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CIRCULAR CUBICS IN PSEUDO-EUCLIDEAN PLANE

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Abstract. A curve in the pseudo-Euclidean plane is circular if it passes through at least one of the absolute points.

A cubic can be obtained as a locus of the intersections of a conic and the corresponding line of the projectively linked pencil of conics and pencil of lines. In this paper the conditions that the pencils and the projectivity have to fulfill in order to obtain a circular cubic of a certain type of circularity are determined analytically. The cubics of all types (depending on their position with respect to the absolute figure) can be constructed by using these results. The results are first stated for any projective plane and then their pseudo-Euclidean interpretation is given.

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1. Introduction

The *pseudo-Euclidean plane* is a real projective plane where the metric is induced by a real line f and two real points F_1 and F_2 incidental with it, [6, 7]. The line f is called the *absolute line* and the points F_1, F_2 are the *absolute points*. All straight lines through F_1, F_2 are called *isotropic lines*, and all points lying on f are called *isotropic points*.

If an algebraic curve k of order n passes through at least one of the absolute points, the curve is said to be *circular*. If F_1 is an intersection point of k and f with the intersection multiplicity r and F_2 is an intersection point of k and f with the intersection multiplicity t, the curve k has the *type of circularity* (r, t) and its *degree of circularity* is defined as $r + t$, [5]. We say that the curve is $(r + t)$-circular. If $n = r + t$, the curve is *entirely circular*.

In further classification we will not distinguish the type (r, t) from the type (t, r) since the possibility of constructing one of them implies the possibility of constructing the other.

The conics are classified in the paper [5] into: non-circular conics (ellipses, hyperbolas, parabolas), special hyperbolas (circularity of type (1, 0)), special parabolas (circularity of type (2, 0)) and circles (circularity of type (1, 1)).

The circular cubics can be of the following types:

- Type of circularity (1,0)

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  - The cubic passes through the absolute point F_1.

- Type of circularity (1,1)
  - The cubic passes through both absolute points F_1 and F_2.

- Type of circularity (2,0)
  - The cubic touches the absolute line f at F_1.
  - The cubic has a double point (node, isolated double point or cusp) in F_1.

- Type of circularity (2, 1)
  - The cubic touches the absolute line f at F_1 and passes through F_2.
  - The cubic has a double point (node, isolated double point or cusp) in F_1 and passes through F_2.

- Type of circularity (3, 0)
  - The cubic osculates f at F_1.
  - The cubic touches f at its double point F_1.

A curve of order four in the projective plane can be defined as a locus of the intersections of pairs of corresponding conics in projectively linked pencils of conics. That projectivity has been studied in [1]. The results were stated for the projective plane and then their isotropic interpretation was given. The pseudo-Euclidean interpretation of the same projective results was presented in [2], where it was shown that by using this method the entirely circular quartics of all types of circularity can be obtained.

The aim of this paper is to construct the circular cubics in the pseudo-Euclidean plane by using projectively linked pencil of conics and pencil of lines, and to classify them according to their position with respect to the absolute figure. Different methods of obtaining circular cubics and quartics where studied in [4] and [9].

2. Projective plane

A curve of order three in the projective plane can be defined as a locus of the intersections of pairs of a conic and its corresponding line of the projectively linked pencil of conics and pencil of lines, [10]. Let A, B be the conics (described by symmetric $3\times 3-$matrices with the same name) and let c, d be the lines (described by $3\times 1-$matrices with the same name) in the projective plane. Let $\pi : [A, B] \to [c, d]$ be the projective mapping of the conics $A + \lambda B$ to the lines $c + \lambda d$ for all $\lambda \in \mathbb{R} \cup \infty$. Some calculations deliver the following equation of the cubic k:

$$\mathrm{F}(\vec{x}) \equiv \vec{x}^\top A\vec{x} \cdot d\vec{x} - \vec{x}^\top B\vec{x} \cdot c\vec{x} = 0.$$

The cubic k in the algebraic sense passes through the following nine points: four basic points of the pencil $[A, B]$, vertex of the pencil $[c, d]$, two intersection points of A and c and two intersection points of B and d. According to [8] the number of points required for the determination of a cubic is nine, but nine points do not in every case determine a single cubic.

Remark 2.1. We have to be aware of the fact that proportional matrices A, B, c, d and $\alpha A, \beta B, \gamma c, \delta d$ represent the same two conics and two lines, but the consequential cubics are different. For defining the projectivity beside A, c and B, d we need one more pair of elements. Although the corresponding cubics are different, their properties that are of our interest stay the same.

We will now search for the conditions that the pencils and the projectivity have to fulfill in order to obtain a circular cubic of a certain type. The results will be first stated for a projective plane and then their pseudo-Euclidean interpretation will be given.

Let us start with a point $\vec{y}$ on the cubic k. We can assume that it is an intersection point of the basic conic A and the basic line c: we have $\vec{y}^\top A\vec{y} = 0$, $\vec{y}c = 0$. The tangential behavior in this point on k is usually being studied by observing the intersections of k with arbitrary straight lines through $\vec{y}$. Such a line q will be spanned by $\vec{y}$ and a further point $\vec{z}$. It can be parametrized by

$$q \quad ... \quad \vec{y} + t\vec{z}, \quad t \in \mathbb{R} \cup \infty.$$

The intersections of k and q belong to the zeros of the following polynomial of degree 3 in t:

$$\mathrm{p}(t) = \mathrm{F}(\vec{y} + t\vec{z}) = t\mathrm{F}_1(\vec{y}, \vec{z}) + t^2\mathrm{F}_2(\vec{y}, \vec{z}) + t^3\mathrm{F}_3(\vec{y}, \vec{z}),$$

where

$$\begin{aligned} \mathrm{F}_1(\vec{y}, \vec{z}) &= 2d\vec{y} \cdot \vec{y}^\top A\vec{z} - c\vec{z} \cdot \vec{y}^\top B\vec{y}, \\ \mathrm{F}_2(\vec{y}, \vec{z}) &= d\vec{y} \cdot \vec{z}^\top A\vec{z} + 2d\vec{z} \cdot \vec{y}^\top A\vec{z} - 2c\vec{z} \cdot \vec{y}^\top B\vec{z}, \\ \mathrm{F}_3(\vec{y}, \vec{z}) &= d\vec{z} \cdot \vec{z}^\top A\vec{z} - c\vec{z} \cdot \vec{z}^\top B\vec{z}. \end{aligned}$$

The equation of the tangent of k at the **regular point** $\vec{y}$ is

$$\mathrm{F}_1(\vec{y}, \vec{z}) = 0 \tag{2.1}$$

as $t = 0$ has to be a zero of multiplicity 2 of the polynomial p, and $\vec{y}$ is the intersection of k and q with intersection multiplicity 2.

A necessary condition to gain $\vec{y}$ as a **double point** of k is

$$\mathrm{F}_1(\vec{y}, \vec{z}) = 0 \quad \text{for every point } \vec{z} \tag{2.2}$$

as then $t = 0$ is a zero of multiplicity 2 of the polynomial p for all $\vec{z}$ (and for all lines q passing through $\vec{y}$). The equations of the tangents of k at such a double point $\vec{y}$ are determined by

$$\mathrm{F}_2(\vec{y}, \vec{z}) = 0. \tag{2.3}$$

One can also wonder if it is possible to get an osculation of the tangent q and the cubic k at the regular point $\vec{y}$ and the answer is positive. In that case $\mathrm{F}_2(\vec{y},\vec{z})$ should equal zero, but not for all points $\vec{z}$ of the plane, only for all points $\vec{z}$ lying on the tangent q.

Remark 2.2. Before we continue with our study, let us recall a fact that will play an important role in our observations. If A is a conic and $\vec{y}$ is a point lying on A, then three cases are possible:

- A is a regular conic.
  The equation $\vec{y}^\top A\vec{z} = 0$ is the equation of the tangent to A at $\vec{y}$.
- A is a singular conic, but $\vec{y}$ is not its singular point, i.e. $A = a_1 \cup a_2$, $\vec{y} \in a_1$, $\vec{y} \notin a_2$.
  The equation $\vec{y}^\top A\vec{z} = 0$ is the equation of the line a_1. This line plays the role of the tangent to A at $\vec{y}$.
- A is a singular conic with a singular point $\vec{y}$, i.e. $A = a_1 \cup a_2$, $\vec{y} \in a_1, a_2$.
  In this case $\vec{y}^\top A\vec{z} = 0$ for every point $\vec{z}$ of the plane.

Let us now go back to the cubic k and the point $\vec{y}$ lying on it, $\vec{y} \in A, c$. We will discuss the following four cases:

- $\vec{y} \notin B, d$
  $\vec{y}$ is not the basic point of $[A,B]$ neither the vertex of $[c,d]$.
- $\vec{y} \in B$, $\vec{y} \notin d$
  $\vec{y}$ is the basic point of $[A,B]$, but not the vertex of $[c,d]$.
- $\vec{y} \notin B$, $\vec{y} \in d$
  $\vec{y}$ is the vertex of $[c,d]$, but not the basic point of $[A,B]$.
- $\vec{y} \in B, d$
  $\vec{y}$ is at the same time the basic point of $[A,B]$ and the vertex of $[c,d]$.

We will first study the case when $\vec{y} \in A, c$, $\vec{y} \notin B, d$.
If A is a singular conic with a singular point $\vec{y}$ then the tangent (2.1) to k at $\vec{y}$ is identical to the line c. Therefore, we can conclude:

Theorem 2.3. *Let $[A,B]$ be a pencil of conics and $[c,d]$ be a pencil of lines. Let the cubic k be the result of the projective mapping $\pi : [A,B] \to [c,d]$. If the line corresponding to the singular conic passes through its singular point $\vec{y}$, then that line is the tangent of k at $\vec{y}$.*

If A is a singular conic, but $\vec{y}$ is not its singular point ($A = a_1 \cup a_2$, $\vec{y} \in a_1$, $\vec{y} \notin a_2$), the tangent (2.1) to k at $\vec{y}$ is a linear combination of the lines a_1 and c.

If A is a regular conic, then the tangent to k at $\vec{y}$ is given by the equation

$$2d\vec{y} \cdot \vec{y}^\top A\vec{z} - c\vec{z} \cdot \vec{y}^\top B\vec{y} = 0.$$

The tangent to k is identical to the tangent to A precisely when c touches A. Thus,

Theorem 2.4. *Let $[A, B]$ be a pencil of conics and $[c, d]$ be a pencil of lines. Let the cubic k be the result of the projective mapping $\pi : [A, B] \to [c, d]$. If there is a line touching its corresponding conic, then that line also touches k.*

Let us now observe the case when $\vec{y}$ is the basic point of $[A, B]$, but not the vertex of $[c, d]$ ($\vec{y} \in A, B, c$, $\vec{y} \notin d$).
If A is a singular conic with a singular point $\vec{y}$ (i.e. $A = a_1 \cup a_2$, $\vec{y} \in a_1, a_2$), it follows that the conics of the pencil $[A, B]$ touch each other at the basic point $\vec{y}$. The condition (2.2), which characterize $\vec{y}$ as a singular point of k, is fulfilled. The tangents of k at $\vec{y}$ are determined by (2.3). This expression is reduced to

$$d\vec{y} \cdot \vec{z}^\top A\vec{z} - 2c\vec{z} \cdot \vec{y}^\top B\vec{z} = 0.$$

If B touches a_1, the pencil $[A, B]$ contains conics osculating each other at $\vec{y}$. The equation above becomes

$$a_1\vec{z} \cdot [d\vec{y} \cdot a_2\vec{z} - 2\omega c\vec{z}] = 0$$

for some $\omega \in \mathbb{R}$. This assumptions provide a_1 as the tangent of one branch of the cubic k at its double point $\vec{y}$. We conclude

Theorem 2.5. *Let $[A, B]$ be a pencil of conics touching each other at the basic point $\vec{y}$ and let $[c, d]$ be a pencil of lines. Let the cubic k be the result of the projective mapping $\pi : [A, B] \to [c, d]$.*
If the singular conic with the singular point $\vec{y}$ is mapped onto the line passing through $\vec{y}$, the cubic k has the double point at $\vec{y}$. Furthermore, if the conics of the pencil $[A, B]$ osculate (or hyperosculate) each other at $\vec{y}$, one of the tangents to k at $\vec{y}$ is the common tangent of the conics of $[A, B]$.

If A is not the singular conic with the singular point $\vec{y}$, the tangent (2.1) to k at $\vec{y}$ is given by the equation

$$\vec{y}^\top A\vec{z} = 0$$

which characterizes the tangent to the conic A. It follows immediately

Theorem 2.6. *Let $[A, B]$ be a pencil of conics with the basic point $\vec{y}$ and let $[c, d]$ be a pencil of lines. Let the cubic k be the result of the projective mapping $\pi : [A, B] \to [c, d]$.*
The tangent to k at $\vec{y}$ is identical to the tangent to the conic of $[A, B]$ with its corresponding line of $[c, d]$ passing through $\vec{y}$.

Particularly, if the conics touch each other, their common tangent is the tangent to the constructed cubic.

The tangent $\vec{y}^\top A\vec{z} = 0$ can also osculate the cubic k. To gain osculation the following condition has to be fulfilled

$$d\vec{y} \cdot \vec{z}^\top A\vec{z} - 2c\vec{z} \cdot \vec{y}^\top B\vec{z} = 0$$

for all $\vec{z}$ such that $\vec{y}^\top A\vec{z} = 0$. If A is a singular conic with a non-singular point $\vec{y}$, ($A = a_1 \cup a_2$, $\vec{y} \in a_1$ $\vec{y} \notin a_2$), and B touches a_1, the line a_1 is an inflexion tangent to k.

The previous study leads us to

Theorem 2.7. *Let $[A, B]$ be a pencil of conics touching each other at the basic point $\vec{y}$ and let $[c, d]$ be a pencil of lines. Let the cubic k be the result of the projective mapping $\pi : [A, B] \to [c, d]$.*
The common tangent of the conics of $[A, B]$ is the tangent to k at $\vec{y}$.
If the singular conic with non-singular point $\vec{y}$ is linked to the line through $\vec{y}$, the tangent osculates k.

Let us now study the case when $\vec{y} \in A, c, d$, $\vec{y} \notin B$, i.e. $\vec{y}$ is the vertex of $[c, d]$, but not the basic point of $[A, B]$. The equation (2.1) is reduced to

$$c\vec{z} \cdot \vec{y}^\top B\vec{y} = 0$$

which provides c as the tangent to k at $\vec{y}$.
In other words: There is exactly one conic of the pencil $[A, B]$ passing through the vertex of the pencil $[c, d]$. The line of $[c, d]$ corresponding to that conic is the tangent to k at the vertex.

The line c osculates k if and only if $\vec{y}^\top A\vec{z} = 0$ for every $\vec{z}$ on c ($c\vec{z} = 0$). This is true exactly when A touches c or A is the singular conic with the singular point $\vec{y}$.

Now we can state the following:

Theorem 2.8. *Let $[A, B]$ be a pencil of conics and $[c, d]$ be a pencil of lines. Let the cubic k be the result of the projective mapping $\pi : [A, B] \to [c, d]$.*
The tangent to k at the vertex of $[c, d]$ is identical to the line of $[c, d]$ whose corresponding conic passes through the vertex. If that line touches the corresponding conic or the conic is the singular conic with the singular point at the vertex, the tangent osculates k.

The case to be considered is when $\vec{y} \in A, B, c, d$, i.e. $\vec{y}$ is at the same time the basic point of $[A, B]$ and the vertex of $[c, d]$. $\vec{y}$ is obviously a double point of k at which tangents are determined by

$$d\vec{z} \cdot \vec{y}^\top A\vec{z} - c\vec{z} \cdot \vec{y}^\top B\vec{z}.$$

One tangent to k at $\vec{y}$ coincides with the tangent to A iff c touches A or B touches A. If c touches A, then one tangent is c while the other tangent is given by the equation $d\vec{z} - \omega \cdot \vec{y}^\top B\vec{z} = 0$, for some $\omega \in \mathbb{R}$. Thus, if d also touches B, then the other tangent coincides with d. We can now state:

Theorem 2.9. *Let $[c, d]$ be a pencil of lines with the vertex $\vec{y}$ and let $[A, B]$ be a pencil of conics with the basic point in $\vec{y}$. Let the cubic k be the result of the projective mapping $\pi : [A, B] \to [c, d]$.*
The vertex $\vec{y}$ is the double point of k. If there is a line of $[c, d]$ touching the corresponding conic, that line is one tangent of the cubic at its double point.

If the conics A and B touch each other, their common tangent is also the tangent of the cubic k. The pencil $[A, B]$ is now the pencil of conics touching each other at $\vec{y}$. Therefore, there is a singular conic of $[A, B]$ with the singular point at $\vec{y}$. Without loss of generality we can assume that A is that conic. The tangents at $\vec{y}$ are now determined by the equation $c\vec{z} \cdot \vec{y}^\top B\vec{z} = 0$. Obviously, one tangent at $\vec{y}$ is the common tangent of the conics of the pencil, while the other tangent is the line linked to the singular conic with the singular point at the contact point. These two tangents can coincide and that results with the cusp in $\vec{y}$. These observations lead us to

Theorem 2.10. *Let $[c, d]$ be a pencil of lines with the vertex $\vec{y}$ and let $[A, B]$ be a pencil of conics touching each other at $\vec{y}$. Let the cubic k be the result of the projective mapping $\pi : [A, B] \to [c, d]$.*
The vertex $\vec{y}$ is the double point of the cubic k at which one tangent coincides with the common tangent of the conics of $[A, B]$. The other tangent at $\vec{y}$ is the line of $[c, d]$ linked to the singular conic with the singular point in $\vec{y}$.

The studies similar to the studies presented here were done in [1] and [3] for obtaining entirely circular quartics in an isotropic and a hyperbolic plane. In [2] the authors continued with observations in the third projective-metric plane, a pseudo-Euclidean plane. In the present paper we switch from quartics to cubics, and therefore from the projectivity between two pencils of conics to the projectivity between a pencil of conics and a pencil of lines.

3. Pseudo-Euclidean plane

In this section we will give the pseudo-Euclidean interpretation of the obtained projective results. By choosing suitable types of pencils and by setting projectivity in a proper way we can obtain different types of circular cubics in the pseudo-Euclidean plane. For example, we can state:

Theorem 3.1. *Let $[A, B]$ be a pencil of circles and let $[c, d]$ be a pencil of lines. The result of the projective mapping $\pi : [A, B] \to [c, d]$ is a 2-circular cubic k passing through both absolute points.*
Furthermore, if $[c, d]$ is a pencil of isotropic lines, k is entirely circular.

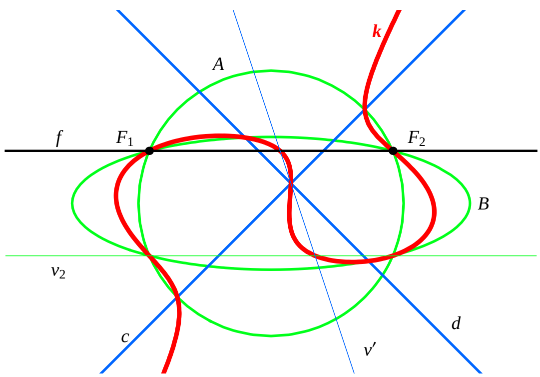

Figure 1: A cubic of type of circularity $(1, 1)$

All figures in this section are produced using the program *Mathematica*. Figure 1 displays a 2-circular cubic k which passes through both absolute points F_1, F_2. It is generated by the projectively linked pencil of circles $[A, B]$ and pencil of lines $[c, d]$. The projectivity is defined by three corresponding pairs: $A \leftrightarrow c$, $B \leftrightarrow d$, $V \leftrightarrow v'$ where V is a singular conic formed by two lines, and the absolute line f is one of them.

By using this method we can construct cubics of all types of circularity. Here we will indicate only some of the most interesting results.

A direct consequence of Theorem 2.4 is

Theorem 3.2. *Let $[A, B]$ be a pencil of conics and $[c, d]$ be a pencil of isotropic lines with the vertex at F_1.*
The result of the projective mapping $\pi : [A, B] \to [c, d]$ is a 1-circular cubic k passing through F_1.
If $[A, B]$ contains a special parabola touching the absolute line f at F_2 and that conic is mapped onto f, k is an entirely circular cubic passing through F_1 and touching f at F_2.

By giving a pseudo-Euclidean interpretation to Theorem 2.5 we obtain

Theorem 3.3. *Let $[A, B]$ be a pencil of special parabolas with the basic point F_1 and let $[c, d]$ be a pencil of lines. Let the cubic k be the result of the projective mapping $\pi : [A, B] \to [c, d]$.*
If the singular conic with the singular point F_1 is mapped onto the isotropic line through F_1, the cubic k is 2-circular with the double point at F_1. Furthermore, if the conics of the pencil $[A, B]$ osculate (or hyperosculate) each other, the cubic k is entirely circular touching the absolute line at the double point F_1.

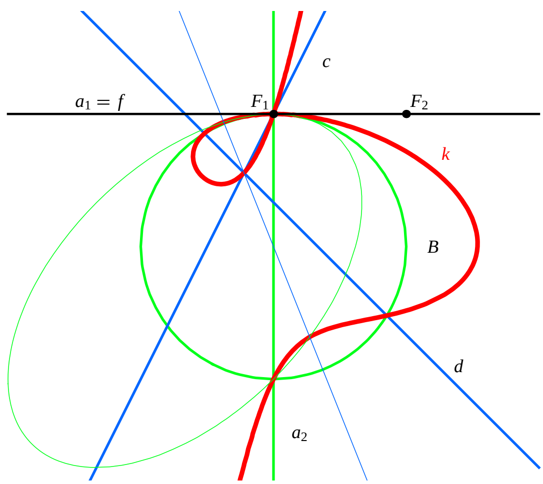

Figure 2: A cubic of type of circularity $(3, 0)$

The cubic k shown in Figure 2 is obtained by a projectively linked pencil of special parabolas osculating each other at the absolute point F_1 and a pencil of lines. Since the singular conic $A = f \cup a_2$ is mapped onto the isotropic line c through F_1, the cubic is entirely circular.

If we place the point $\vec{y}$ from Theorem 2.6 in an absolute point of the pseudo-Euclidean plane, we get

Theorem 3.4. *Let $[A, B]$ be a pencil of special hyperbolas with the basic point F_1 and let $[c, d]$ be a pencil of lines. Let the cubic k be the result of the projective mapping $\pi : [A, B] \to [c, d]$.*
k is a 1-circular cubic passing through F_1. If the special parabola of $[A, B]$ is mapped onto the isotropic line through F_1, k is a 2-circular cubic touching f at F_1.

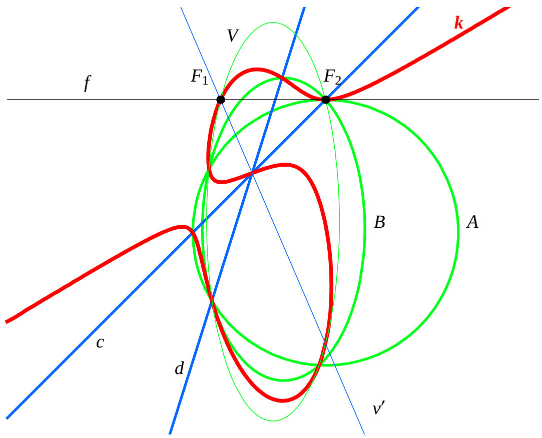

Figure 3: A cubic of type of circularity $(1, 2)$

The cubic k in Figure 3 is the result of the projective mapping between the pencil of special hyperbolas passing through the absolute point F_2 and the pencil of lines. The mapping is defined by three pairs: $A \leftrightarrow c$, $B \leftrightarrow d$, $V \leftrightarrow v'$. The pencil $[A, B]$, besides special hyperbolas, contains one special parabola and one circle. Since the special parabola A is linked to the isotropic line c through F_1, the cubic k touches f at F_2. The circle V is linked to the isotropic line v' through F_1 and therefore k passes through F_1. Thus, k is entirely circular.

From Theorem 2.7 we get

Theorem 3.5. *Let $[A, B]$ be a pencil of special parabolas with the basic point F_1 and let $[c, d]$ be a pencil of lines. Let the cubic k be the result of the projective mapping $\pi : [A, B] \to [c, d]$.*
k is a 2-circular cubic touching the absolute line at F_1.
If the singular conic containing the absolute line as its part is mapped onto the isotropic line through F_1, the cubic k is entirely circular osculating the absolute line at F_1.

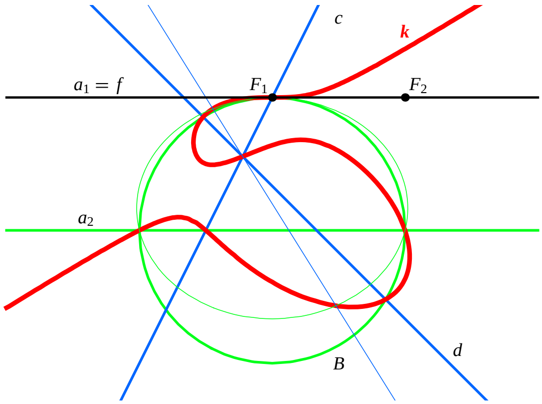

Figure 4: A cubic of type of circularity $(3, 0)$

Figure 4 displays an entirely circular cubic k of type (3,0). It is generated by the projectivity linking the pencil of special parabolas $[A, B]$ and the pencil of lines $[c, d]$ such that the singular conic A (formed by two lines, the absolute line f is one of them) is linked to the isotropic line c.

On the basis of the Theorem 2.10 by putting the vertex of the pencil into the absolute point we can state the following two theorems.

Theorem 3.6. *Let the pencil $[c, d]$ of isotropic lines through F_1 be given and let $[A, B]$ be a pencil of conics touching each other at F_1. Let the cubic k be the result of the projective mapping $\pi : [A, B] \to [c, d]$.*
If $[A, B]$ is a pencil of special hyperbolas, k is 2-circular cubic with the double point at F_1 at which one tangent coincides with the common tangent of the conics of $[A, B]$.
Furthermore, if $[A, B]$ is a pencil of circles touching each other at F_1, k is an entirely circular cubic having the double point at F_1 and passing through F_2.

The cubic k in Figure 5 is constructed as the result of the mapping between the pencil of circles $[A, B]$ touching each other at the absolute point F_1 and the pencil of isotropic lines $[c, d]$ through F_1. The obtained cubic has the double point at F_1 and passes through F_2. Therefore, it is entirely circular. The common tangent to the conics from $[A, B]$ touches one branch of k at F_1. The other tangent coincides with the line c since that line is linked to the singular conic $A = a_1 \cup a_2$.

Theorem 3.7. *Let $[A, B]$ be a pencil of special parabolas touching f at F_1 and let $[c, d]$ be a pencil of isotropic lines through F_1. Let the cubic k be the result of the projective mapping $\pi : [A, B] \to [c, d]$.*
k is an entirely circular cubic with the double point at F_1. The absolute line touches one branch of k, while the other tangent is the isotropic line linked to the singular conic with the singular point in F_1.

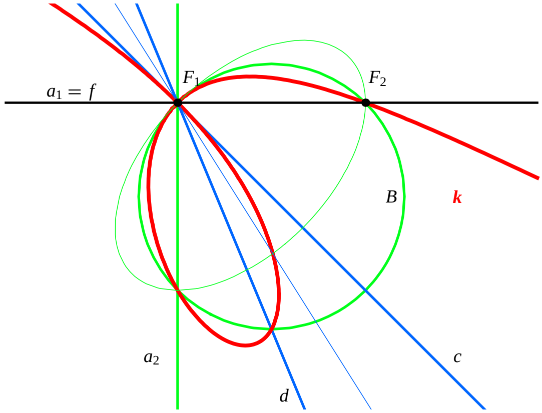

Figure 5: A cubic of type of circularity $(2, 1)$

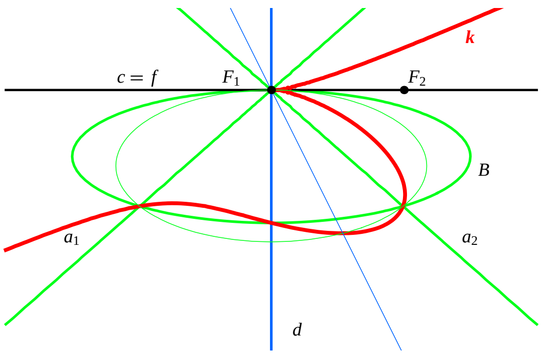

Figure 6: A cubic of type of circularity $(3, 0)$

The cubic k in Figure 6 is entirely circular. It is the result of the mapping between the pencil of special parabolas $[A, B]$ and the pencil of isotropic line $[c, d]$. Since the absolute line $c = f$ is linked to the singular conic $A = a_1 \cup a_2$, both tangents at the absolute point F_1 coincide with f. Therefore, k has a cusp in F_1.

4. Conclusion

In this paper we have studied the cubics obtained as the results of the projectively linked pencils of conics and lines. The pseudo-Euclidean interpretations of the projective situations have been presented. It has been shown that by using this method it is possible to construct circular cubics of all types (regarding their position with respect to the absolute figure).

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