \pi_{j\,k} - \pi\ g_{j\,k} + \pi_{j\,k}. \tag{1.10}$$

The equation (1.10) implies

$$\overline{R}_{j\,k} = R_{j\,k} - (n-2)\ \pi_{j\,k} - \pi\ g_{j\,k}. \tag{1.11}$$

Transvecting (1.11) by $g^{j\,k}$ and using $g_{j\,k}\ g^{j\,k} = n$, we get

$$\overline{R} = R - 2(n-1)\pi. \tag{1.12}$$

Let $\overline{R}_{i\,j\,k\,h} = R_{i\,j\,k\,h}$ then from (1.8), we get

$$g_{i\,h}\ \pi_{j\,k} - g_{j\,h}\ \pi_{i\,k} + g_{j\,k}\ \pi_{i\,h} - g_{i\,k}\ \pi_{j\,h} = 0. \tag{1.13}$$

Transvecting (1.13) by $g^{i\,h}$, we get

$$(n-2)\pi_{j\,k} + \pi\, g_{j\,k} = 0. \tag{1.14}$$

Transvecting (1.14) by $g^{j\,k}$, we get

$$2(n-1)\,\pi = 0, \tag{1.15}$$

this implies

$$\pi = 0, \quad \textit{since} \quad n \neq 1, \quad (n > 2) \tag{1.16}$$

Thus we conclude :

Theorem 1.1. *In a Riemannian manifold (M^n, g), $(n > 2)$, equipped with the semi-symmetric metric connection ∇ if the curvature tensor of type(0,4) with respect to semi-symmetric metric connection is equal to the curvature tensor of type(0,4) with respect to Riemannian connection D then $\pi = 0$, i.e. $\pi_{i\,h}\, g^{i\,h} = 0$.*

2. Conformal curvature tensor

We know that the conformal curvature tensor $C_{i\,j\,k\,h}$ in Riemannian manifold is defined by

$$\begin{aligned} C_{i\,j\,k\,h} =& R_{i\,j\,k\,h} - \frac{1}{n-2}(R_{j\,k}\, g_{i\,h} - R_{i\,k}\, g_{j\,h} + R_{i\,h}\, g_{j\,k} - R_{j\,h}\, g_{i\,k}) \\ &+ \frac{R}{(n-1)(n-2)}(g_{i\,h}\, g_{j\,k} - g_{j\,h}\, g_{i\,k}), \end{aligned} \tag{2.1}$$

where $R_{i\,j\,k\,h}$, $R_{i\,j}$ and R are the curvature tensor, Ricci tensor and scalar curvature tensor of connection respectevely.

Now we propose :

Theorem 2.1. *In a Riemannian manifold (M^n, g), $(n > 2)$, equipped with semi-symmetric metric connection the conformal curvature tensor with respect to semi-symmetric metric connection ∇ and the conformal curvature tensor with respect to Riemannian connection D are equal.*

Proof. Now, the conformal curvature tensor with respect to semi-symmetric metric connection is given by

$$\begin{aligned} \overline{C}_{i\,j\,k\,h} =& \overline{R}_{i\,j\,k\,h} - \frac{1}{n-2}(\overline{R}_{j\,k}\, g_{i\,h} - \overline{R}_{i\,k}\, g_{j\,h} + \overline{R}_{i\,h}\, g_{j\,k} - \overline{R}_{j\,h}\, g_{i\,k}) \\ &+ \frac{\overline{R}}{(n-1)(n-2)}(g_{i\,h}\, g_{j\,k} - g_{j\,h}\, g_{i\,k}). \end{aligned} \tag{2.2}$$

Using (1.8), (1.11) and (1.12) in (2.2), we get

$$
\begin{aligned}
\overline{C}_{i\,j\,k\,h} =& R_{i\,j\,k\,h} - g_{i\,h}\,\pi_{j\,k} + g_{j\,h}\,\pi_{i\,k} - g_{j\,k}\,\pi_{i\,h} + g_{i\,k}\,\pi_{j\,h} \\
& - \frac{1}{n-2}\Big(\, g_{i\,h}(R_{j\,k} - (n-2)\pi_{j\,k} - \pi\, g_{j\,k}) \\
& - g_{j\,h}(R_{i\,k} - (n-2)\pi_{i\,k} - \pi\, g_{i\,k}) \\
& + g_{j\,k}(R_{i\,h} - (n-2)\pi_{i\,h} - \pi\, g_{i\,h}) \\
& - g_{i\,k}(R_{j\,h} - (n-2)\pi_{j\,h} - \pi\, g_{j\,h})\,\Big) \\
& + \frac{R - 2(n-1)\,\pi}{(n-1)\,(n-2)}(g_{i\,h}\,g_{j\,k} - g_{j\,h}\,g_{i\,k}),
\end{aligned}
\tag{2.3}
$$

$$
\begin{aligned}
\overline{C}_{i\,j\,k\,h} =& R_{i\,j\,k\,h} - \frac{1}{(n-2)}(g_{i\,h}\,R_{j\,k} - g_{j\,h}\,R_{i\,k} + g_{j\,k}\,R_{i\,h} - g_{i\,k}\,R_{j\,h}) \\
& + \frac{R}{(n-1)(n-2)}(g_{i\,h}\,g_{j\,k} - g_{j\,h}\,g_{i\,k}).
\end{aligned}
\tag{2.4}
$$

Using (2.1) in (2.4), we get

$$
\overline{C}_{i\,j\,k\,h} = C_{i\,j\,k\,h}. \tag{2.5}
$$

□

3. Conharmonic curvature tensor

Definition 3.1. The conharmonic curvature tensor L of type $(0,4)$ on Riemannian manifold is defined by

$$
L_{i\,j\,k\,h} = R_{i\,j\,k\,h} - \frac{1}{n-2}\Big(R_{j\,k}\,g_{i\,h} - R_{i\,k}\,g_{j\,h} + R_{i\,h}\,g_{j\,k} - R_{j\,h}g_{i\,k}\Big). \tag{3.1}
$$

Now the conharmonic curvature tensor with respect to semi- symmetric metric connection is given by

$$
\overline{L}_{i\,j\,k\,h} = \overline{R}_{i\,j\,k\,h} - \frac{1}{n-2}(\overline{R}_{j\,k}\,g_{i\,h} - \overline{R}_{i\,k}\,g_{j\,h} + \overline{R}_{i\,h}\,g_{j\,k} - \overline{R}_{j\,h}\,g_{i\,k}). \tag{3.2}
$$

Using (1.8) and (1.11) in (3.2), we get

$$
\begin{aligned}
&\overline{L}_{i\,j\,k\,h} \\
=\;& R_{i\,j\,k\,h} - g_{i\,h}\,\pi_{j\,k} + g_{j\,h}\,\pi_{i\,k} - g_{j\,k}\,\pi_{i\,h} + g_{i\,k}\,\pi_{j\,h} \\
& - \frac{1}{(n-2)}\Big(g_{i\,h}\,(R_{j\,k} - (n-2)\pi_{j\,k} - \pi\, g_{j\,k}) \\
& - g_{j\,h}\,(R_{i\,k} - (n-2)\pi_{i\,k} - \pi\, g_{k\,i}) \\
& + g_{j\,k}\,(R_{i\,h} - (n-2)\pi_{i\,h} - \pi\, g_{i\,h}) - g_{i\,k}\,(R_{j\,h} - (n-2)\pi_{j\,h} - \pi\, g_{j\,h})\Big).
\end{aligned}
\tag{3.3}
$$

The equation (3.3) implies

$$\overline{L}_{ijkh} = R_{ijkh} - \frac{1}{(n-2)}(R_{jk}\, g_{ih} - R_{ik}\, g_{jh} + R_{ih}\, g_{jk} - R_{jh}\, g_{ik}) + \frac{2\pi}{n-2}\,(g_{jk}\, g_{ih} - g_{jh}\, g_{ik}). \tag{3.4}$$

Using (3.1) in (3.4), we have

$$\overline{L}_{ijkh} = L_{ijkh} + \frac{2\pi}{(n-2)}(g_{jk}\, g_{ih} - g_{jh}\, g_{ik}). \tag{3.5}$$

From (3.5), we have $\overline{L}_{ijkh} = L_{ijkh}$, if and only if $\pi = 0$ or $g_{jk}\, g_{ih}$ is symmetric in i and j or k and h.

Thus we conclude :

Theorem 3.2. *In a Riemannian manifold (M^n, g), $(n > 2)$, equipped with a semi-symmetric metric connection, the conharmonic curvature tensor with respect to the semi-symmetric metric connection is equal to the conharmonic curvature tensor with respect to Riemannian connection if and only if at least one of the following conditions holds:*

(i) $\pi = 0$

(ii) $g_{jk}\, g_{ih}$ *is symmetric in i and j*

(iii) $g_{jk}\, g_{ih}$ *is symmetric in k and h*

Now we propose:

Theorem 3.3. *In Riemannian manifold (M^n, g), $n > 2$, equipped with the semi-symmetric metric connection ∇, the conharmonic curvature tensor with respect to the semi-symmetric metric connection has the following properties:*

(i) $\overline{L}_{ijkh} = -\overline{L}_{jikh}$ *i.e. skew symmetric in first two indices*

(ii) $\overline{L}_{ijkh} = -\overline{L}_{ijhk}$ *i.e. skew symmetric in last two indices*

(iii) $\overline{L}_{ijkh} + \overline{L}_{jkih} + \overline{L}_{kijh} = 0.$

Proof. Interchanging i and j in (3.5), we get

$$\overline{L}_{jikh} = L_{jikh} + \frac{2\pi}{(n-2)}(g_{ik}\, g_{jh} - g_{ih}\, g_{jk}). \tag{3.6}$$

Adding (3.5) and (3.6), we get

$$\overline{L}_{ijkh} + \overline{L}_{jikh} = L_{ijkh} + L_{jikh}. \tag{3.7}$$

Since in a Riemannian manifold

$$L_{ijkh} + L_{jikh} = 0. \tag{3.8}$$

Then from (3.7) and (3.8), we have (i)

Now interchanging h and k in (3.5), we have

$$\overline{L}_{ijhk} = L_{ijhk} + \frac{2\pi}{(n-2)}\left(g_{jh}\, g_{ik} - g_{jk}\, g_{ih}\right). \tag{3.9}$$

Adding (3.5) and (3.9), we have

$$\overline{L}_{ijkh} + \overline{L}_{ijhk} = L_{ijkh} + L_{ijhk}. \tag{3.10}$$

Since in a Riemannian manifold

$$L_{ijhk} + L_{ijkh} = 0. \tag{3.11}$$

Then from (3.10) and (3.11), we get (ii)

Again interchaging i , j and k in cyclic order in the equation (3.5), we get

$$\overline{L}_{ijkh} = L_{ijkh} + \frac{2\pi}{(n-2)}(g_{jk}\, g_{ih} - g_{jh}\, g_{ik}), \tag{3.12}$$

$$\overline{L}_{jkih} = L_{jkih} + \frac{2\pi}{(n-2)}(g_{ki}\, g_{jh} - g_{kh}\, g_{ji}), \tag{3.13}$$

and

$$\overline{L}_{kijh} = L_{kijh} + \frac{2\pi}{(n-2)}(g_{ij}\, g_{kh} - g_{ih}\, g_{kj}). \tag{3.14}$$

Adding (3.12), (3.13) and (3.14) , we have

$$\overline{L}_{ijkh} + \overline{L}_{jkih} + \overline{L}_{kijh} = L_{ijkh} + L_{jkih} + L_{kijh}. \tag{3.15}$$

In a Riemannian manifold, the conharmonic curvature tensor with respect to the Riemannian connection satisfies the relation

$$L_{ijkh} + L_{jkih} + L_{kijh} = 0. \tag{3.16}$$

Therefore, from (3.15) and (3.16), we get (*iii*). □

4. Weyl projective curvature tensor

Definition 4.1. The Weyl projective curvature tensor P of type (0,4) on a Riemannian manifold is defined by

$$P_{ijkh} = R_{ijkh} - \frac{1}{(n-1)}\left(R_{jk}\, g_{ih} - R_{ik}\, g_{jh}\right). \tag{4.1}$$

Now the Weyl projective curvature tensor of type (0,4) with respect to a semi-symmetric metric connection is given by

$$\overline{P}_{ijkh} = \overline{R}_{ijkh} - \frac{1}{(n-1)}(\overline{R}_{jk}\, g_{ih} - \overline{R}_{ik}\, g_{jh}). \tag{4.2}$$

Using (1.8) and (1.11) in (4.2), we get

$$\begin{aligned}\overline{P}_{ijkh} =&R_{ijkh} - g_{ih}\,\pi_{jk} + g_{jh}\,\pi_{ik} - g_{jk}\,\pi_{ih} + g_{ik}\,\pi_{jh}\\ &- \frac{1}{(n-1)}\big(g_{ih}\,(R_{jk} - (n-2)\,\pi_{jk} - \pi\,g_{jk})\\ &- g_{jh}\,(R_{ik} - (n-2)\pi_{ik} - \pi\,g_{ik})\big).\end{aligned} \tag{4.3}$$

Using (4.1) and $\pi = \pi_{ih}\,g^{ih}$ in (4.3), we get

$$\overline{P}_{ijkh} = P_{ijkh} + \frac{1}{(n-1)}(\pi_{ik}\,g_{jh} - \pi_{jk}\,g_{ih} + \pi_{ih}\,g_{jk} - \pi_{jh}\,g_{ik}), \tag{4.4}$$

Thus we conclude:

Theorem 4.2. *In the Riemannian manifold* (M^n, g)*,* $(n > 2)$*, equipped with the semi-symmetric metric connection* ∇*, the Weyl projective curvature tensor with respect to the semi-symmetric metric connection has the following properties.*

$$(i)\overline{P}_{ijkh} = -\overline{P}_{jikh}$$
$$(ii)\overline{P}_{ijkh} + \overline{P}_{jkih} + \overline{P}_{kijh} = 0, \quad \textit{if } \pi_{ik} = \pi_{ki}$$

Proof. Interchanging i and j in (4.4), we get

$$\overline{P}_{jikh} = P_{jikh} + \frac{1}{n-1}(\pi_{jk}\,g_{ih} - \pi_{ik}\,g_{jh} + \pi_{jh}\,g_{ik} - \pi_{ih}\,g_{jk}). \tag{4.5}$$

Adding (4.4) and (4.5), we get

$$\overline{P}_{ijkh} + \overline{P}_{jikh} = P_{ijkh} + P_{jikh}. \tag{4.6}$$

The Weyl projective curvature tensor with respect to Riemannian connection has the property

$$P_{ijkh} = -P_{jikh}. \tag{4.7}$$

Using (4.7) in (4.6), we get (i)

Interchanging i, j, k in cyclic order in the equation (4.4), we get

$$\overline{P}_{ijkh} = P_{ijkh} + \frac{1}{(n-1)}(\pi_{ik}\,g_{jh} - \pi_{jk}\,g_{ih} + \pi_{ih}\,g_{jk} - \pi_{jh}\,g_{ik}), \tag{4.8}$$

$$\overline{P}_{jkih} = P_{jkih} + \frac{1}{(n-1)}(\pi_{ji}\,g_{kh} - \pi_{ki}\,g_{jh} + \pi_{jh}\,g_{ki} - \pi_{kh}\,g_{ji}), \tag{4.9}$$

and

$$\overline{P}_{kijh} = P_{kijh} + \frac{1}{(n-1)}(\pi_{kj}\,g_{ih} - \pi_{ij}\,g_{kh} + \pi_{kh}\,g_{ij} - \pi_{ih}\,g_{kj}). \tag{4.10}$$

Adding (4.8), (4.9) and (4.10), we get

$$\overline{P}_{i\,j\,k\,h} + \overline{P}_{j\,k\,i\,h} + \overline{P}_{k\,i\,j\,h} = P_{i\,j\,k\,h} + P_{j\,k\,i\,h} + P_{k\,i\,j\,h}, \text{ if } \pi_{i\,k} = \pi_{k\,i} \tag{4.11}$$

The Weyl projective curvature tensor with respect to a Riemannian manifold has the property

$$P_{i\,j\,k\,h} + P_{j\,k\,i\,h} + P_{k\,i\,j\,h} = 0. \tag{4.12}$$

Therefore from (4.11) and (4.12), we get *(ii)* □

5. Einstein space

Definition 5.1. The space in which the Ricci tensor satisfies the relation

$$R_{(i\,j)} = \lambda\, g_{i\,j}\ , \tag{5.1}$$

is called an Einstein space. Where $R_{(i\,j)}$ is symmetric part of Ricci tensor and λ is scalar funtion.

Definition 5.2. The space in which the Ricci tensor satisfies the relation

$$R_{i\,j} = \gamma\, g_{i\,j}\ , \tag{5.2}$$

is called an Einstein Riemannian space, where γ is a scalar function.

The symmetric part of the Ricci tensor with respect to a semi-symmetric metric connection is given by

$$\overline{R}_{(i\,j)} = \frac{1}{2}(\overline{R}_{i\,j} + \overline{R}_{j\,i}), \tag{5.3}$$

using (1.11) in (5.3), we have

$$\overline{R}_{(i\,j)} = \frac{1}{2}(R_{i\,j} - (n-2)\,\pi_{i\,j} - \pi\, g_{i\,j} + R_{j\,i} - (n-2)\,\pi_{j\,i} - \pi\, g_{j\,i}). \tag{5.4}$$

Equation (5.4) implies

$$\overline{R}_{(i\,j)} = \frac{1}{2}(R_{i\,j} + R_{j\,i} - (n-2)\,(\pi_{i\,j} + \pi_{j\,i}) - 2\pi\, g_{i\,j}). \tag{5.5}$$

Using (1.9) in (5.5), we get

$$\overline{R}_{(i\,j)} = \frac{1}{2}(R_{i\,j} + R_{j\,i} - (n-2)(\nabla_i\,\omega_j + \nabla_j\,\omega_i - 2\omega_i\,\omega_j + g_{i\,j}\,\omega) - 2\pi\, g_{i\,j}). \tag{5.6}$$

Using (5.3) and $\omega = \omega^i\,\omega_i$ in (5.6), we get

$$\overline{R}_{(i\,j)} = R_{(i\,j)} - \frac{(n-2)}{2}(\nabla_i\,\omega_j + \nabla_j\,\omega_i - \omega_i\,\omega_j - 2\pi\, g_{i\,j}). \tag{5.7}$$

Transvecting by g^{ij} in (5.7), we have

$$\overline{R}_{(ij)}\, g^{ij} = R_{(ij)}\, g^{ij} - \frac{(n-2)}{2}((\nabla_i\, \omega_j + \nabla_j\, \omega_i - \omega_i\, \omega_j)\, g^{ij} - 2\pi\, g_{ij}\, g^{ij}). \tag{5.8}$$

Using (5.1) in (5.8), we get

$$\lambda\, g_{ij}\, g^{ij} = \gamma\, g_{ij}\, g^{ij} - \frac{(n-2)}{2}(2\nabla_i\, \omega^i - \omega - 2\pi\, g_{ij}\, g^{ij}). \tag{5.9}$$

Using $g_{ij}\, g^{ij} = \delta^i_i = n$ in (5.9), we get

$$(\lambda - \gamma)\, n = -(n-2)(\nabla_i\, \omega^i - \frac{\omega}{2} - n\pi\,). \tag{5.10}$$

Thus we conclude:

Theorem 5.3. *If Riemannian manifold (M^n, g) , $(n > 2)$, equipped with a semi-symmetric metric connection is an Einstein space with respect to the Riemannian connection then the Riemannian manifold equipped with the semi-symmetric metric connection will be an Einstein space with respect to the semi-symmetric metric connection if*

$$(\lambda - \gamma)\, n = -(n-2)(\nabla_i\, \omega^i - \frac{\omega}{2} - n\pi\,)\, . \tag{5.11}$$

6. Recurrent space

Definition 6.1. A torsion tensor T on a Riemannain manifold is said to be recurrent if

$$\nabla_k\, T = \mu_k\, T, \tag{6.1}$$

where μ_k is a recurrent vector field.

Now we propose:

Theorem 6.2. *In a Riemannian manifold (M^n, g), $(n > 2)$, equipped with a semi- symmetric metric connection if torsion tensor T is recurrent with respect to the Riemannian connection then it will also be recurrent with respect to the semi- symmetric metric connection.*

Proof. The covariant derivatives of the torsion tensor T^h_{ij} with respect to the connections ∇ and D are given by

$$\nabla_k\, T^h_{ij} = \partial_k\, T^h_{ij} + T^r_{ij}\, \Gamma^h_{rk} - T^h_{rj}\, \Gamma^r_{ik} - T^h_{ir}\, \Gamma^r_{jk}, \tag{6.2}$$

and

$$D_k\, T^h_{ij} = \partial_k\, T^h_{ij} + T^r_{ij}\, \left\{ {h \atop r\,k} \right\} - T^h_{rj}\, \left\{ {r \atop i\,k} \right\} - T^h_{ir}\, \left\{ {r \atop j\,k} \right\}. \tag{6.3}$$

Subtracting (6.3) from (6.2), we get

$$\nabla_k T^h_{ij} - D_k T^h_{ij} = T^r_{ij} (\Gamma^h_{rk} - \{{}^{\,h}_{r\,k}\}) - T^h_{rj} (\Gamma^r_{ik} - \{{}^{\,r}_{i\,k}\}) - T^h_{ir} (\Gamma^r_{jk} - \{{}^{\,r}_{j\,k}\}). \tag{6.4}$$

Using (1.6) in (6.4), we get

$$\nabla_k T^h_{ij} - D_k T^h_{ij} = T^r_{ij} (\delta^h_r \omega_k - g_{rk} \omega^h) - T^h_{rj} (\delta^r_i \omega_k - g_{ik} \omega^r) - T^h_{ir} (\delta^r_j \omega_k - g_{jk} \omega^r). \tag{6.5}$$

Using $\omega^h = \omega_t g^{th}$ in (6.5), we get

$$\nabla_k T^h_{ij} - D_k T^h_{ij} = T^r_{ij} (\delta^h_r \omega_k - g_{rk} \omega_k g^{kh}) - T^h_{rj} (\delta^r_i \omega_k - g_{ik} \omega_k g^{kr}) - T^h_{ir} (\delta^r_j \omega_k - g_{jk} \omega_k g^{kr}). \tag{6.6}$$

Using $g_{rk} g^{kh} = \delta^h_r$ in (6.6), we have

$$\nabla_k T^h_{ij} = D_k T^h_{ij}. \tag{6.7}$$

□

Now we propose:

Theorem 6.3. *In a Riemannian manifold (M^n, g), $(n > 2)$, equipped with a semi -symmetric metric connection ∇ if the Ricci tensor is recurrent with respect to the Riemannian connection then it will also be recurrent with respect to the semi-symmetric metric connection.*

Proof. The covariant derivatives of the Ricci tensor with respect to the connections ∇ and D are given by

$$\nabla_k \overline{R}_{ij} = \partial_k \overline{R}_{ij} - \overline{R}_{rj} \Gamma^r_{ik} - \overline{R}_{ir} \Gamma^r_{jk} \tag{6.8}$$

and

$$D_k \overline{R}_{ij} = \partial_k \overline{R}_{ij} - \overline{R}_{rj} \{{}^{\,r}_{i\,k}\} - \overline{R}_{ir} \{{}^{\,r}_{j\,k}\}. \tag{6.9}$$

Subtracting (6.9) from (6.8), we get

$$\nabla_k \overline{R}_{ij} - D_k \overline{R}_{ij} = -\overline{R}_{rj} (\Gamma^r_{ik} - \{{}^{\,r}_{i\,k}\}) - \overline{R}_{ir} (\Gamma^r_{jk} - \{{}^{\,r}_{j\,k}\}). \tag{6.10}$$

Using (1.6) in (6.10), we get

$$\nabla_k \overline{R}_{ij} - D_k \overline{R}_{ij} = -\overline{R}_{rj} (\delta^r_i \omega_k - g_{ik} \omega^r) - \overline{R}_{ir} (\delta^r_j \omega_k - g_{jk} \omega^r). \tag{6.11}$$

Using $\omega^h = \omega_t g^{th}$ (6.11), we get

$$\nabla_k \overline{R}_{ij} - D_k \overline{R}_{ij} = -\overline{R}_{rj} (\delta^r_i \omega_k - g_{ik} \omega_k g^{rk}) - \overline{R}_{ir} (\delta^r_j \omega_k - g_{jk} \omega_k g^{rk}). \tag{6.12}$$

Using $g_{ik} g^{jk} = \delta^j_i$ in (6.12), we have

$$\nabla_k \overline{R}_{ij} = D_k \overline{R}_{ij}. \tag{6.13}$$

□

Now the covariant derivatives of conformal curvatures tensor with respect to connections ∇ and D are given by

(6.14)
$$\nabla_l \overline{C}_{ijkh} = \partial_l \overline{C}_{ijkh} - \overline{C}_{rjkh} \Gamma^r_{li} - \overline{C}_{irkh} \Gamma^r_{lj} - \overline{C}_{ijrh} \Gamma^r_{lk} - \overline{C}_{ijkr} \Gamma^r_{lh},$$

and

(6.15)
$$D_l \overline{C}_{ijkh} = \partial_l \overline{C}_{ijkh} - \overline{C}_{rjkh} \{^{\,r}_{l\,i}\} - \overline{C}_{irkh} \{^{\,r}_{l\,j}\} - \overline{C}_{ijrh} \{^{\,r}_{l\,k}\} - \overline{C}_{ijkr} \{^{\,r}_{l\,h}\}.$$

Subtracting (6.15) from (6.14), we get

$$\begin{aligned}\nabla_l \overline{C}_{ijkh} - D_l \overline{C}_{ijkh} = &- \overline{C}_{rjkh} (\Gamma^r_{li} - \{^{\,r}_{l\,i}\}) - \overline{C}_{irkh} (\Gamma^r_{lj} - \{^{\,r}_{l\,j}\}) \\ &- \overline{C}_{ijrh} (\Gamma^r_{lk} - \{^{\,r}_{l\,k}\}) - \overline{C}_{ijkr} (\Gamma^r_{lh} - \{^{\,r}_{l\,h}\}),\end{aligned} \tag{6.16}$$

using (1.6) in (6.16), we get

(6.17)
$$\begin{aligned}\nabla_l \overline{C}_{ijkh} - D_l \overline{C}_{ijkh} = &- \overline{C}_{rjkh} (\delta^r_l \omega_i - g_{il} \omega^r) - \overline{C}_{irkh} (\delta^r_l \omega_j - g_{lj} \omega^r) \\ &- \overline{C}_{ijrh} (\delta^r_l \omega_k - g_{lk} \omega^r) - \overline{C}_{ijkr} (\delta^r_l \omega_h - g_{lh} \omega^r).\end{aligned}$$

Using $\omega^h = \omega_t g^{th}$ and $g_{ik} g^{jk} = \delta^j_i$ in (6.17), we have

$$\nabla_l \overline{C}_{ijkh} = D_l \overline{C}_{ijkh}. \tag{6.18}$$

Thus we conclude:

Theorem 6.4. *In a Riemannian manifold (M^n, g), $n > 2$, equipped with a semi-symmetric metric connection, if the conformal curvature tensor is recurrent with respect to the Riemannian connection then it will also be recurrent with respect to the semi-symmetric metric connection.*

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