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CHAOS EXPANSION METHODS IN MALLIAVIN CALCULUS: A SURVEY OF RECENT RESULTS

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Dedicated to Professor Bogoljub Stanković on the occasion of his 90th birthday and to Professor James Vickers on the occasion of his 60th birthday

Abstract. We present a review of the most important historical as well as recent results of Malliavin calculus in the framework of the Wiener-Itô chaos expansion.

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1. Introduction

The Malliavin derivative $\mathbb{D}$, the Skorokhod integral δ and the Ornstein-Uhlenbeck operator $\mathcal{R}$ are three operators that play a crucial role in the stochastic calculus of variations, an infinite-dimensional differential calculus on white noise spaces [2, 7, 36, 42, 43, 48]. These operators correspond respectively to the annihilation, the creation and the number operator in quantum operator theory.

- The Malliavin derivative, as a modification of Gâteaux derivatives, represents a stochastic gradient in direction of the white noise process [3, 36, 43]. Originally, it was invented by Paul Malliavin in order to provide a probabilistic proof of Hörmander's sum of squares theorem for hypoelliptic operators and to study the existence and regularity of a density for the solution of stochastic differential equations [29], but nowadays it has found significant applications in stochastic control and mathematical finance [8, 30, 47].

- The Skorokhod integral, as the adjoint operator of the Malliavin derivative, is a standard tool in classical $(L)^2$ theory of non-adapted stochastic

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differential equations. It represents an extension of the Itô integral from the space of adapted processes to the space of non-anticipative processes [6, 12, 16]. Sometimes it is referred to as the stochastic divergence operator.

- The Ornstein-Uhlenbeck operator, as the composition of the stochastic gradient and divergence, is a stochastic analogue of the Laplacian.

It is of great importance to manage solving different classes of equations which involve the operators of Malliavin calculus. In particular, we consider the following basic equations involving the operators of Malliavin calculus:

$$\mathcal{R}\, u = g, \qquad \mathbb{D} u = h, \qquad \delta\, u = f. \tag{1.1}$$

In the classical setting, the domain of these operators is a strict subset of the set of processes with finite second moments [7, 27, 36] leading to Sobolev type normed spaces. A more general characterization of the domain of these operators in Kondratiev generalized function spaces has been derived in [19, 23, 24], while in [25] we considered their domains within Kondratiev test function spaces. The three equations in (1.1), that have been considered in [21] and [25] provide a full characterization of the range of all three operators. Moreover, the solutions to equations (1.1) are obtained in an explicit form, which is highly useful for computer modelling that involves polynomial chaos expansion simulation methods used in numerical stochastic analysis [9, 31, 49].

After a short revision of the results on uniqueness of the solutions to equations (1.1) (Theorem 3.1, Theorem 4.1, Theorem 5.1) obtained in [21] and [25] we proceed by proving some properties such as the duality relationship between the Malliavin derivative and the Skorokhod integral (Theorem 6.1) and the chain rule (Theorem 6.11) as well as many others such as the product rule (Theorem 6.6, Theorem 6.8), partial integration etc.

A special emphasis is put on the characterization of Gaussian processes and Gaussian solutions of equations (1.1). As an important consequence and application of our results we obtain a connection between the Wick product and the ordinary product (Theorem 4.6 and Theorem 5.10). We also provide several illustrative examples to facilitate comprehension of our results. These examples can be considered as supplementary material to [21] and [25].

A recent discovery made in [33]-[35] made a nice connection between the Malliavin calculus and Stein's method, which is used to measure the distance to Gaussian distributions. In Theorem 7.10 we review this relationship using the chaos expansion method.

The method of chaos expansions is used to illustrate several known results in Malliavin calculus and thus provide a comprehensive insight into its capabilities. For example, we prove using the chaos expansion method some well-known results such as the commutator relationship between $\mathbb{D}$ and δ (Theorem 5.8), the relation between Itô integration and Riemann integration (Remark 5.9) as well as the Itô representation theorem (Corollary 5.3).

We strongly emphasize the *methodology* of the chaos expansion technique for solving singular SDEs. This method has been applied successfully to several classes of SPDEs (e.g. [20, 22, 26, 27, 28, 40, 46]) to obtain an explicit form of the solution. Therefore, we have chosen to write an expository survey with detailed step-by-step proofs and comprehensive examples that illustrate the full advantage of this technique. Some advantages of the chaos expansion technique are the following:

- It provides an explicit form of the solution. The solution is obtained in form of a series expansion.
- It is easy to apply, since it uses orthogonal bases and series expansions, applying the method of undetermined coefficients. Note that we avoid using the Hermite transform [13] or the S-transform [12], since these methods depend on the ability to apply their inverse transforms. Our method requires only to find an appropriate weight factor to make the resulting series convergent.
- It can be adapted to create numerical approximations and model simulations (e.g. by stochastic Galerkin methods). Polynomial chaos expansion approximations are known to be more efficient than Monte Carlo methods. Moreover, for non-Gaussian processes, convergence can be easily improved by changing the Hermite basis to another family of orthogonal polynomials (Charlier, Laguerre, Meixner, etc.).

2. Preliminaries

Consider the Gaussian white noise probability space $(S'(\mathbb{R}), \mathcal{B}, \mu)$, where $S'(\mathbb{R})$ denotes the space of tempered distributions, $\mathcal{B}$ the Borel $\sigma-$algebra generated by the weak topology on $S'(\mathbb{R})$ and μ the Gaussian white noise measure corresponding to the characteristic function

$$\int_{S'(\mathbb{R})} e^{i\langle\omega,\phi\rangle} d\mu(\omega) = e^{-\frac{1}{2}\|\phi\|^2_{L^2(\mathbb{R})}}, \qquad \phi \in S(\mathbb{R}), \tag{2.1}$$

given by the Bochner-Minlos theorem.

Denote by $h_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n}(e^{-\frac{x^2}{2}})$, $n \in \mathbb{N}_0$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, the family of Hermite polynomials and $\xi_n(x) = \frac{1}{\sqrt[4]{\pi}\sqrt{(n-1)!}} e^{-\frac{x^2}{2}} h_{n-1}(\sqrt{2}x)$, $n \in \mathbb{N}$, the family of Hermite functions. The family of Hermite functions forms a complete orthonormal system in $L^2(\mathbb{R})$. For a complete preview of properties of h_n and ξ_n a comprehensive reference is [10]. We follow the characterization of the Schwartz spaces in terms of the Hermite basis: The space of rapidly decreasing functions as a projective limit space $S(\mathbb{R}) = \bigcap_{l\in\mathbb{N}_0} S_l(\mathbb{R})$ and the space of tempered distributions as an inductive limit space $S'(\mathbb{R}) = \bigcup_{l\in\mathbb{N}_0} S_{-l}(\mathbb{R})$ where

$$S_l(\mathbb{R}) = \{f = \sum_{k=1}^{\infty} a_k\, \xi_k : \ \|f\|_l^2 = \sum_{k=1}^{\infty} a_k^2 (2k)^l < \infty\},\ l \in \mathbb{Z},\ \mathbb{Z} = -\mathbb{N} \cup \mathbb{N}_0.$$

Note that $S_p(\mathbb{R})$ is a Hilbert space endowed with the scalar product $\langle \cdot, \cdot \rangle_p$ given by

$$\langle \xi_k, \xi_l \rangle_p = \begin{cases} 0, & k \neq l \\ \|\xi_k\|_p^2 = (2k)^p, & k = l. \end{cases}, \quad p \in \mathbb{Z}.$$

Moreover, the functions $\tilde{\xi}_k = \xi_k (2k)^{-\frac{p}{2}}$, $k \in \mathbb{N}$, constitute an orthonormal basis for $S_p(\mathbb{R})$. Indeed,

$$\langle \tilde{\xi}_k, \tilde{\xi}_l \rangle_p = \begin{cases} 0, & k \neq l \\ \|\tilde{\xi}_k\|_p^2 = \|\xi_k\|_{(L)^2}^2 = 1, & k = l. \end{cases}, \quad p \in \mathbb{Z}.$$

2.1. The Wiener chaos spaces

Let $\mathcal{I} = (\mathbb{N}_0^{\mathbb{N}})_c$ denote the set of sequences of nonnegative integers which have only finitely many nonzero components $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m, 0, 0 \ldots)$, $\alpha_i \in \mathbb{N}_0$, $i = 1, 2, ..., m$, $m \in \mathbb{N}$. The kth unit vector $\varepsilon^{(k)} = (0, \cdots, 0, 1, 0, \cdots)$, $k \in \mathbb{N}$ is the sequence of zeros with the only entry 1 as its kth component. The multi-index $\mathbf{0} = (0, 0, 0, 0, \ldots)$ has all zero entries. The length of a multi-index $\alpha \in \mathcal{I}$ is defined as $|\alpha| = \sum_{k=1}^{\infty} \alpha_k$.

Operations with multi-indices are carried out componentwise e.g. $\alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \ldots)$, $\alpha! = \alpha_1! \alpha_2! \alpha_3! \cdots$, $\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!}$. Note that $\alpha > \mathbf{0}$ (equivalently $|\alpha| > 0$) if there is at least one component $\alpha_k > 0$. We adopt the convention that $\alpha - \beta$ exists only if $\alpha - \beta > \mathbf{0}$ and otherwise it is not defined.

Let $(2\mathbb{N})^{\alpha} = \prod_{k=1}^{\infty} (2k)^{\alpha_k}$. Note that $\sum_{\alpha \in \mathcal{I}} (2\mathbb{N})^{-p\alpha} < \infty$ for $p > 1$ (see e.g. [13]).

Let $(L)^2 = L^2(S'(\mathbb{R}), \mathcal{B}, \mu)$ be the Hilbert space of random variables with finite second moments. We define by

$$H_\alpha(\omega) = \prod_{k=1}^{\infty} h_{\alpha_k}(\langle \omega, \xi_k \rangle), \quad \alpha \in \mathcal{I},$$

the Fourier-Hermite orthogonal basis of $(L)^2$ such that $\|H_\alpha\|_{(L)^2}^2 = \alpha!$. In particular, for the kth unit vector $H_{\varepsilon^{(k)}}(\omega) = \langle \omega, \xi_k \rangle$, $k \in \mathbb{N}$.

The prominent *Wiener-Itô chaos expansion theorem* states that each element $F \in (L)^2$ has a unique representation of the form

$$F(\omega) = \sum_{\alpha \in \mathcal{I}} c_\alpha \, H_\alpha(\omega),$$

$\omega \in S'(\mathbb{R})$, $c_\alpha \in \mathbb{R}$, $\alpha \in \mathcal{I}$, such that $\|F\|_{(L)^2}^2 = \sum_{\alpha \in \mathcal{I}} c_\alpha^2 \, \alpha! < \infty$.

Definition 2.1. The spaces

$$\mathcal{H}_k = \{F \in (L)^2 \, : \, F = \sum_{\alpha \in \mathcal{I}, |\alpha| = k} c_\alpha \, H_\alpha\}, \quad k \in \mathbb{N}_0,$$

that are obtained by closing the linear span of the kth order Hermite polynomials in $(L)^2$ are called the Wiener chaos spaces of order k.

For example, $\mathcal{H}_0$ is the set of constant random variables, $\mathcal{H}_1$ is a set of Gaussian random variables, $\mathcal{H}_2$ is a space of quadratic Gaussian random variables and so on. We will show that $\mathcal{H}_1$ contains only Gaussian random variables and the most important processes: Brownian motion and white noise belong to $\mathcal{H}_1$.

Each $\mathcal{H}_k$, $k \in \mathbb{N}_0$ is a closed subspace of $(L)^2$. Moreover, the Wiener-Itô chaos expansion theorem can be stated in the form:

$$(L)^2 = \bigoplus_{k=0}^{\infty} \mathcal{H}_k.$$

Hence, every $F \in (L)^2$ can be represented in the form $F(\omega) = \sum_{k=0}^{\infty} \sum_{\substack{\alpha\in\mathcal{I} \\ |\alpha|=k}} c_\alpha H_\alpha(\omega)$, $\omega \in S'(\mathbb{R})$, where $\sum_{|\alpha|=k} c_\alpha\, H_\alpha(\omega) \in \mathcal{H}_k$, $k = 0, 1, 2, \ldots$.

Theorem 2.2. *All random variables which belong to $\mathcal{H}_1$ are Gaussian random variables.*

Proof. Random variables that belong to the space $\mathcal{H}_1$ are linear combinations of elements $\langle\omega, \xi_k\rangle$, $k \in \mathbb{N}$, $\omega \in S'(\mathbb{R})$. From the definition of the Gaussian measure (2.1) it follows that $E_\mu(\langle\omega, \xi_k\rangle) = 0$ and $Var(\langle\omega, \xi_k\rangle) = E_\mu(\langle\omega, \xi_k\rangle^2) = \|\xi_k\|^2_{L^2(\mathbb{R})} = 1$. Thus, from the form of the characteristic function we conclude that $\langle\omega, \xi_k\rangle : \mathcal{N}(0, 1)$, $k \in \mathbb{N}$. Thus, every finite linear combination of Gaussian random variables $\sum_{k=1}^{n} a_k\, \langle\omega, \xi_k\rangle$ is a Gaussian random variable and the limit of Gaussian random variables $\sum_{k=1}^{\infty} a_k\, \langle\omega, \xi_k\rangle = \lim_{n\to+\infty} \sum_{k=1}^{n} a_k\, \langle\omega, \xi_k\rangle$ is also Gaussian. □

After Example 2.13 it will be also clear that $\mathcal{H}_1$ is the closed Gaussian space generated by the random variables $B_t(\omega)$, $t \geq 0$, where B_t is Brownian motion (see also [42]).

Remark 2.3. We note the following important facts:

1) Although the space $(L)^2$ is constructed with respect to Gaussian measure, it contains all (square integrable) random variables, not just those with Gaussian distribution but also all absolutely continuous, singularly continuous, discrete and mixed type distributions.

2) All elements in $\mathcal{H}_0 \oplus \mathcal{H}_1$ are Gaussian (those with zero expectation are strictly in $\mathcal{H}_1$), but the converse is not true. In Example 7.16 we show that there exist Gaussian random variables with higher order chaos expansions. Representative elements of $\mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$ are for example quadratic Gaussian random variables and the Chi-square distribution as a finite sum of independent quadratic Gaussian variables.

3) Discrete random variables (with finite variance) belong to $\bigoplus_{k=0}^{\infty} \mathcal{H}_k$, i.e. their chaos expansions forms consist of multi-indices of all lengths.

4) All finite sums i.e. partial sums of a chaos expansion correspond to absolutely continuous distributions or almost surely constant distributions. There is no possibility to obtain discrete random variables by using finite sums in the Wiener-Itô expansion. This is a consequence of Theorem 7.8.

In the next section we introduce suitable spaces, called Kondratiev spaces, that will contain random variables with infinite variances.

2.2. Kondratiev spaces

The stochastic analogue of Schwartz spaces as generalized function spaces are the Kondratiev spaces of generalized random variables.

Definition 2.4. The space of the Kondratiev test random variables $(S)_1$ consists of elements $f = \sum_{\alpha \in \mathcal{I}} c_\alpha H_\alpha \in (L)^2$, $c_\alpha \in \mathbb{R}$, $\alpha \in \mathcal{I}$, such that

$$\|f\|^2_{1,p} = \sum_{\alpha \in \mathcal{I}} c_\alpha^2 (\alpha!)^2 (2\mathbb{N})^{p\alpha} < \infty, \qquad \text{for all } p \in \mathbb{N}_0.$$

The space of the Kondratiev generalized random variables $(S)_{-1}$ consists of formal expansions of the form $F = \sum_{\alpha \in \mathcal{I}} b_\alpha H_\alpha$, $b_\alpha \in \mathbb{R}$, $\alpha \in \mathcal{I}$, such that

$$\|F\|^2_{-1,-p} = \sum_{\alpha \in \mathcal{I}} b_\alpha^2 (2\mathbb{N})^{-p\alpha} < \infty, \qquad \text{for some } p \in \mathbb{N}_0.$$

Definition 2.5. The space of the Hida test random variables $(S)_0^+$ consists of elements $f = \sum_{\alpha \in \mathcal{I}} c_\alpha H_\alpha \in (L)^2$, $c_\alpha \in \mathbb{R}$, $\alpha \in \mathcal{I}$, such that

$$\|f\|^2_{0,p} = \sum_{\alpha \in \mathcal{I}} c_\alpha^2 \alpha! (2\mathbb{N})^{p\alpha} < \infty, \qquad \text{for all } p \in \mathbb{N}_0.$$

The space of the Hida generalized random variables $(S)_0^-$ consists of formal expansions of the form $F = \sum_{\alpha \in \mathcal{I}} b_\alpha H_\alpha$, $b_\alpha \in \mathbb{R}$, $\alpha \in \mathcal{I}$, such that

$$\|F\|^2_{0,-p} = \sum_{\alpha \in \mathcal{I}} b_\alpha^2 \alpha! (2\mathbb{N})^{-p\alpha} < \infty, \qquad \text{for some } p \in \mathbb{N}_0.$$

This provides a sequence of spaces $(S)_{\rho,p} = \{f \in (L)^2 : \|f\|_{\rho,p} < \infty\}$, $\rho \in \{-1, 0, 1\}$, $p \in \mathbb{Z}$, such that

$$(S)_{1,p} \subseteq (S)_{0,p} \subseteq (L)^2 \subseteq (S)_{0,-p} \subseteq (S)_{-1,-p},$$

$$(S)_{1,p} \subseteq (S)_{1,q} \subseteq (L)^2 \subseteq (S)_{-1,-q} \subseteq (S)_{-1,-p},$$

for all $p \geq q \geq 0$ and the inclusions denote continuous embeddings and $(S)_{0,0} = (L)^2$. Thus, $(S)_1 = \bigcap_{p \in \mathbb{N}_0} (S)_{1,p}$ and $(S)_0^+ = \bigcap_{p \in \mathbb{N}_0} (S)_{0,p}$ can be equipped with the projective topology and $(S)_{-1} = \bigcup_{p \in \mathbb{N}_0} (S)_{-1,-p}$, $(S)_0^- = \bigcup_{p \in \mathbb{N}_0} (S)_{0,-p}$ as their duals with the inductive topology. Note that $(S)_1$, $(S)_0^+$ are nuclear and the following Gel'fand triples

$$(S)_1 \subseteq (L)^2 \subseteq (S)_{-1}, \qquad (S)_0^+ \subseteq (L)^2 \subseteq (S)_0^-$$

are obtained.

From the estimate $\alpha! \leq (2\mathbb{N})^\alpha$ it follows that

$$(2\mathbb{N})^{-p\alpha} \leq \alpha!(2\mathbb{N})^{-p\alpha} \leq (2\mathbb{N})^{-(p-1)\alpha},$$

thus

$$(S)_{-1,-(p-1)} \subseteq (S)_{0,-p} \subseteq (S)_{-1,-p}, \quad \text{for all } \ p \in \mathbb{N},$$

and similarly

$$(S)_{1,p+1} \subseteq (S)_{0,p} \subseteq (S)_{1,p}, \quad \text{for all } \ p \in \mathbb{N}_0.$$

We will denote by $\ll \cdot,\cdot \gg$ the dual pairing between $(S)_{0,-p}$ and $(S)_{0,p}$. Its action is given by $\ll A, B \gg = \ll \sum_{\alpha\in\mathcal{I}} a_\alpha H_\alpha, \sum_{\alpha\in\mathcal{I}} b_\alpha H_\alpha \gg = \sum_{\alpha\in\mathcal{I}} \alpha! a_\alpha b_\alpha$. In case of random variables with finite variance it reduces to the scalar product $\ll A, B \gg_{(L)^2} = E(AB)$. For any fixed $p \in \mathbb{Z}$, $(S)_{0,p}$, $p \in \mathbb{Z}$, is a Hilbert space (we identify the case $p = 0$ with $(L)^2$) endowed with the scalar product

$$\ll H_\alpha, H_\beta \gg_p = \begin{cases} 0, & \alpha \neq \beta, \\ \alpha!(2\mathbb{N})^{p\alpha}, & \alpha = \beta, \end{cases}, \qquad \text{for } \ p \in \mathbb{Z},$$

extended by linearity and continuity to

$$\ll A, B \gg_p = \sum_{\alpha\in\mathcal{I}} \alpha! a_\alpha b_\alpha (2\mathbb{N})^{p\alpha}, \quad p \in \mathbb{Z}.$$

In the framework of white noise analysis, the problem of pointwise multiplication of generalized functions is overcome by introducing the Wick product. It is well defined in the Kondratiev spaces of test and generalized stochastic functions $(S)_1$ and $(S)_{-1}$; see for example [12, 13].

Definition 2.6. Let $F, G \in (S)_{-1}$ be given by their chaos expansions $F(\omega) = \sum_{\alpha\in\mathcal{I}} f_\alpha H_\alpha(\omega)$ and $G(\omega) = \sum_{\beta\in\mathcal{I}} g_\beta H_\beta(\omega)$, for unique $f_\alpha, g_\beta \in \mathbb{R}$. The Wick product of F and G is the element denoted by $F\lozenge G$ and defined by

$$\begin{aligned} F\lozenge G(\omega) &= \sum_{\gamma\in\mathcal{I}} \left(\sum_{\alpha+\beta=\gamma} f_\alpha g_\beta \right) H_\gamma(\omega) \\ &= \sum_{\alpha\in\mathcal{I}} \sum_{\beta\in\mathcal{I}} f_\alpha \, g_\beta \, H_{\alpha+\beta}(\omega). \end{aligned}$$

The same definition is provided for the Wick product of test random variables belonging to $(S)_1$.

Note that the Kondratiev spaces $(S)_1$ and $(S)_{-1}$ are closed under the Wick multiplication [13], while the space $(L)^2$ is not closed under it.

Example 2.7. The random variable defined by the chaos expansion $F = \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n!}} H_{n\varepsilon^{(n)}}$ belongs to $(L)^2$ since $\|F\|^2_{(L)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$, but $F \lozenge F$ is not in $(L)^2$. Clearly,

$$\begin{aligned} \|F\lozenge F\|^2_{(L)^2} &= \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} \frac{1}{k(n-k)\sqrt{k!(n-k)!}} \right)^2 n! \\ &\geq \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} \frac{1}{k(n-k)} \right)^2 = \infty. \end{aligned}$$

The most important property of the Wick multiplication is its relation to the Itô-Skorokhod integration [12, 13], since it reproduces the fundamental theorem of calculus. This fact will be revisited in Remark 5.9.

In the sequel we will need the notion of *Wick-versions of analytic functions.* For this purpose note that the nth Wick power is defined by $F^{\lozenge n} = F^{\lozenge (n-1)} \lozenge F$, $F^{\lozenge 0} = 1$. Note that $H_{n\varepsilon_k} = H_{\varepsilon_k}^{\lozenge n}$ for $n \in \mathbb{N}_0$, $k \in \mathbb{N}$.

Definition 2.8. If $\varphi : \mathbb{R} \to \mathbb{R}$ is a real analytic function at the origin represented by the power series

$$\varphi(x) = \sum_{n=0}^{\infty} a_n\, x^n, \quad x \in \mathbb{R},$$

then its Wick version $\varphi^{\lozenge} : (S)_{-1} \to (S)_{-1}$ is given by

$$\varphi^{\lozenge}(F) = \sum_{n=0}^{\infty} a_n\, F^{\lozenge n}, \quad F \in (S)_{-1}.$$

2.3. Generalized stochastic processes

Let $\tilde{X}$ be a Banach space endowed with the norm $\|\cdot\|_{\tilde{X}}$ and let $\tilde{X}'$ denote its dual space. In this section we describe $\tilde{X}-$valued random variables. Most notably, if $\tilde{X}$ is a space of functions on $\mathbb{R}$, e.g. $\tilde{X} = C^k([a,b]))$, $-\infty < a < b < \infty$ or $\tilde{X} = L^2(\mathbb{R})$, we obtain the notion of a stochastic process. We will also define processes where $\tilde{X}$ is not a normed space, but a nuclear space topologized by a family of seminorms, e.g. $\tilde{X} = S(\mathbb{R})$ (see e.g. [39]).

Definition 2.9. Let f have the formal expansion $f = \sum_{\alpha\in\mathcal{I}} f_\alpha \otimes H_\alpha$, where $f_\alpha \in X$, $\alpha \in \mathcal{I}$. Define the following spaces:

$$\begin{aligned} X \otimes (S)_{1,p} &= \{f : \|f\|^2_{X\otimes(S)_{1,p}} = \sum_{\alpha\in\mathcal{I}} \alpha!^2 \|f_\alpha\|^2_X (2\mathbb{N})^{p\alpha} < \infty\}, \\ X \otimes (S)_{-1,-p} &= \{f : \|f\|^2_{X\otimes(S)_{-1,-p}} = \sum_{\alpha\in\mathcal{I}} \|f_\alpha\|^2_X (2\mathbb{N})^{-p\alpha} < \infty\}, \\ X \otimes (S)_{0,p} &= \{f : \|f\|^2_{X\otimes(S)_{0,p}} = \sum_{\alpha\in\mathcal{I}} \alpha! \|f_\alpha\|^2_X (2\mathbb{N})^{p\alpha} < \infty\}, \\ X \otimes (S)_{0,-p} &= \{f : \|f\|^2_{X\otimes(S)_{0,-p}} = \sum_{\alpha\in\mathcal{I}} \alpha! \|f_\alpha\|^2_X (2\mathbb{N})^{-p\alpha} < \infty\}, \end{aligned}$$

where X denotes an arbitrary Banach space (allowing both possibilities $X = \tilde{X}$, $X = \tilde{X}'$).

Especially, for $p = 0$, $X \otimes (S)_{0,0}$ will be denoted by

$$X \otimes (L)^2 = \{f : \|f\|^2_{X\otimes(S)_{0,-p}} = \sum_{\alpha\in\mathcal{I}} \alpha! \|f_\alpha\|^2_X < \infty\}.$$

We will denote by $E(F) = f_{(0,0,0,\dots)}$ the generalized expectation of the process F.

Definition 2.10. Generalized stochastic processes and test stochastic processes in Kondratiev sense are elements of the spaces

$$X \otimes (S)_{-1} = \bigcup_{p\in\mathbb{N}} X \otimes (S)_{-1,-p}, \quad X \otimes (S)_1 = \bigcap_{p\in\mathbb{N}} X \otimes (S)_{1,p},$$

respectively.

Generalized stochastic processes and test stochastic processes in Hida sense are elements of the spaces

$$X \otimes (S)_0^- = \bigcup_{p\in\mathbb{N}} X \otimes (S)_{0,-p}, \quad X \otimes (S)_0^+ = \bigcap_{p\in\mathbb{N}} X \otimes (S)_{0,p},$$

respectively.

Remark 2.11. In this case the symbol $\otimes$ denotes the projective tensor product of two spaces i.e. $\tilde{X}' \otimes (S)_{-1}$ is the completion of the tensor product with respect to the π-topology.

The Kondratiev space $(S)_1$ is nuclear and thus $(\tilde{X} \otimes (S)_1)' \cong \tilde{X}' \otimes (S)_{-1}$. Note that $\tilde{X}' \otimes (S)_{-1}$ is isomorphic to the space of linear bounded mappings $\tilde{X} \to (S)_{-1}$, and it is also isomporphic to the space of linear bounded mappings $(S)_{+1} \to \tilde{X}'$. The same holds for the Hida spaces, too.

In [44] and [45] a general setting of S'-valued generalized stochastic process is provided (we restrict our attention to the Kondratiev setting): $S'(\mathbb{R})$-valued generalized stochastic processes are elements of $X \otimes S'(\mathbb{R}) \otimes (S)_{-1}$ and they are given by chaos expansions of the form

$$f = \sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} a_{\alpha,k} \otimes \xi_k \otimes H_\alpha = \sum_{\alpha\in\mathcal{I}} b_\alpha \otimes H_\alpha = \sum_{k\in\mathbb{N}} c_k \otimes \xi_k, \tag{2.2}$$

where $b_\alpha = \sum_{k\in\mathbb{N}} a_{\alpha,k} \otimes \xi_k \in X \otimes S'(\mathbb{R})$, $c_k = \sum_{\alpha\in\mathcal{I}} a_{\alpha,k} \otimes H_\alpha \in X \otimes (S)_{-1}$ and $a_{\alpha,k} \in X$. Thus,

$$X \otimes S_{-l}(\mathbb{R}) \otimes (S)_{-1,-p} = \left\{ f : \|f\|^2_{X\otimes S_{-l}(\mathbb{R})\otimes(S)_{-1,-p}} = \sum_{\alpha\in\mathcal{I},k\in\mathbb{N}} \|a_{\alpha,k}\|^2_X (2k)^{-l} (2\mathbb{N})^{-p\alpha} < \infty \right\}$$

and

$$X \otimes S'(\mathbb{R}) \otimes (S)_{-1} = \bigcup_{p,l\in\mathbb{N}} X \otimes S_{-l}(\mathbb{R}) \otimes (S)_{-1,-p}.$$

The generalized expectation of an S'-valued stochastic process f is given by $E(f) = \sum\limits_{k\in\mathbb{N}} a_{(0,0,\dots),k} \otimes \xi_k = b_{(0,0,\dots)}$.

In an analogue way, we define S-valued test processes as elements of $X \otimes S(\mathbb{R}) \otimes (S)_1$, which are given by chaos expansions of the form (2.2), where $b_\alpha = \sum_{k\in\mathbb{N}} a_{\alpha,k} \otimes \xi_k \in X \otimes S(\mathbb{R})$, $c_k = \sum_{\alpha\in\mathcal{I}} a_{\alpha,k} \otimes H_\alpha \in X \otimes (S)_1$ and $a_{\alpha,k} \in X$. Thus,

$$X \otimes S_l(\mathbb{R}) \otimes (S)_{1,p} = \left\{ f : \|f\|^2_{X\otimes S_l(\mathbb{R})\otimes(S)_{1,p}} = \sum_{\alpha\in\mathcal{I},k\in\mathbb{N}} \alpha!^2 \|a_{\alpha,k}\|^2_X (2k)^l (2\mathbb{N})^{p\alpha} < \infty \right\}$$

and

$$X \otimes S(\mathbb{R}) \otimes (S)_1 = \bigcap_{p,l\in\mathbb{N}} X \otimes S_l(\mathbb{R}) \otimes (S)_{1,p}.$$

One can define the Hida spaces in a similar way. Especially, for $p = l = 0$, one obtains the space of processes with finite second moments and square integrable trajectories $X \otimes L^2(\mathbb{R}) \otimes (L)^2$. It is isomporphic to $X \otimes L^2(\mathbb{R} \times \Omega)$ and if X is a separable Hilbert space, then it is also isomorphic to $L^2(\mathbb{R} \times \Omega; X)$.

Remark 2.12. In the sequel we will use the notation $\mathcal{H}_k$, $k \in \mathbb{N}_0$, to denote not just $(L)^2$-random variables, but also generalized stochastic processes and test processes which have a chaos expansion of the form (2.2) only with multi-indices of length $|\alpha| = k$.

Example 2.13. Brownian motion as an element of $S'(\mathbb{R}) \otimes (L)^2$, is defined by

$$B_t(\omega) := \langle \omega, \kappa_{[0,t]} \rangle, \quad \omega \in S'(\mathbb{R}),$$

where $\kappa_{[0,t]}$ is the characteristic function of the interval $[0,t]$, $t > 0$. It is a Gaussian process with zero expectation and covariance function $E_\mu(B_t(\omega)\, B_s(\omega)) = \min\{t,s\}$. The chaos expansion of Brownian motion is given by

$$B_t(\omega) = \sum_{k=1}^{\infty} \left(\int_0^t \xi_k(s)ds \right) H_{\varepsilon^{(k)}}(\omega).$$

For all $k \in \mathbb{N}$, its coefficients $\int\limits_0^t \xi_k(s)ds$ are in $C^\infty(\mathbb{R})$.

Singular white noise is defined by the chaos expansion

$$W_t(\omega) = \sum_{k=1}^{\infty} \xi_k(t) H_{\varepsilon^{(k)}}(\omega),$$

and it is an element of the space $S_k(\mathbb{R}) \otimes (S)_{-1,-p}$ for $k, p \geq 1$. With weak derivatives in the $(S)_{-1}$ sense $\frac{d}{dt} B_t = W_t$ holds. Both Brownian motion and singular white noise belong to the Wiener chaos space of order one.

2.4. Multiplication of stochastic processes

We generalize the definition of the Wick product of random variables to the set of generalized stochastic processes in the way as it is done in [20, 40] and [41]. From now on we assume that X is closed under multiplication, i.e. $x \cdot y \in X$ for all $x, y \in X$.

Definition 2.14. Let $F, G \in X \otimes (S)_{\pm 1}$ be generalized (resp. test) stochastic processes given by chaos expansions $f = \sum_{\alpha \in \mathcal{I}} f_\alpha \otimes H_\alpha$, $g = \sum_{\alpha \in \mathcal{I}} g_\alpha \otimes H_\alpha$, where $f_\alpha, g_\alpha \in X$, $\alpha \in \mathcal{I}$. Then the Wick product $F \lozenge G$ is defined by

$$F \lozenge G = \sum_{\gamma \in \mathcal{I}} \left(\sum_{\alpha+\beta=\gamma} f_\alpha g_\beta \right) \otimes H_\gamma. \tag{2.3}$$

Theorem 2.15. *Let the stochastic processes F and G be given in their chaos expansion forms $F = \sum\limits_{\alpha \in \mathcal{I}} f_\alpha \otimes H_\alpha$ and $G = \sum\limits_{\alpha \in \mathcal{I}} g_\alpha \otimes H_\alpha$.*

1. *If $F \in X \otimes (S)_{-1,-p_1}$ and $G \in X \otimes (S)_{-1,-p_2}$ for some $p_1, p_2 \in \mathbb{N}_0$, then $F \lozenge G$ is a well defined element in $X \otimes (S)_{-1,-q}$, for $q \geq p_1 + p_2 + 2$.*

2. *If $F \in X \otimes (S)_{1,p_1}$ and $G \in X \otimes (S)_{1,p_2}$ for $p_1, p_2 \in \mathbb{N}_0$, then $F \lozenge G$ is a well defined element in $X \otimes (S)_{1,q}$, for $q \leq \min\{p_1, p_2\} - 2$.*

Proof. 1. By the Cauchy-Schwartz inequality, the following holds

$$\begin{aligned}
&\|F \lozenge G\|^2_{X \otimes (S)_{-1,-q}} \\
&\quad = \sum_{\gamma \in \mathcal{I}} \| \sum_{\alpha+\beta=\gamma} f_\alpha g_\beta \|^2_X \, (2\mathbb{N})^{-q\gamma} \\
&\quad \leq \sum_{\gamma \in \mathcal{I}} \| \sum_{\alpha+\beta=\gamma} f_\alpha g_\beta \|^2_X \, (2\mathbb{N})^{-(p_1+p_2+2)\gamma} \\
&\quad \leq \sum_{\gamma \in \mathcal{I}} \left(\sum_{\alpha+\beta=\gamma} \|f_\alpha\|^2_X (2\mathbb{N})^{-p_1\gamma} \right) \left(\sum_{\alpha+\beta=\gamma} \|g_\beta\|^2_X (2\mathbb{N})^{-p_2\gamma} \right) (2\mathbb{N})^{-2\gamma} \\
&\quad \leq \left(\sum_{\gamma \in \mathcal{I}} (2\mathbb{N})^{-2\gamma} \right) \left(\sum_{\alpha \in \mathcal{I}} \|f_\alpha\|^2_X (2\mathbb{N})^{-p_1\alpha} \right) \left(\sum_{\beta \in \mathcal{I}} \|g_\beta\|^2_X (2\mathbb{N})^{-p_2\beta} \right) \\
&\quad = M \cdot \|F\|^2_{X \otimes (S)_{-1,-p_1}} \cdot \|G\|^2_{X \otimes (S)_{-1,-p_2}} < \infty,
\end{aligned}$$

since $M = \sum_{\gamma \in \mathcal{I}} (2\mathbb{N})^{-2\gamma} < \infty$ by the nuclearity of $(S)_{-1}$.

2. Let now $F \in X \otimes (S)_{1,p_1}$ and $G \in X \otimes (S)_{1,p_2}$ for all $p_1, p_2 \in \mathbb{N}_0$. Then the chaos expansion form of $F \lozenge G$ is given by (2.3) and

$$
\begin{aligned}
&\|F\lozenge G\|^2_{X\otimes(S)_{1,q}}\\
&= \sum_{\gamma\in\mathcal{I}}\gamma!^2\|\sum_{\alpha+\beta=\gamma} f_\alpha g_\beta\|^2_X(2\mathbb{N})^{q\gamma}\cdot(2\mathbb{N})^{2\gamma}(2\mathbb{N})^{-2\gamma}\\
&= \sum_{\gamma\in\mathcal{I}}(2\mathbb{N})^{-2\gamma}\|\sum_{\alpha+\beta=\gamma}\gamma! f_\alpha g_\beta(2\mathbb{N})^{\frac{q+2}{2}\gamma}\|^2_X\\
&\le \sum_{\gamma\in\mathcal{I}}(2\mathbb{N})^{-2\gamma}\|\sum_{\alpha+\beta=\gamma}\alpha!\beta!(2\mathbb{N})^{\alpha+\beta} f_\alpha g_\beta(2\mathbb{N})^{\frac{q+2}{2}(\alpha+\beta)}\|^2_X\\
&\le M\left(\sum_{\alpha\in\mathcal{I}}\alpha!^2\|f_\alpha\|^2_X(2\mathbb{N})^{2(\frac{q+2}{2}+1)\alpha}\right)\left(\sum_{\beta\in\mathcal{I}}\beta!^2\|g_\beta\|^2_X(2\mathbb{N})^{2(\frac{q+2}{2}+1)\beta}\right)\\
&\le M\left(\sum_{\alpha\in\mathcal{I}}\alpha!^2\|f_\alpha\|^2_X(2\mathbb{N})^{p_1\alpha}\right)\left(\sum_{\beta\in\mathcal{I}}\beta!^2\|g_\beta\|^2_X(2\mathbb{N})^{p_2\beta}\right)\\
&= M\cdot\|F\|^2_{X\otimes(S)_{1,p_1}}\cdot\|G\|^2_{X\otimes(S)_{1,p_2}}<\infty,
\end{aligned}
$$

if $q\le p_1-2$ and $q\le p_2-2$. We used beside the Cauchy-Schwartz inequality the estimate $(\alpha+\beta)!\le\alpha!\,\beta!\,(2\mathbb{N})^{\alpha+\beta}$, for all $\alpha,\beta\in\mathcal{I}$. □

Applying the well-known formula for the Fourier-Hermite polynomials (see [13])

$$
H_\alpha\cdot H_\beta=\sum_{\gamma\le\min\{\alpha,\beta\}}\gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma}H_{\alpha+\beta-2\gamma} \tag{2.4}
$$

one can define the ordinary product $F\cdot G$ of two stochastic processes F and G. Thus, by applying formally (2.4) we obtain

$$
\begin{aligned}
F\cdot G &= \sum_{\alpha\in\mathcal{I}} f_\alpha\otimes H_\alpha\cdot\sum_{\beta\in\mathcal{I}} g_\beta\otimes H_\beta\\
&= \sum_{\alpha\in\mathcal{I}}\sum_{\beta\in\mathcal{I}} f_\alpha\, g_\beta\otimes H_\alpha\cdot H_\beta\\
&= \sum_{\alpha\in\mathcal{I}}\sum_{\beta\in\mathcal{I}} f_\alpha\, g_\beta\otimes\sum_{\mathbf{0}\le\gamma\le\min\{\alpha,\beta\}}\gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma}H_{\alpha+\beta-2\gamma}\\
&= F\lozenge G+\sum_{\alpha\in\mathcal{I}}\sum_{\beta\in\mathcal{I}} f_\alpha\, g_\beta\otimes\sum_{\mathbf{0}<\gamma\le\min\{\alpha,\beta\}}\gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma}H_{\alpha+\beta-2\gamma}\\
&= F\lozenge G+\sum_{\tau\in\mathcal{I}}\sum_{\alpha\in\mathcal{I}}\sum_{\beta\in\mathcal{I}} f_\alpha g_\beta\sum_{\substack{\gamma>\mathbf{0},\,\delta\le\tau\\ \gamma+\tau-\delta=\beta,\,\gamma+\delta=\alpha}}\frac{\alpha!\beta!}{\gamma!\delta!(\tau-\delta)!}\;H_\tau.
\end{aligned}
$$

For example, for Brownian motion we have

$$
B_{t_1}\cdot B_{t_2}=B_{t_1}\lozenge B_{t_2}+\min\{t_1,t_2\},\quad B_t^2=B_t^{\lozenge 2}+t.
$$

Note also that, $E(F\Diamond G) = f_{\mathbf{0}}\, g_{\mathbf{0}} = EF\cdot EG$, without the assumption of independence of F and G as opposed to $E(F\cdot G)\neq EF\cdot EG$.

Particularly, it is clear that the following identities hold for the Fourier-Hermite polynomials:

$$H_{\varepsilon^{(k)}}\cdot H_{\varepsilon^{(l)}} = \begin{cases} H_{2\varepsilon^{(k)}}+1 & , \quad k=l \\ H_{\varepsilon^{(k)}+\varepsilon^{(l)}} & , \quad k\neq l \end{cases} = \begin{cases} H^{\Diamond 2}_{\varepsilon^{(k)}}+1 & , \quad k=l \\ H_{\varepsilon^{(k)}}\Diamond H_{\varepsilon^{(l)}} & , \quad k\neq l \end{cases}.$$

In Section 4 we will use the Malliavin derivative operator to express the difference between the ordinary product and the Wick product of a generalized stochastic process from $X\otimes(S)_{-1}$ and singular white noise W_t (Theorem 4.6). Here we state some general cases when the ordinary product is well defined.

Theorem 2.16. *The following holds:*

1. *If $F,G\in X\otimes(S)_1$ then the product $F\cdot G$ is a well defined element in $X\otimes(S)_1$. Moreover, for every $m\in\mathbb{N}_0$ there exist $r,s\in\mathbb{N}_0$ and $C(m)>0$ such that*

$$\|F\cdot G\|_{X\otimes(S)_{1,m}}\leq C(m)\|F\|_{X\otimes(S)_{1,r}}\|G\|_{X\otimes(S)_{1,s}}$$

   holds.

2. *If $F\in X\otimes(S)_1$ and $G\in X\otimes(S)_{-1}$ then their product $F\cdot G$ is well defined and belongs to $X\otimes(S)_{-1}$.*

The proof is similar to the one for multiplication of Schwartz test functions and multiplication of tempered distributions with test functions.

Note, for $F,G\in X\otimes(L)^2$ the ordinary product $F\cdot G$ not necessarily belongs to $X\otimes(L)^2$.

2.5. Operators of the Malliavin calculus

In [2, 7, 26, 27, 36, 43] the Malliavin derivative and the Skorokhod integral are defined on a subspace of $(L)^2$ so that the resulting process after application of these operators necessarily remains in $(L)^2$. We will recall of these classical results and denote the corresponding domains with a "zero" in order to retain a nice symmetry between test and generalized processes. In [19, 20, 23, 24] we allowed values in $(S)_{-1}$ and thus obtained larger domains for all operators. These domains will be denoted by a "minus" sign to reflect the fact that they correspond to generalized processes. In [25] we introduced also domains for test processes. These domains will be denoted by a "plus" sign.

Definition 2.17. Let a generalized stochastic process $u\in X\otimes(S)_{-1}$ be of the form $u=\sum_{\alpha\in\mathcal{I}}u_\alpha\otimes H_\alpha$. If there exists $p\in\mathbb{N}$ such that

$$\sum_{\alpha\in\mathcal{I}}|\alpha|^2\,\|u_\alpha\|_X^2(2\mathbb{N})^{-p\alpha}<\infty, \tag{2.5}$$

then the Malliavin derivative of u is defined by

$$\mathbb{D}u = \sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} \alpha_k \, u_\alpha \otimes \xi_k \otimes H_{\alpha-\varepsilon^{(k)}}, \tag{2.6}$$

where by convention $\alpha - \varepsilon^{(k)}$ does not exist if $\alpha_k = 0$, i.e.

$$H_{\alpha-\varepsilon^{(k)}} = \begin{cases} 0, & \alpha_k = 0 \\ H_{(\alpha_1,\alpha_2,\dots,\alpha_{k-1},\alpha_k-1,\alpha_{k+1},\dots,\alpha_m,0,0,\dots)}, & \alpha_k \geq 1 \end{cases}.$$

for $\alpha = (\alpha_1, \alpha_2, ..., \alpha_{k-1}, \alpha_k, \alpha_{k+1}, ..., \alpha_m, 0, 0, ...) \in \mathcal{I}$.

The set of generalized stochastic processes $u \in X \otimes (S)_{-1}$ which satisfy (2.5) constitutes the domain of the Malliavin derivative, denoted by $Dom_-(\mathbb{D})$. Thus the domain of the Malliavin derivative is given by

$$\begin{aligned} Dom_-(\mathbb{D}) &= \bigcup_{p\in\mathbb{N}} Dom_{-p}(\mathbb{D}) \\ &= \bigcup_{p\in\mathbb{N}} \left\{ u \in X \otimes (S)_{-1} : \sum_{\alpha\in\mathcal{I}} |\alpha|^2 \, \|u_\alpha\|_X^2 (2\mathbb{N})^{-p\alpha} < \infty \right\}. \end{aligned}$$

A process $u \in Dom_-(\mathbb{D}) \subset X \otimes (S)_{-1}$ is called a Malliavin differentiable process. Note that (2.6) can also be expressed in the form

$$\mathbb{D}u = \sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} (\alpha_k + 1) \, u_{\alpha+\varepsilon^{(k)}} \otimes \xi_k \otimes H_\alpha. \tag{2.7}$$

For stochastic test processes from $X \otimes (S)_1$ the Malliavin derivative is always defined i.e.

$$Dom_p(\mathbb{D}) = \{u \in X \otimes (S)_1 : \sum_{\alpha\in\mathcal{I}} \alpha!^2 \, \|u_\alpha\|_X^2 (2\mathbb{N})^{p\alpha} < \infty\} = X \otimes (S)_{1,p}.$$

In order to retain symmetry in notation, we denote

$$Dom_+(\mathbb{D}) = \bigcap_{p\in\mathbb{N}} Dom_p(\mathbb{D}) = \bigcap_{p\in\mathbb{N}} (X \otimes (S)_{1,p}) = X \otimes (S)_1.$$

In the classical literature it is usual to define the Malliavin derivative only for the $(L)^2$ case:

Definition 2.18. Let a square integrable stochastic process $u \in X \otimes (L)^2$ be of the form $u = \sum_{\alpha\in\mathcal{I}} u_\alpha \otimes H_\alpha$. If the condition

$$\sum_{\alpha\in\mathcal{I}} |\alpha| \, \alpha! \, \|u_\alpha\|_X^2 < \infty \tag{2.8}$$

holds, then u is a Malliavin differentiable process and the Malliavin derivative of u is defined by (2.6). All processes u satisfying condition (2.8) belong to the domain of $\mathbb{D}$ denoted by $Dom_0(\mathbb{D})$, i.e. the domain is given by

$$Dom_0(\mathbb{D}) = \left\{ u \in X \otimes (L)^2 : \sum_{\alpha\in\mathcal{I}} |\alpha|\, \alpha!\, \|u_\alpha\|_X^2 \, < \infty \right\}.$$

Theorem 2.19. ([19, 25])

a) The Malliavin derivative of a generalized process $u \in X \otimes (S)_{-1}$ is a linear and continuous mapping

$$\mathbb{D}: \quad Dom_{-p}(\mathbb{D}) \to X \otimes S_{-l}(\mathbb{R}) \otimes (S)_{-1,-p},$$

for $l > p+1$ and $p \in \mathbb{N}$.

b) The Malliavin derivative of a test stochastic process $v \in X \otimes (S)_1$ is a linear and continuous mapping

$$\mathbb{D}: \quad Dom_p(\mathbb{D}) \to X \otimes S_l(\mathbb{R}) \otimes (S)_{1,p},$$

for $l < p-1$ and $p \in \mathbb{N}$.

c) The Malliavin derivative of a square integrable process $u \in Dom_0(\mathbb{D})$ is a linear and continuous mapping

$$\mathbb{D}: \ Dom_0(\mathbb{D}) \ \to \ X \otimes L^2(\mathbb{R}) \otimes (L)^2.$$

Proof. *a*) Let u be as in Definition 2.17. Then,

$$\begin{aligned}
\|\mathbb{D}u\|^2_{X\otimes S_{-l}(\mathbb{R})\otimes(S)_{-1,-p}} &= \sum_{\alpha\in\mathcal{I}} \|\sum_{k=1}^{\infty} \alpha_k u_\alpha \otimes \xi_k\|^2_{X\otimes S_{-l}} (2\mathbb{N})^{-p(\alpha-\varepsilon^{(k)})} \\
&= \sum_{\alpha\in\mathcal{I}} \left(\sum_{k\in\mathbb{N}} \alpha_k^2 \|u_\alpha\|_X^2 (2k)^{-l} \right) (2k)^p (2\mathbb{N})^{-p\alpha} \\
&\le \sum_{\alpha\in\mathcal{I}} |\alpha|^2 \|u_\alpha\|_X^2 (2\mathbb{N})^{-p\alpha} \sum_{k\in\mathbb{N}} (2k)^{-l+p} \\
&\le C \sum_{\alpha\in\mathcal{I}} |\alpha|^2 \|u_\alpha\|_X^2 (2\mathbb{N})^{-p\alpha} < \infty,
\end{aligned}$$

where $\sum_{k\in\mathbb{N}} (2k)^{-l+p} = C < \infty$ for $l > p+1$. We also used the generalized Minkowski inequality to obtain that

$$\sum_{k\in\mathbb{N}} \alpha_k^2 (2k)^{p-l} \le (\sum_{k\in\mathbb{N}} \alpha_k^4)^{\frac{1}{2}} \cdot \sum_{k\in\mathbb{N}} (2k)^{p-l}$$

and the fact that $(\sum\limits_{k\in\mathbb{N}} \alpha_k^4)^{\frac{1}{2}} \le \sum\limits_{k\in\mathbb{N}} \alpha_k^2 \le (\sum\limits_{k\in\mathbb{N}} \alpha_k)^2 = |\alpha|^2$.

b) Let $v = \sum\limits_{\alpha\in\mathcal{I}} v_\alpha \otimes H_\alpha \in X \otimes (S)_{1,p}$ for all $p \geq 0$, i.e. let the condition $\sum\limits_{\alpha\in\mathcal{I}} \|v_\alpha\|_X^2 \, \alpha!^{\,2} \, (2\mathbb{N})^{p\alpha} < \infty$ hold. Then, from (2.6) and

$$\begin{aligned}
\|\mathbb{D}v\|^2_{X\otimes S_l(\mathbb{R})\otimes(S)_{1,p}} &= \sum_{\alpha\in\mathcal{I}} \|\sum_{k\in\mathbb{N}} \alpha_k \, v_\alpha \otimes \xi_k\|^2_{X\otimes S_l(\mathbb{R})} \, (\alpha-\varepsilon^{(k)})!^2 \, (2\mathbb{N})^{p(\alpha-\varepsilon^{(k)})} \\
&= \sum_{\alpha\in\mathcal{I}}\sum_{k\in\mathbb{N}} \alpha_k^2 \, (\alpha-\varepsilon^{(k)})!^{\,2} \, \|v_\alpha\|_X^2 \, (2k)^l \, (2\mathbb{N})^{p(\alpha-\varepsilon^{(k)})} \\
&= \sum_{\alpha\in\mathcal{I}}\sum_{k\in\mathbb{N}} \alpha!^{\,2} \, \|v_\alpha\|_X^2 \, (2k)^{l-p} \, (2\mathbb{N})^{p\alpha} \\
&\leq C \sum_{\alpha\in\mathcal{I}} \alpha!^{\,2} \, \|v_\alpha\|_X^2 \, (2\mathbb{N})^{p\alpha} \; = C \, \|v\|^2_{X\otimes(S)_{1,\,p}} < \infty,
\end{aligned}$$

the assertion follows, where $C = \sum\limits_{k\in\mathbb{N}} (2k)^{l-p} < \infty$ for $p > l+1$. We also used $\alpha_k \, (\alpha-\varepsilon^{(k)})! \, = \alpha!$, $k\in\mathbb{N}$, $\alpha\in\mathcal{I}$ and $(2\mathbb{N})^{\varepsilon^{(k)}} = (2k)$, $k \in \mathbb{N}$.

c) Let $u \in Dom_0(\mathbb{D})$, i.e. $\sum\limits_{\alpha\in\mathcal{I}} |\alpha|\alpha!\|u_\alpha\|_X^2 < \infty$. Then,

$$\begin{aligned}
\|\mathbb{D}u\|^2_{X\otimes L^2(\mathbb{R})\otimes(L)^2} &= \sum_{\alpha\in\mathcal{I}}\sum_{k\in\mathbb{N}} \alpha_k^2 \, (\alpha-\varepsilon^{(k)})! \; \|w_\alpha\|_X^2 \\
&= \sum_{\alpha\in\mathcal{I}}\sum_{k\in\mathbb{N}} \alpha_k \, \alpha! \; \|w_\alpha\|_X^2 \; = \sum_{\alpha\in\mathcal{I}} |\alpha| \, \alpha! \; \|w_\alpha\|_X^2 \; < \infty.
\end{aligned}$$

□

Note that $Dom_p(\mathbb{D}) \subseteq Dom_0(\mathbb{D}) \subseteq Dom_{-p}(\mathbb{D})$ for all $p\in\mathbb{N}$. Therefore

$$Dom_+(\mathbb{D}) \;\subseteq\; Dom_0(\mathbb{D}) \;\subseteq\; Dom_-(\mathbb{D}).$$

Moreover, using the estimate $|\alpha| \leq (2\mathbb{N})^\alpha$ it follows that

$$\sum_{\alpha\in\mathcal{I}} \|u_\alpha\|_X^2 (2\mathbb{N})^{-p\alpha} \leq \sum_{\alpha\in\mathcal{I}} |\alpha|^2 \|u_\alpha\|_X^2 (2\mathbb{N})^{-p\alpha} \leq \sum_{\alpha\in\mathcal{I}} \|u_\alpha\|_X^2 (2\mathbb{N})^{-(p-2)\alpha}, \text{ i.e.,}$$

$$X\otimes(S)_{-1,-(p-2)} \subseteq Dom_{-p}(\mathbb{D}) \subseteq X\otimes(S)_{-1,-p}, \quad p>3.$$

Remark 2.20. For $u \in Dom_+(\mathbb{D})$ and $u \in Dom_0(\mathbb{D})$ it is usual to write

$$\mathbb{D}_t u = \sum_{\alpha\in\mathcal{I}}\sum_{k\in\mathbb{N}} \alpha_k \, u_\alpha \; \otimes \; \xi_k(t) \; \otimes H_{\alpha-\varepsilon^{(k)}},$$

in order to emphasise that the Malliavin derivative takes a random variable into a process i.e. that $\mathbb{D}u$ is a function of t. Moreover, the formula

$$\mathbb{D}_t F(\omega) = \lim_{h\to 0} \frac{1}{h} \left(F(\omega + h\cdot\kappa_{[t,\infty)}) - F(\omega) \right), \quad \omega \in S'(\mathbb{R}),$$

justifies the name *stochastic derivative* for the Malliavin operator. Since generalized functions do not have point values, this notation would be somewhat misleading for $u \in Dom_-(\mathbb{D})$. Therefore, for notational uniformity, we omit the index t in $\mathbb{D}_t$ that usually appears in the literature and write $\mathbb{D}$.

The Skorokhod integral, as an extension of the Itô integral for non-adapted processes, can be regarded as the adjoint operator of the Malliavin derivative in $(L)^2$-sense. In [19] we have extended the definition of the Skorokhod integral from Hilbert space valued processes to the class of S'-valued generalized processes.

Definition 2.21. Let $F = \sum_{\alpha\in\mathcal{I}} f_\alpha \otimes H_\alpha \in X \otimes S'(\mathbb{R}) \otimes (S)_{-1}$, be a generalized $S'(\mathbb{R})$-valued stochastic process and let $f_\alpha \in X \otimes S'(\mathbb{R})$ be given by the expansion $f_\alpha = \sum_{k\in\mathbb{N}} f_{\alpha,k} \otimes \xi_k$, $f_{\alpha,k} \in X$. Then the process F is integrable in the Skorokhod sense and the chaos expansion of its stochastic integral is given by

$$\delta(F) = \sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} f_{\alpha,k} \otimes H_{\alpha+\varepsilon^{(k)}}. \tag{2.9}$$

In [21] we proved that the domain $Dom_-(\delta)$ of the Skorokhod integral is

$$\begin{aligned} Dom_-(\delta) &= X \otimes S'(\mathbb{R}) \otimes (S)_{-1} \\ &= \bigcup_{(l,p)\in\mathbb{N}^2} Dom_{(-l,-p)}(\delta) = \bigcup_{(l,p)\in\mathbb{N}^2} (X \otimes S_{-l}(\mathbb{R}) \otimes (S)_{-1,-p}). \end{aligned}$$

In [25] we characterized the domains $Dom_+(\delta)$ and $Dom_0(\delta)$ of the Skorokhod integral for test processes from $X \otimes S(\mathbb{R}) \otimes (S)_1$ and square integrable processes from $X \otimes L^2(\mathbb{R}) \otimes (L)^2$. The form of the derivative is in all cases given by the expression (2.9).

The domain $Dom_+(\delta)$ of the Skorokhod integral is

$$Dom_+(\delta) = \bigcap_{(l,p)\in\mathbb{N}^2} Dom_{(l,p)}(\delta),$$

$$Dom_{(l,p)}(\delta) = \left\{ F \in X \otimes S_l(\mathbb{R}) \otimes (S)_{1,p} : \sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} (\alpha_k+1)^2 \alpha!^2 \|f_{\alpha,k}\|_X^2 (2k)^l (2\mathbb{N})^{p\alpha} < \infty \right\}.$$

For square integrable stochastic processes $T \in X \otimes L^2(\mathbb{R}) \otimes (L)^2$ of the form $T = \sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} t_{\alpha,k} \otimes \xi_k \otimes H_\alpha$, $t_{\alpha,k} \in X$, the classical definition is:

$$Dom_0(\delta) = \left\{ T \in X \otimes L^2(\mathbb{R}) \otimes (L)^2 \; : \; \sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} (\alpha_k+1)\alpha! \|t_{\alpha,k}\|_X^2 < \infty \right\}.$$

Theorem 2.22. ([19, 25])

a) The Skorokhod integral δ of an $S_{-l}(\mathbb{R})$-valued generalized stochastic process is a linear and continuous mapping

$$\delta : X \otimes S_{-l}(\mathbb{R}) \otimes (S)_{-1,-p} \to X \otimes (S)_{-1,-q}, \quad q \geq p, \; q > l+1, \; l \in \mathbb{N}.$$

b) The Skorokhod integral δ of a $S_l(\mathbb{R})$-valued stochastic test process is a linear and continuous mapping

$$\delta : Dom_{(l,p)}(\delta) \to X \otimes (S)_{1,\,q}, \qquad q \leq \min\{l,p\},$$

for all $l, p \in \mathbb{N}$.

c) The Skorokhod integral δ of an $L^2(\mathbb{R})$-valued stochastic process is a linear and continuous mapping

$$\delta : \quad Dom_0(\delta) \ \to \ X \otimes (L)^2.$$

Proof. a) Let F be as in Definition 2.21. Clearly,

$$\begin{aligned}
\|\delta(F)\|^2_{X\otimes(S)_{-1,-q}} &= \sum_{\alpha\in\mathcal{I}} \| \sum_{k\in\mathbb{N}} f_{\alpha,k}\|^2_X (2\mathbb{N})^{-q(\alpha+\varepsilon^{(k)})} \\
&= \sum_{\alpha\in\mathcal{I}} \| \sum_{k\in\mathbb{N}} f_{\alpha,k}(2k)^{-\frac{q}{2}}\|^2_X (2\mathbb{N})^{-q\alpha} \\
&\leq \sum_{\alpha\in\mathcal{I}} \left(\sum_{k\in\mathbb{N}} |f_{\alpha,k}|(2k)^{-\frac{l}{2}}(2k)^{-\frac{(q-l)}{2}} \right)^2 (2\mathbb{N})^{-q\alpha} \\
&\leq \sum_{\alpha\in\mathcal{I}} \left(\sum_{k\in\mathbb{N}} |f_{\alpha,k}|^2(2k)^{-l} \sum_{k\in\mathbb{N}} (2k)^{-(q-l)} \right) (2\mathbb{N})^{-q\alpha} \\
&\leq \sum_{\alpha\in\mathcal{I}} \|f_\alpha\|^2_{-l}(2\mathbb{N})^{-p\alpha} \cdot \sum_{k\in\mathbb{N}} (2k)^{-(q-l)} \\
&\leq M \, \|F\|^2_{X\otimes S_{-l}(\mathbb{R})\otimes(S)_{-1,-p}} < \infty,
\end{aligned}$$

for $q \geq p$, where we used the Cauchy-Schwarz inequality and the fact that $M = \sum_{k\in\mathbb{N}}(2k)^{-(q-l)} < \infty$, for $q > l+1$.

b) Let $U = \sum_{\alpha\in\mathcal{I}} u_\alpha \otimes H_\alpha \in X \otimes S_l(\mathbb{R}) \otimes (S)_{1,p}$, $u_\alpha = \sum_{k=1}^\infty u_{\alpha,k} \otimes \xi_k \in X \otimes S_l(\mathbb{R})$, $u_{\alpha,k} \in X$, for $p, l \geq 1$. Then we obtain

$$\begin{aligned}
\|\delta(U)\|^2_{X\otimes(S)_{1,\,q}} &= \sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} \|u_{\alpha,k}\|^2_X \, (\alpha+\varepsilon^{(k)})!^2 \, \| \, (2\mathbb{N})^{q(\alpha+\varepsilon^{(k)})} \\
&= \sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} \|u_{\alpha,k}\|^2_X \, \alpha!^{\,2} \, (\alpha_k+1)^2 \, (2k)^q \, (2\mathbb{N})^{q\alpha} \\
&\leq \|U\|^2_{Dom_{(l,p)}(\delta)} < \infty,
\end{aligned}$$

for $q \leq p$, $q \leq l$.

c) Let $T = \sum\limits_{\alpha\in\mathcal{I}} \sum\limits_{k\in\mathbb{N}} t_{\alpha,k} \, \otimes \xi_k \otimes H_\alpha \in Dom_0(\delta)$. Then,

$$\begin{aligned}
\|\delta(T)\|^2_{X\otimes(L)^2} &= \sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} \|t^2_{\alpha,k}\|^2_X \, (\alpha+\varepsilon^{(k)})! \\
&= \sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} \|t_{\alpha,k}\|^2_X (\alpha_k+1) \, \alpha! < \infty,
\end{aligned}$$

where we used $(\alpha+\varepsilon^{(k)})! = (\alpha_k+1)\,\alpha!$, for $\alpha \in \mathcal{I}$, $k \in \mathbb{N}$. □

Using the estimate $\alpha_k+1 \le 2|\alpha|$, which holds for all $\alpha \in \mathcal{I}$ except for $\alpha = \mathbf{0}$, we obtain

$$\begin{aligned}\sum_{\alpha\in\mathcal{I}}\sum_{k\in\mathbb{N}} \alpha!^2 \ \|f_{\alpha,k}\|_X^2(2k)^l(2\mathbb{N})^{p\alpha} &\le \sum_{\alpha\in\mathcal{I}}\sum_{k\in\mathbb{N}}(\alpha_k+1)^2\alpha!^2\|f_{\alpha,k}\|_X^2(2k)^l(2\mathbb{N})^{p\alpha} \\ &\le \sum_{k\in\mathbb{N}}\|f_{0,k}\|_X^2(2k)^l + 4\sum_{\alpha>\mathbf{0}}\sum_{k\in\mathbb{N}}|\alpha|^2\alpha!^2\|f_{\alpha,k}\|_X^2(2k)^l(2\mathbb{N})^{p\alpha} \\ &\le \|f_{\mathbf{0}}\|^2_{X\otimes S_l(\mathbb{R})} + 4\sum_{\alpha>\mathbf{0}}\sum_{k\in\mathbb{N}}\alpha!^2\|f_{\alpha,k}\|_X^2(2k)^l(2\mathbb{N})^{(p+2)\alpha} \\ &\le 4\|F\|^2_{X\otimes S_l(\mathbb{R})\otimes(S)_{1,p+2}}.\end{aligned}$$

Thus,

$$X\otimes S_l(\mathbb{R})\otimes(S)_{1,p+2} \subseteq Dom_{(l,p)}(\delta) \subseteq X\otimes S_l(\mathbb{R})\otimes(S)_{1,p}, \quad p\in\mathbb{N}.$$

The third main operator of the Malliavin calculus is the Ornstein-Uhlenbeck operator. We describe the domain of the Ornstain-Uhlenbeck operator for different classes of generalized stochastic processes.

Definition 2.23. The composition of the Malliavin derivative and the Skorokhod integral is denoted by $\mathcal{R} = \delta\circ\mathbb{D}$ and called the *Ornstein-Uhlenbeck operator.*

Therefore, for a generalized process $u \in X\otimes(S)_{-1}$ given in the chaos expansion form $u = \sum_{\alpha\in\mathcal{I}} u_\alpha\otimes H_\alpha$, the Ornstein-Uhlenbeck operator is given by

$$\mathcal{R}(u) = \sum_{\alpha\in\mathcal{I}}|\alpha|u_\alpha\otimes H_\alpha. \tag{2.10}$$

Let

$$\begin{aligned}Dom_-(\mathcal{R}) &= \bigcup_{p\in\mathbb{N}} Dom_{-p}(\mathcal{R}) \\ &= \bigcup_{p\in\mathbb{N}}\left\{u\in X\otimes(S)_{-1} : \sum_{\alpha\in\mathcal{I}}|\alpha|^2\ \|u_\alpha\|_X^2(2\mathbb{N})^{-p\alpha}<\infty\right\}.\end{aligned}$$

For test processes, we define

$$\begin{aligned}Dom_+(\mathcal{R}) &= \bigcap_{p\in\mathbb{N}} Dom_p(\mathcal{R}) \\ &= \bigcap_{p\in\mathbb{N}}\left\{v\in X\otimes(S)_1 : \sum_{\alpha\in\mathcal{I}}\alpha!^{\,2}\ |\alpha|^2\ \|v_\alpha\|_X^2\ (2\mathbb{N})^{p\alpha}\ <\infty\right\}.\end{aligned}$$

For square integrable processes the classical definition is:

$$Dom_0(\mathcal{R}) = \left\{w\in X\otimes(L)^2 :\ \sum_{\alpha\in\mathcal{I}}\alpha!\ |\alpha|^2\ \|w_\alpha\|_X^2\ <\infty\right\}.$$

Theorem 2.24. ([23], [25])

a) The operator $\mathcal{R}$ *is a linear and continuous mapping*

$$\mathcal{R} : Dom_{-p}(\mathcal{R}) \to X \otimes (S)_{-1,-p},\ p \in \mathbb{N}.$$

In this case the domains of $\mathbb{D}$ *and* $\mathcal{R}$ *coincide, i.e.* $Dom_{-}(\mathcal{R}) = Dom_{-}(\mathbb{D})$.

b) The operator $\mathcal{R}$ *is a linear and continuous mapping*

$$\mathcal{R} : \quad Dom_{p}(\mathcal{R}) \to X \otimes (S)_{1,\,p}, \qquad p \in \mathbb{N}.$$

In this case the domains of the operators $\mathbb{D}$ *and* $\mathcal{R}$ *do not coincide, i.e.* $Dom_{+}(\mathbb{D}) \supsetneq Dom_{+}(\mathcal{R})$.

c) The operator $\mathcal{R}$ *is a linear and continuous operator*

$$\mathcal{R} : \quad Dom_0(\mathcal{R}) \ \to\ X \otimes (L)^2.$$

In this case the domains of the operators $\mathbb{D}$ *and* $\mathcal{R}$ *also do not coincide and* $Dom_0(\mathbb{D}) \supsetneq Dom_0(\mathcal{R})$.

Proof. a) Let $u = \sum_{\alpha\in\mathcal{I}} u_\alpha \otimes H_\alpha \in X \otimes (S)_{-1,\,-p}$, for some $p \in \mathbb{N}$. Clearly,

$$\|\mathcal{R}u\|^2_{X\otimes(S)_{-1,\,-p}} = \sum_{\alpha\in\mathcal{I}} \|u_\alpha\|^2_X\, |\alpha|^2\, (2\mathbb{N})^{-p\alpha} = \|u\|^2_{Dom_{-p}(\mathcal{R})} < \infty.$$

b) Let a stochastic process $v = \sum_{\alpha\in\mathcal{I}} v_\alpha \otimes H_\alpha \in X \otimes (S)_{1,\,p}$, for all $p \in \mathbb{N}$, i.e. $\sum_{\alpha\in\mathcal{I}} \|v_\alpha\|^2_X\, \alpha!^{\,2}\, (2\mathbb{N})^{p\alpha} < \infty$, for all $p \in \mathbb{N}$. Then,

$$\|\mathcal{R}v\|^2_{X\otimes(S)_{1,\,p}} = \sum_{\alpha\in\mathcal{I}} \|v_\alpha\|^2_X\, |\alpha|^2\, \alpha!^2\, (2\mathbb{N})^{p\alpha} = \|v\|^2_{Dom_p(\mathcal{R})} < \infty,$$

and the statement follows.

c) Let $w = \sum_{\alpha\in\mathcal{I}} w_\alpha \otimes H_\alpha \in Dom_0(\mathcal{R})$. Then $\mathcal{R}(w) = \sum_{\alpha\in\mathcal{I}} |\alpha|\, w_\alpha \otimes H_\alpha$ and

$$\|\mathcal{R}(w)\|^2_{X\otimes(L)^2} = \sum_{\alpha\in\mathcal{I}} |\alpha|^2\, \|w_\alpha\|^2_X \ < \infty,$$

by the assumption $w \in Dom_0(\mathcal{R})$. □

Note also that

$$\sum_{\alpha\in\mathcal{I}} \alpha!^2 \|u_\alpha\|^2_X (2\mathbb{N})^{p\alpha} \le \sum_{\alpha\in\mathcal{I}} \alpha!^2 |\alpha|^2 \|u_\alpha\|^2_X (2\mathbb{N})^{p\alpha} \le \sum_{\alpha\in\mathcal{I}} \alpha!^2 \|u_\alpha\|^2_X (2\mathbb{N})^{(p+2)\alpha},$$

$$\text{i.e., } X \otimes (S)_{1,p+2} \subseteq Dom_p(\mathcal{R}) \subseteq X \otimes (S)_{1,p}, \quad p \in \mathbb{N}.$$

Remark 2.25. Note that $\mathbb{D}:\mathcal{H}_k\to\mathcal{H}_{k-1}$ reduces the Wiener chaos space order and therefore Malliavin differentiation corresponds to the annihilation operator, while $\delta:\mathcal{H}_k\to\mathcal{H}_{k+1}$ increases the chaos order and thus Skorokhod integration corresponds to the creation operator. Clearly, $\mathcal{R}:\mathcal{H}_k\to\mathcal{H}_k$ and the Ornstein-Uhlenbeck operator corresponds to the number operator in quantum theory.

In the following sections we prove that the mappings $\mathbb{D}:\ Dom_{\pm}(\mathbb{D})\to X\otimes S'(\mathbb{R})\otimes(S)_{\pm1}$, $\delta:\ Dom_{\pm}(\delta)\to X\otimes(S)_{\pm1}$, $\mathcal{R}:\ Dom_{\pm}(\mathcal{R})\to X\otimes(S)_{\pm1}$, given in this section are surjective.

3. The Ornstein-Uhlenbeck operator

Theorem 3.1. ([21, 25]) *Let g have zero generalized expectation. The equation*

$$\mathcal{R}u=g,\qquad Eu=\tilde{u}_0\in X,$$

has a unique solution u represented in the form

$$u=\tilde{u}_0+\sum_{\alpha\in\mathcal{I},|\alpha|>0}\frac{g_\alpha}{|\alpha|}\otimes H_\alpha.$$

Moreover, the following holds:

1. *If $g\in X\otimes(S)_{-1,-p}$, $p\in\mathbb{N}$, then $u\in Dom_{-p}(\mathcal{R})$.*
2. *If $g\in X\otimes(S)_{1,p}$, $p\in\mathbb{N}$, then $u\in Dom_{p}(\mathcal{R})$.*
3. *If $g\in X\otimes(L)^2$, then $u\in Dom_0(\mathcal{R})$.*

Proof. Let us seek for a solution in form of $u=\sum_{\alpha\in\mathcal{I}}u_\alpha\otimes H_\alpha$. From $\mathcal{R}u=g$ it follows that

$$\sum_{\alpha\in\mathcal{I}}|\alpha|u_\alpha\otimes H_\alpha=\sum_{\alpha\in\mathcal{I}}g_\alpha\otimes H_\alpha,$$

i.e., $u_\alpha=\frac{g_\alpha}{|\alpha|}$ for all $\alpha\in\mathcal{I}$, $|\alpha|>0$. From the initial condition we obtain $u_{(0,0,0,0,\dots)}=Eu=\tilde{u}_0$.

1. Assume that $g\in X\otimes(S)_{-1,-p}$. Then, $u\in Dom_{-p}(\mathcal{R})$ since

$$\begin{aligned}\|u\|^2_{Dom_{-p}(\mathcal{R})}&=\sum_{|\alpha|>0}|\alpha|^2\|u_\alpha\|^2_X(2\mathbb{N})^{-p\alpha}=\sum_{|\alpha|>0}\|g_\alpha\|^2_X(2\mathbb{N})^{-p\alpha}\\&=\|g\|^2_{X\otimes(S)_{-1,-p}}<\infty.\end{aligned}$$

2. In this case $u\in Dom_p(\mathcal{R})$ since

$$\begin{aligned}\|u\|^2_{Dom_p(\mathcal{R})}&=\sum_{|\alpha|>0}\alpha!^2\,|\alpha|^2\,\|u_\alpha\|^2_X(2\mathbb{N})^{p\alpha}\\&=\sum_{|\alpha|>0}\alpha!^2\,\|g_\alpha\|^2_X\,(2\mathbb{N})^{p\alpha}=\|f\|^2_{X\otimes(S)_{1,p}}<\infty.\end{aligned}$$

3. If g is square integrable, then $u \in Dom_0(\mathcal{R})$ since

$$\|u\|^2_{Dom_0(\mathcal{R})} = \sum_{|\alpha|>0} |\alpha|^2 \, \alpha! \, \|u_\alpha\|^2_X = \sum_{|\alpha|>0} \alpha! \, \|h_\alpha\|^2_X = \|h\|^2_{X\otimes(L)^2} < \infty.$$

□

Remark 3.2. Note that $\mathcal{R}u = u$ if and only if $u \in \mathcal{H}_1$ i.e. Gaussian processes with zero expectation and first order chaos are the only fixed points for the Ornstein-Uhlenbeck operator. For example, $\mathcal{R}(B_t) = B_t$ and $\mathcal{R}(W_t) = W_t$.

Also, it is clear that $\mathcal{H}_m$ is the eigenspace corresponding to the eigenvalue m ($m \in \mathbb{N}$) of the Ornstein-Uhlenbeck operator.

Remark 3.3. If $Eu = 0$, one can define the pseudo-inverse $\mathcal{R}^{-1}$ as in [33, 36], given by

$$\mathcal{R}^{-1}u = \mathcal{R}^{-1}\left(\sum_{\alpha\in\mathcal{I},|\alpha|>0} u_\alpha \otimes H_\alpha\right) = \sum_{\alpha\in\mathcal{I},|\alpha|>0} \frac{u_\alpha}{|\alpha|} \otimes H_\alpha.$$

Thus,

$$\mathcal{R}\mathcal{R}^{-1}(u) = u \quad \text{and} \quad \mathcal{R}^{-1}\mathcal{R}(u) = u. \tag{3.1}$$

In general case, for $Eu \neq 0$, we have

$$\mathcal{R}\mathcal{R}^{-1}(u - Eu) = u \quad \text{and} \quad \mathcal{R}^{-1}\mathcal{R}(u) = u.$$

Corollary 3.4. *Each process $g \in X \otimes (S)_{\pm 1}$, resp. $g \in X \otimes (L)^2$, can be represented as*

$$g = Eg + \mathcal{R}(u),$$

for some $u \in Dom_{\pm}(\mathcal{R})$, resp. $u \in Dom_0(\mathcal{R})$.

Proof. The assertion follows for $u = \mathcal{R}^{-1}(g - Eg)$. □

Remark 3.5. We note that if a stochastic process f belongs to the Wiener chaos space $\bigoplus_{i=0}^m \mathcal{H}_i$ for some $m \in \mathbb{N}$, then the solution u of the equation $\mathcal{R}u = f$ belongs also to the Wiener chaos space $\bigoplus_{i=0}^m \mathcal{H}_i$.

4. The Malliavin derivative

Theorem 4.1. ([21, 25]) *Let a process h have a chaos expansion representation $h = \sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} h_{\alpha,k} \otimes \xi_k \otimes H_\alpha$. Then the equation*

$$\begin{cases} \mathbb{D}u = h, \\ Eu = \widetilde{u}_0, \qquad \widetilde{u}_0 \in X, \end{cases} \tag{4.1}$$

has a unique solution u represented in the form

$$u = \widetilde{u}_0 + \sum_{\alpha\in\mathcal{I},|\alpha|>0} \frac{1}{|\alpha|} \sum_{k\in\mathbb{N}} h_{\alpha-\varepsilon^{(k)},k} \otimes H_\alpha. \tag{4.2}$$

Moreover, the following holds:

1. *If* $h \in X \otimes S_{-p}(\mathbb{R}) \otimes (S)_{-1,-q}$, $p, q \in \mathbb{N}$, *then* $u \in Dom_{-q}(\mathbb{D})$.
2. *If* $h \in X \otimes L^2(\mathbb{R}) \otimes (L)^2$, *then* $u \in Dom_0(\mathbb{D})$.
3. *If* $h \in X \otimes S_p(\mathbb{R}) \otimes (S)_{1,q}$, $p, q \in \mathbb{N}$, *then* $u \in Dom_q(\mathbb{D})$.

Proof. 1. Applying the Skorokhod integral on both sides of (4.1) one obtains

$$\mathcal{R}u = \delta(h),$$

for a given $h \in X \otimes S'(\mathbb{R}) \otimes (S)_{-1} = Dom_-(\delta)$. From the initial condition it follows that the solution u is given in the form $u = \widetilde{u}_0 + \sum_{\alpha \in \mathcal{I}, |\alpha|>0} u_\alpha \otimes H_\alpha$ and its coefficients are obtained from the system

$$|\alpha|\, u_\alpha = \sum_{k \in \mathbb{N}} h_{\alpha - \varepsilon^{(k)},k}, \qquad |\alpha| > 0, \tag{4.3}$$

where by convention $\alpha - \varepsilon^{(k)}$ does not exist if $\alpha_k = 0$. Hence, the solution u is given in the form (4.2). Now, we prove that the solution u belongs to the space $Dom_{-q}(\mathbb{D})$. Clearly,

$$\begin{aligned}
\|u\|^2_{Dom_{-q}(\mathbb{D})} &= \sum_{\alpha \in \mathcal{I}} |\alpha|^2 \, \|u_\alpha\|^2_X \, (2\mathbb{N})^{-q\alpha} \\
&= \sum_{\alpha \in \mathcal{I}, |\alpha|>0} \| \sum_{k \in \mathbb{N}} h_{\alpha-\varepsilon^{(k)},k} \|^2_X (2\mathbb{N})^{-q\alpha} \\
&= \sum_{\alpha \in \mathcal{I}} \| \sum_{k \in \mathbb{N}} h_{\alpha,k} \|^2_X \, (2\mathbb{N})^{-q\alpha} \, (2\mathbb{N})^{-q\varepsilon^{(k)}} \\
&\le \sum_{\alpha \in \mathcal{I}} \left(\sum_{k \in \mathbb{N}} \|h_{\alpha,k}\|_X \, (2k)^{-\frac{p}{2}} \, (2k)^{-\frac{(q-p)}{2}} \right)^2 (2\mathbb{N})^{-q\alpha} \\
&\le \sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} \|h_{\alpha,k}\|^2_X \, (2k)^{-p} \, (2\mathbb{N})^{-q\alpha} \sum_{k \in \mathbb{N}} (2k)^{-(q-p)} \\
&= C \|h\|^2_{X \otimes S_{-p}(\mathbb{R}) \otimes (S)_{-1,-q}} < \infty,
\end{aligned}$$

since $C = \sum_{k \in \mathbb{N}} (2k)^{-(q-p)} < \infty$, for $q > p + 1$.

2. In this case we have that

$$\begin{aligned}
\|u\|^2_{Dom_0(\mathbb{D})} &= \sum_{\alpha \in \mathcal{I}} |\alpha| \, \alpha! \, \|u_\alpha\|^2_X = \sum_{\alpha \in \mathcal{I}, |\alpha|>0} \frac{\alpha!}{|\alpha|} \, \| \sum_{k \in \mathbb{N}} h_{\alpha-\varepsilon^{(k)},k} \|^2_X \\
&= \sum_{\alpha \in \mathcal{I}} \| \sum_{k \in \mathbb{N}} h_{\alpha,k} \|^2_X \, \frac{(\alpha + \varepsilon^{(k)})!}{|\alpha + \varepsilon^{(k)}|} \\
&\le \sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} \|h_{\alpha,k}\|^2_X \alpha! \\
&= \sum_{\alpha \in \mathcal{I}} \alpha! \|f_\alpha\|^2_{X \otimes L^2(\mathbb{R})} = \|f\|^2_{X \otimes L^2(\mathbb{R}) \otimes (L)^2} < \infty.
\end{aligned}$$

We have made use of the fact $\frac{(\alpha+\varepsilon^{(k)})!}{|\alpha+\varepsilon^{(k)}|} \leq \alpha!$.

3. Clearly, δ can again be applied onto h, since $h \in X \otimes S_p(\mathbb{R}) \otimes (S)_{1,q} \subseteq Dom_{(p,q-2)}(\delta)$. It remains to prove that the solution u given in the form (4.2) belongs to $Dom_q(\mathbb{D})$. Clearly,

$$\begin{aligned}
\|u\|^2_{Dom_q(\mathbb{D})} &= \sum_{\alpha\in\mathcal{I}} \alpha!^2\, \|u_\alpha\|^2_X\, (2\mathbb{N})^{q\alpha} \\
&= \sum_{\alpha\in\mathcal{I},|\alpha|>0} \frac{\alpha!^2}{|\alpha|^2}\ \|\sum_{k\in\mathbb{N}} h_{\alpha-\varepsilon^{(k)},k}\|^2_X\ (2\mathbb{N})^{q\alpha} \\
&= \sum_{\alpha\in\mathcal{I}} \|\sum_{k\in\mathbb{N}} h_{\alpha,k}\|^2_X\ \frac{(\alpha+\varepsilon^{(k)})!^2}{|\alpha+\varepsilon^{(k)}|^2}\ (2\mathbb{N})^{q\alpha}\, (2\mathbb{N})^{q\varepsilon^{(k)}} \\
&\leq \sum_{\alpha\in\mathcal{I}} \|\sum_{k\in\mathbb{N}} h_{\alpha,k}\|^2_X\ \alpha!^2\ (2\mathbb{N})^{q\alpha}\, (2k)^q \\
&= \sum_{\alpha\in\mathcal{I}} \|\sum_{k=1}^{\infty} h_{\alpha,k}(2k)^{\frac{q}{2}}\|^2_X \alpha!^2 (2\mathbb{N})^{q\alpha} \\
&= \sum_{\alpha\in\mathcal{I}} \|\sum_{k=1}^{\infty} h_{\alpha,k}(2k)^{\frac{p}{2}}(2k)^{\frac{q-p}{2}}\|^2_X \alpha!^2 (2\mathbb{N})^{q\alpha} \\
&\leq \sum_{\alpha\in\mathcal{I}}\sum_{k=1}^{\infty} \|h_{\alpha,k}\|^2_X (2k)^p \sum_{k=1}^{\infty}(2k)^{q-p}\ \alpha!^2 (2\mathbb{N})^{q\alpha} \\
&= C\cdot \|h\|^2_{X\otimes S_p(\mathbb{R})\otimes (S)_{1,q}} < \infty,
\end{aligned}$$

since $C = \sum_{k\in\mathbb{N}} (2k)^{q-p} < \infty$, for $p > q+1$. In the fourth step of the estimation we used again that $\frac{(\alpha+\varepsilon^{(k)})!}{|\alpha+\varepsilon^{(k)}|} \leq \alpha!$. □

Corollary 4.2. *If $\mathbb{D}(u) = 0$, then $u = Eu$ i.e. u is constant almost surely.*

In other words, the kernel of the operator $\mathbb{D}$ is $\mathcal{H}_0$.

Corollary 4.3. *For every $h \in X \otimes S'(\mathbb{R}) \otimes (S)_{\pm 1}$, resp. $h \in X \otimes L^2(\mathbb{R}) \otimes (L)^2$, there exists a unique $u \in Dom_{\pm}(\mathbb{D})$, resp. $u \in Dom_0(\mathbb{D})$, such that $Eu = 0$ and $h = \mathbb{D}(u)$ holds.*

Proof. The assertion follows for $u = \mathcal{R}^{-1}(\delta(h))$. □

Example 4.4. Let $t \geq 0$. Consider now the following examples which illustrate the results of Theorem 4.1.

1. Denote by $\kappa_{[0,t_0]} = \begin{cases} 1, & t \in [0,t_0] \\ 0, & t \notin [0,t_0] \end{cases}$ the characteristic function of the interval $[0,t_0]$. It is an element of $L^2(\mathbb{R})$ and thus its expansion representation is $\kappa_{[0,t_0]}(t) = \sum_{k=1}^{\infty} \left(\int_0^{t_0} \xi_k(t)dt\right) \xi_k(t)$. Consider the initial value

problem

$$\mathbb{D}u = \kappa_{[0,t_0]}(t), \quad Eu = \widetilde{u}_0. \tag{4.4}$$

Recall that $H_{(0,0,0,\dots)} = 1$ and then we may regard h in (4.1) as $h = \kappa_{[0,t_0]}(t) \in L^2(\mathbb{R}) \otimes \mathcal{H}_0$. Therefore, $u \in L^2(\mathbb{R}) \otimes (\mathcal{H}_0 \oplus \mathcal{H}_1)$. From (4.3) we obtain the form of the coefficients of the solution $u_{\varepsilon^{(k)}} = h_{0,k} = \int_0^{t_0} \xi_k(t)\, dt$. Then the solution of the equation (4.4) is of the form

$$u(t_0, \omega) = \widetilde{u}_0 + \sum_{k=1}^{\infty} \int_0^{t_0} \xi_k(t) dt\, H_{\varepsilon^{(k)}}(\omega) = \widetilde{u}_0 + B_{t_0}(\omega),$$

i.e. it is Brownian motion with drift parameter $\tilde{u}_0$.

2. Consider the equation

$$\mathbb{D}u = d_{t_0}(t), \quad Eu = \widetilde{u}_0, \tag{4.5}$$

where $d_{t_0}(t)$ denotes the Dirac delta function concentrated at t_0, represented in the chaos expansion form

$$d_{t_0}(t) = \sum_{k=1}^{\infty} \xi_k(t_0)\, \xi_k(t) = \sum_{k=1}^{\infty} \xi_k(t_0)\, \xi_k(t)\, H_0(\omega).$$

The solution to (4.5) belongs to the space $S'(\mathbb{R}) \otimes (\mathcal{H}_0 \oplus \mathcal{H}_1)$ because $d_{t_0}(t) \in S'(\mathbb{R}) \otimes \mathcal{H}_0$. The chaos expansion form of the solution is given by

$$u = \widetilde{u}_0 + \sum_{k=1}^{\infty} \xi_k(t_0)\, H_{\varepsilon^{(k)}}(\omega) = \widetilde{u}_0 + W_{t_0}(\omega),$$

i.e. it represents singular white noise.

3. Consider now an equation with singular white noise

$$\mathbb{D}u = W_t(\omega), \quad Eu = 0.$$

W_t belongs to the Wiener chaos space of order one and (since we assumed $Eu = 0$) the solution u will belong to the Wiener chaos space of order two. From $W_t = \sum_{k=1}^{\infty} \xi_k H_{\varepsilon^{(k)}}$ it follows that $h_{\alpha,k} = 1$ only for $\alpha = \varepsilon^{(k)}$ and $h_{\alpha,k} = 0$ for all $\alpha \neq \varepsilon^{(k)}$. Thus, $h_{\alpha-\varepsilon^{(k)},k} = h_{\varepsilon^{(k)},k} = 1$ only for $\alpha = 2\varepsilon^{(k)}$ and is equal to zero for all other $\alpha \in \mathcal{I}$. Thus, with $|\alpha| = 2$ we obtain u_α from (4.3), and the form of the solution is

$$u(\omega) = \frac{1}{2} \sum_{k=1}^{\infty} H_{2\varepsilon^{(k)}}(\omega).$$

4. Consider the equation

$$\mathbb{D}u = B_{t_0}(\omega)\kappa_{[0,t_0]}(t), \qquad Eu = 0.$$

The chaos expansion of the right hand side is

$$\begin{aligned} B_{t_0}(\omega)\kappa_{[0,t_0]}(t) &= \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\left(\int_0^{t_0}\left(\int_0^{t_0}\xi_k(s)ds\right)\xi_j(t)dt\right)\xi_j(t)H_{\varepsilon^{(k)}}(\omega) \\ &= \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\left(\int_0^{t_0}\xi_k(s)ds\right)\left(\int_0^{t_0}\xi_j(s)ds\right)\xi_j(t)H_{\varepsilon^{(k)}}(\omega). \end{aligned}$$

This implies $h_{\varepsilon^{(k)},j} = \frac{1}{2}\left(\int_0^{t_0}\xi_k(s)ds\right)\left(\int_0^{t_0}\xi_j(s)ds\right)$. Again, $h_{\alpha-\varepsilon^{(l)},l}$ is nonzero only for α of the form $\alpha = \varepsilon^{(l)} + \varepsilon^{(k)}$ and in this case we have with $|\alpha| = 2$ that

$$u_{\varepsilon^{(k)},l} = \frac{1}{2}h_{\varepsilon^{(k)},l} = \frac{1}{2}\left(\int_0^{t_0}\xi_k(s)ds\right)\left(\int_0^{t_0}\xi_l(s)ds\right).$$

Thus, the solution belongs to the space $L^2(\mathbb{R})\otimes\mathcal{H}_2$ and is of the form

$$u = \frac{1}{2}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\left(\int_0^{t_0}\xi_k(t)dt\right)\left(\int_0^{t_0}\xi_l(s)ds\right)H_{\varepsilon^{(k)}+\varepsilon^{(l)}}(\omega).$$

Note that the solution can be represented in terms of the Wick product

$$u = \frac{1}{2}B_{t_0}(\omega)^{\lozenge 2}.$$

5. Consider now the equation

$$\mathbb{D}(u) = B_{t_1}\kappa_{[0,t_2]}(t), \quad Eu = 0.$$

Similarly as in the previous case it can be shown by symmetry of t_1 and t_2 that it is equivalent to the equation

$$\mathbb{D}(u) = B_{t_2}\kappa_{[0,t_1]}(t), \quad Eu = 0,$$

and that both equations have the solution

$$u = \frac{1}{2}B_{t_1}\lozenge B_{t_2} = \frac{1}{2}(B_{t_1}B_{t_2} - \min\{t_1,t_2\}).$$

6. Similarly to the previous cases, $u = \frac{1}{2}W_{t_1}\lozenge W_{t_2}$ solves the equation

$$\mathbb{D}u = W_{t_1}(\omega)d_{t_2}(t) = W_{t_2}(\omega)\delta_{t_1}(t),$$

while $u = \frac{1}{2}\, W_{t_0}(\omega)^{\lozenge 2}$ is the solution to the equation

$$\mathbb{D}u = W_{t_0}(\omega)d_{t_0}(t).$$

Remark 4.5. If a stochastic process h belongs to the Wiener chaos space $\bigoplus_{i=0}^{m} \mathcal{H}_i$ for some $m \in \mathbb{N}$, then the unique solution u of the equation (4.1) belongs to the Wiener chaos space $\bigoplus_{i=0}^{m+1} \mathcal{H}_i$. Especially, if the input function h is a constant random variable i.e. an element of $\mathcal{H}_0$, then the solution u to (4.1) is a Gaussian process.

Theorem 4.6. ([21]) *Let $h \in X \otimes (S)_{-1}$ and W_t, B_t denote white noise and Brownian motion respectively. Then,*

$$h \cdot W_t - h\Diamond W_t = \mathbb{D}(h),$$

i.e. $\frac{d}{dt}(h \cdot B_t - h\Diamond B_t) = \mathbb{D}(h)$ *in weak* $S'(\mathbb{R})$*-sense.*

Proof. Let h be of the form $h = \sum_{\alpha\in\mathcal{I}} h_\alpha H_\alpha$ and $W_t = \sum_{n=1}^{\infty} \xi_n(t) H_{\varepsilon^{(n)}}$. Then,

$$h\Diamond W_t = \sum_{\gamma\in\mathcal{I}} \sum_{\alpha+\varepsilon^{(n)}=\gamma} h_\alpha \xi_n(t) H_\gamma = \sum_{\gamma\in\mathcal{I}} \sum_{n=1}^{\infty} h_{\gamma-\varepsilon^{(n)}} \xi_n(t) H_\gamma$$

and

$$h \cdot W_t = \sum_{\alpha\in\mathcal{I}} \sum_{n=1}^{\infty} h_{\alpha-\varepsilon^{(n)}} \xi_n(t) H_{\alpha-\varepsilon^{(n)}} H_{\varepsilon^{(n)}}.$$

Now applying the well-known formula (2.4) for Hermite polynomials one obtains

$$H_{\alpha-\varepsilon^{(n)}} \cdot H_{\varepsilon^{(n)}} = H_\alpha + (\alpha - \varepsilon^{(n)})_n H_{\alpha-2\varepsilon^{(n)}},$$

where we used $\binom{\alpha}{\varepsilon^{(k)}} = \alpha_k$, $k \in \mathbb{N}$. Hence,

$$h \cdot W_t = \sum_{\alpha\in\mathcal{I}} \sum_{n=1}^{\infty} h_{\alpha-\varepsilon^{(n)}} \xi_n(t) (H_\alpha + (\alpha_n - 1) H_{\alpha-2\varepsilon^{(n)}}),$$

which implies

$$\begin{aligned} h \cdot W_t - h\Diamond W_t &= \sum_{\alpha\in\mathcal{I}} \sum_{n=1}^{\infty} h_{\alpha-\varepsilon^{(n)}} \xi_n(t) (\alpha_n - 1) H_{\alpha-2\varepsilon^{(n)}} \\ &= \sum_{\alpha\in\mathcal{I}} \sum_{n=1}^{\infty} h_{\alpha+\varepsilon^{(n)}} \xi_n(t) (\alpha_n + 1) H_\alpha \\ &= \mathbb{D}(h), \end{aligned}$$

using (2.7). Thus the assertion follows. □

Remark 4.7. Note that if $h \in X \otimes (S)_{-1,-p}$, then $\mathbb{D}(h) \in X \otimes S_{-l}(\mathbb{R}) \otimes (S)_{-1,-(p+2)}$, $l > p + 1$. Thus, apart from the Wick product $h\Diamond W_t$ being well-defined, the ordinary product is also well-defined in the generalized sense as an element of $X \otimes S'(\mathbb{R}) \otimes (S)_{-1}$ and it is given by $h \cdot W_t = h\Diamond W_t + \mathbb{D}(h)$.

Example 4.8. Let $X = S'(\mathbb{R})$ and $h = W_{t_0}$. Then

$$W_{t_0} \cdot W_t = W_{t_0} \Diamond W_t + \mathbb{D}(W_{t_0}) = W_{t_0} \Diamond W_t + d_{t_0}(t) \tag{4.6}$$

holds in $S'(\mathbb{R}) \otimes S'(\mathbb{R}) \otimes (S)_{-1}$. Note that (4.6) is well defined for all $(t, t_0) \in \mathbb{R}^2$ except for $t = t_0$ where the Dirac delta distribution $d_{t_0}(t) = d(t - t_0) \in S'(\mathbb{R}) \otimes S'(\mathbb{R})$ has its singularity. It is possible to give meaning to $d_{t_0}(t_0) = \sum_{n=1}^{\infty} \xi_n(t_0)^2$ as the *point value* of a distribution in the sense of Colombeau generalized numbers. Thus, in Colombeau sense, it will be possible to define $W_t^2 = W_t^{\Diamond 2} + d_t(t)$. For the Colombeau theory we refer to [5, 11].

The previous theorem states that the Malliavin derivative indicates the speed of change in time between the ordinary product and the Wick product.

A generalization of Theorem 4.6 can be obtained by replacing white noise with an arbitrary process of first chaos order, i.e. considering $f \in \mathcal{H}_1$ and comparing the difference between $h \cdot f$ and $h \Diamond f$. This will be done in Theorem 5.10 in the next section.

Remark 4.9. Note that if a stochastic process h belongs to the Wiener chaos space $\bigoplus_{i=0}^{m} \mathcal{H}_i$ for some $m \in \mathbb{N}$, then the unique solution u of the equation $\mathbb{D}(u) = h$ belongs to the Wiener chaos space $\bigoplus_{i=0}^{m+1} \mathcal{H}_i$.

Remark 4.10. It is easy to check that if $\psi \in S_{-l}(\mathbb{R})$ is given by $\psi = \sum_{i=1}^{\infty} \psi_i \xi_i$, then $\delta(\psi) \in (S)_{-1,-l}$ and it is given by $\delta(\psi) = \sum_{i=1}^{\infty} \psi_i H_{\varepsilon^{(i)}}$. Moreover, one can define the Wick version of the stochastic exponential:

$$\exp^{\Diamond} \delta(\psi) = \sum_{k=0}^{\infty} \frac{\delta(\psi)^{\Diamond k}}{k!} = \sum_{\alpha \in \mathcal{I}} \frac{\psi^{\alpha}}{\alpha!} H_{\alpha}, \quad \text{where } \psi^{\alpha} = \prod_{i=1}^{\infty} \psi_i^{\alpha_i}.$$

In [19] we have proven that the stochastic exponentials are eigenvectors of the Malliavin derivative corresponding to the eigenvalue ψ, i.e. the process $u = \tilde{u}_0 \otimes \exp^{\Diamond} \delta(\psi) \in X \otimes S_{-l}(\mathbb{R}) \otimes (S)_{-1,-l}$ is the unique solution to the equation

$$\begin{cases} \mathbb{D}u = \psi \otimes u, & \psi \in S'(\mathbb{R}) \\ Eu = \widetilde{u}_0, & \widetilde{u}_0 \in X \end{cases}.$$

5. The Skorokhod integral

Theorem 5.1. ([21, 25]) *Let f be a process with zero expectation and chaos expansion representation of the form $f = \sum_{\alpha \in \mathcal{I}, |\alpha| \geq 1} f_{\alpha} \otimes H_{\alpha}$, $f_{\alpha} \in X$. Then the integral equation*

$$\delta(u) = f, \tag{5.1}$$

has a unique solution u given by

$$u = \sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} (\alpha_k + 1) \frac{f_{\alpha + \varepsilon^{(k)}}}{|\alpha + \varepsilon^{(k)}|} \otimes \xi_k \otimes H_{\alpha}. \tag{5.2}$$

Moreover, the following holds:

1. *If* $f \in X \otimes (S)_{-1,-p}$, $p \in \mathbb{N}$, *then* $u \in X \otimes S_{-l}(\mathbb{R}) \otimes (S)_{-1,-p}$, *for* $l > p+1$.
2. *If* $f \in X \otimes (S)_{1,p}$, $p \in \mathbb{N}$, *then* $u \in Dom_{(l,p)}(\delta)$, *for* $l < p-1$.
3. *If* $f \in X \otimes (L)^2$, *then* $u \in Dom_0(\delta)$.

Proof. 1. We seek for the solution in $Range_-(\mathbb{D})$. It is clear that $u \in Range_-(\mathbb{D})$ is equivalent to $u = \mathbb{D}(\widetilde{u})$, for some $\widetilde{u}$. This approach is general enough, since according to Theorem 4.1, for all $u \in X \otimes S'(\mathbb{R}) \otimes (S)_{-1}$ there exists $\widetilde{u} \in Dom_-(\mathbb{D})$ such that $u = \mathbb{D}(\widetilde{u})$ holds. Thus, equation (5.1) is equivalent to the system of equations

$$\begin{cases} u &= \mathbb{D}(\widetilde{u}), \\ \mathcal{R}(\widetilde{u}) &= f. \end{cases}$$

The solution to $\mathcal{R}(\widetilde{u}) = f$ is given by

$$\widetilde{u} = \widetilde{u}_0 + \sum_{\alpha \in \mathcal{I},\, |\alpha| \geq 1} \frac{f_\alpha}{|\alpha|} \otimes H_\alpha,$$

where $\tilde{u}_{(0,0,0,\dots)} = \widetilde{u}_0$ can be chosen arbitrarily. Now, the solution of the initial equation (5.1) is obtained after applying the operator $\mathbb{D}$ i.e.

$$\begin{aligned} u = \mathbb{D}(\widetilde{u}) &= \sum_{\alpha \in \mathcal{I},\, |\alpha| \geq 1} \sum_{k \in \mathbb{N}} \alpha_k \frac{f_\alpha}{|\alpha|} \otimes \xi_k \otimes H_{\alpha - \varepsilon^{(k)}} \\ &= \sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} (\alpha_k + 1) \frac{f_{\alpha + \varepsilon^{(k)}}}{|\alpha + \varepsilon^{(k)}|} \otimes \xi_k \otimes H_\alpha. \end{aligned}$$

It remains to prove the convergence of the solution (5.2) in $X \otimes S'(\mathbb{R}) \otimes (S)_{-1}$. Under the assumption $f \in X \otimes (S)_{-1,-p}$, for some $p > 0$ first we prove that $\widetilde{u} \in Dom_{-p}(\mathbb{D})$. Clearly,

$$\begin{aligned} \|\widetilde{u}\|^2_{Dom_{-p}(\mathbb{D})} &= \sum_{\alpha \in \mathcal{I}} |\alpha|^2 \, \|\widetilde{u}_\alpha\|^2_X \, (2\mathbb{N})^{-p\alpha} \\ &= \sum_{\alpha \in \mathcal{I},\, |\alpha| > 0} |\alpha|^2 \, \frac{\|f_\alpha\|^2_X}{|\alpha|^2} \, (2\mathbb{N})^{-p\alpha} \\ &= \sum_{\alpha \in \mathcal{I},\, |\alpha| > 0} \|f_\alpha\|^2_X \, (2\mathbb{N})^{-p\alpha} < \infty. \end{aligned}$$

Hence, the convergence of the solution u in the space $X \otimes S_{-l} \otimes (S)_{-1,-p}$, for $l > p+1$ follows from

$$\begin{aligned} \|u\|^2_{X \otimes S_{-l} \otimes (S)_{-1,-p}} &= \sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} \frac{(\alpha_k + 1)^2}{|\alpha + \varepsilon^{(k)}|^2} \, \|f_{\alpha + \varepsilon^{(k)}}\|^2_X \, \|\xi_k\|^2_{-l} \, (2\mathbb{N})^{-p\alpha} \\ &\leq \sum_{\alpha \in \mathcal{I},\, |\alpha| > 0} \sum_{k \in \mathbb{N}} \|f_\alpha\|^2_X \, (2k)^{-l} \, (2\mathbb{N})^{-p(\alpha - \varepsilon^{(k)})} \\ &\leq M \sum_{\alpha \in \mathcal{I}} \|f_\alpha\|^2_X \, (2\mathbb{N})^{-p\alpha} < \infty, \end{aligned}$$

since $M = \sum\limits_{k\in\mathbb{N}} (2k)^{p-l}$ is finite for $l > p+1$.

2. The form of the solution (5.2) is obtained in a similar way as in the previous case. We prove the convergence of the solution u in the space $Dom_{(l,p)}(\delta)$. First we prove that $\widetilde{u} \in Dom_p(\mathbb{D})$ and then $u \in Dom_{(l,p)}(\delta)$ for appropriate $l \in \mathbb{N}$. We obtain

$$\begin{aligned}
\|\widetilde{u}\|^2_{Dom_p(\mathbb{D})} &= \sum_{\alpha\in\mathcal{I}} \alpha!^2 \, \|u_\alpha\|^2_X \, (2\mathbb{N})^{p\alpha} \\
&= \sum_{\alpha\in\mathcal{I},|\alpha|>0} \alpha!^2 \, \frac{\|f_\alpha\|^2_X}{|\alpha|^2} \, (2\mathbb{N})^{p\alpha} \\
&\leq \sum_{\alpha\in\mathcal{I},|\alpha|>0} \alpha!^2 \, \|f_\alpha\|^2_X \, (2\mathbb{N})^{p\alpha} = \|f\|^2_{X\otimes(S)_{1,p}} < \infty
\end{aligned}$$

and thus $\widetilde{u} \in Dom_+(\mathbb{D})$. Now,

$$\begin{aligned}
\|u\|^2_{Dom_{(l,p)}(\delta)} &= \sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} \alpha!^2 \, (\alpha_k+1)^4 \, \frac{\|f_{\alpha+\varepsilon^{(k)}}\|^2_X}{|\alpha+\varepsilon^{(k)}|^2} \, (2k)^l \, (2\mathbb{N})^{p\alpha} \\
&= \sum_{\alpha\in\mathcal{I},|\alpha|>0} \sum_{k\in\mathbb{N}} \alpha!^2 \, \alpha_k^2 \, \frac{\|f_\alpha\|^2_X}{|\alpha|^2} \, (2k)^l \, (2\mathbb{N})^{p(\alpha-\varepsilon^{(k)})} \\
&\leq \sum_{\alpha\in\mathcal{I},|\alpha|>0} \alpha!^2 \, \|f_\alpha\|^2_X \, (2\mathbb{N})^{p\alpha} \sum_{k\in\mathbb{N}} \alpha_k^2 \frac{1}{|\alpha|^2} (2k)^l \, (2k)^{-p} \\
&\leq C \, \|f\|^2_{X\otimes(S)_{1,p}} < \infty,
\end{aligned}$$

since $C = \sum\limits_{k\in\mathbb{N}} (2k)^{l-p} < \infty$ for $p > l+1$. In the second step we used that $(\alpha-\varepsilon^{(k)})! \, \alpha_k^2 = \alpha! \, \alpha_k$, and in the fourth step we used $\alpha_k \leq |\alpha|$.

3. In this case we have

$$\begin{aligned}
\|\widetilde{u}\|^2_{Dom_0(\mathbb{D})} &= \sum_{\alpha\in\mathcal{I}} |\alpha| \, \alpha! \, \|u_\alpha\|^2_X = \sum_{\alpha\in\mathcal{I},|\alpha|>0} |\alpha| \, \alpha! \, \frac{\|f_\alpha\|^2_X}{|\alpha|^2} \\
&\leq \sum_{\alpha\in\mathcal{I},|\alpha|>0} \alpha! \, \|f_\alpha\|^2_X = \|f\|^2_{X\otimes(L)^2} < \infty
\end{aligned}$$

and thus $\widetilde{u} \in Dom_0(\mathbb{D})$. Also,

$$\begin{aligned}
\|u\|^2_{Dom_0(\delta)} &= \sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} \alpha! \, (\alpha_k+1)^3 \, \frac{\|f_{\alpha+\varepsilon^{(k)}}\|^2_X}{|\alpha+\varepsilon^{(k)}|^2} = \sum_{\alpha\in\mathcal{I},|\alpha|>0} \sum_{k\in\mathbb{N}} \alpha! \, \alpha_k^2 \, \frac{\|f_\alpha\|^2_X}{|\alpha|^2} \\
&= \sum_{\alpha\in\mathcal{I},|\alpha|>0} \alpha! \left(\sum_{k\in\mathbb{N}} \frac{\alpha_k^2}{|\alpha|^2} \right) \|f_\alpha\|^2_X \leq \|f\|^2_{X\otimes(L)^2} < \infty,
\end{aligned}$$

since for $|\alpha| > 0$ the estimate $\frac{\sum\limits_{k\in\mathbb{N}} \alpha_k^2}{|\alpha|^2} \leq \frac{(\sum\limits_{k\in\mathbb{N}} \alpha_k)^2}{|\alpha|^2} = 1.$ holds. □

Remark 5.2. If a stochastic process f belongs to the Wiener chaos space $\bigoplus_{i=1}^{m}\mathcal{H}_i$ for some $m\in\mathbb{N}$, then the solution u of the equation (5.1) belongs to the Wiener chaos space $\bigoplus_{i=0}^{m-1}\mathcal{H}_i$. Especially, if f is a quadratic Gaussian random process, i.e. an element of $\mathcal{H}_2$, then the solution u to (5.1) is a Gaussian process.

Corollary 5.3. *Each process $f\in X\otimes(S)_{\pm1}$, resp. $f\in X\otimes(L)^2$, can be represented as*

$$f = Ef + \delta(u)$$

for some $u\in X\otimes S'(\mathbb{R})\otimes(S)_{\pm1}$, resp. $u\in X\otimes L^2(\mathbb{R})\otimes(L)^2$.

Proof. The assertion follows for $u=\mathbb{D}(\mathcal{R}^{-1}(f-Ef))$. □

Note that the latter result reduces to the celebrated Itô representation theorem (see e.g. [13, 42]) in case when f is a square integrable adapted process.

Remark 5.4. In [48] a more general formula appears for the $f\in(L)^2$ case, which is equivalent to the classical Wiener-Itô chaos expansion. For $f\in(L)^2$ there exist u_k, $k\in\mathbb{N}$, such that each u_k is a square integrable function symmetric in all arguments,

$$f = Ef + \sum_{k=1}^{\infty}\delta^{(k)}(u_k),$$

and u_k are given by

$$u_k = \frac{1}{k!}E(\mathbb{D}^{(k)}f).$$

Moreover, if f is given by the chaos expansion $f=\sum_{\alpha\in\mathcal{I}}f_\alpha H_\alpha$, then $u_k=\sum_{|\alpha|=k}f_\alpha\xi^{\hat{\otimes}\alpha}$, where $\xi^{\hat{\otimes}\alpha}=\xi_1^{\hat{\otimes}\alpha_1}\hat{\otimes}\xi_2^{\hat{\otimes}\alpha_2}\hat{\otimes}\cdots$ and $\hat{\otimes}$ denotes the symmetric tensor product.

Remark 5.5. Since Gaussian processes play an important role in white noise analysis, we elaborate the explicit form of solutions in special cases for $m=2$ and for $m=3$.

1. First, assuming that the process f has zero expectation and a chaos expansion in the Wiener chaos space of maximal order two, i.e.

$$f = \sum_{\alpha\in\mathcal{I},1\le|\alpha|\le 2} f_\alpha\otimes H_\alpha\in\mathcal{H}_1\oplus\mathcal{H}_2,\quad f_\alpha\in X,$$

the solution u of the equation (5.1) belongs to the Wiener chaos space of order one $u\in\mathcal{H}_0\oplus\mathcal{H}_1$, i.e. it is a Gaussian process. Clearly, from (5.2) we obtain the coefficients $u_{\alpha,k}$, for lengths $|\alpha|\le 1$ and $k\in\mathbb{N}$. Therefore, for $\alpha=(0,0,....)$ the coefficients are

$$u_{(0,0,\dots),k} = f_{\varepsilon^{(k)}}, \tag{5.3}$$

and for $\alpha = \varepsilon^{(j)}$, $j \in \mathbb{N}$ the coefficients are

$$u_{\varepsilon^{(j)},k} = \begin{cases} f_{2\varepsilon^{(j)}}, & k = j \\ \frac{1}{2} f_{\varepsilon^{(j)}+\varepsilon^{(k)}}, & k \neq j \end{cases}. \tag{5.4}$$

Note that the coefficients of the solution are symmetric, i.e. $u_{\varepsilon^{(j)},k} = u_{\varepsilon^{(k)},j} = \frac{1}{2} f_{\varepsilon^{(j)}+\varepsilon^{(k)}}$, $k \neq j$, $k, j \in \mathbb{N}$. Thus the solution of (5.1) is given by

$$u = u_0 + \sum_{k=1}^{\infty} f_{2\varepsilon^{(j)}} \otimes \xi_j \otimes H_{\varepsilon^{(j)}} + \frac{1}{2} \sum_{j=1}^{\infty} \sum_{\substack{k=1 \\ k \neq j}}^{\infty} f_{\varepsilon^{(j)}+\varepsilon^{(k)}} \otimes \xi_k \otimes H_{\varepsilon^{(j)}}$$

with the generalized expectation

$$u_0 = \sum_{k=1}^{\infty} f_{\varepsilon^{(k)}} \otimes \xi_k.$$

2. If $f \in \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$ then the solution u belongs to the Wiener chaos space of maximal order two $\mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$, i.e. can be expressed in terms of multi-indices of length zero, one and two. The coefficients of the constant part of the solution (the generalized expectation), obtained for $|\alpha| = 0$, are given by (5.3) and of the Gaussian part of the solution, obtained for $|\alpha| = 1$, are represented in the form (5.4). For $|\alpha| = 2$ two cases may occur, $\alpha = 2\varepsilon^{(i)}$, $i \in \mathbb{N}$ or $\alpha = \varepsilon^{(i)} + \varepsilon^{(j)}$, $i \neq j$. Then, the coefficients are represented by

$$u_{2\varepsilon^{(i)},k} = \begin{cases} f_{3\varepsilon^{(i)}}, & k = i \\ \frac{2}{3} f_{2\varepsilon^{(i)}+\varepsilon^{(k)}}, & k \neq i \end{cases}, \qquad k \in \mathbb{N}, \text{ and}$$

$$u_{\varepsilon^{(i)}+\varepsilon^{(j)},k} = \begin{cases} \frac{2}{3} f_{2\varepsilon^{(i)}+\varepsilon^{(j)}}, & k = i \\ \frac{2}{3} f_{\varepsilon^{(i)}+2\varepsilon^{(j)}}, & k = j \\ \frac{1}{3} f_{\varepsilon^{(i)}+\varepsilon^{(j)}+\varepsilon^{(k)}}, & k \neq i, k \neq j \end{cases}, \qquad k \in \mathbb{N}.$$

3. In general, for any $\alpha \in \mathcal{I}$, $|\alpha| = n$ the coefficients are given in the form

$$u_{(n-1)\varepsilon^{(k)},k} = f_{n\varepsilon^{(k)}}, \quad \text{and} \quad u_{\varepsilon^{(i_1)}+\varepsilon^{(i_2)}+\ldots+\varepsilon^{(i_{n-1})},k} =$$

$$= \begin{cases} \frac{1}{n} f_{\varepsilon^{(i_1)}+\varepsilon^{(i_2)}+\ldots+\varepsilon^{(i_{n-1})}+\varepsilon^{(k)}} & , k \notin \{i_1, i_2, \ldots, i_{n-1}\} \\ \frac{2}{n} f_{2\varepsilon^{(i_1)}+\varepsilon^{(i_3)}+\ldots+\varepsilon^{(i_{n-1})}+\varepsilon^{(k)}} & , k = i_1 \notin \{i_2, \ldots, i_{n-1}\} \\ \frac{3}{n} f_{3\varepsilon^{(i_1)}+\varepsilon^{(i_4)}+\ldots+\varepsilon^{(i_{n-1})}+\varepsilon^{(k)}} & , k = i_1 = i_2 \notin \{i_3, \ldots, i_{n-1}\} \\ \vdots & \\ \frac{n-1}{n} f_{(n-1)\varepsilon^{(i_1)}+\varepsilon^{k}} & , k = i_1 = i_2 = \ldots = i_{n-2} \neq i_{n-1} \end{cases},$$

for $k, i_1, i_2, \ldots, i_{n-1}, n \in \mathbb{N}$.

Example 5.6. We provide some examples as illustrations for the integral equation (5.1).

1. The solution of the equation

$$\delta u = B_{t_0}(\omega)$$

belongs to the Wiener chaos space of order zero and it is obtained in the form

$$u(t) = \sum_{k\in\mathbb{N}} \int_0^{t_0} \xi_k(t)dt\, \xi_k(t) = \kappa_{[0,t_0]}(t),$$

i.e. it is the characteristic function of the interval $[0, t_0]$.

2. Consider the equation with singular white noise

$$\delta u = W_{t_0}(\omega), \tag{5.5}$$

where $W_{t_0}(\omega) = \sum_{k=1}^{\infty} \xi_k(t_0)\, H_{\varepsilon^{(k)}}$. It is clear that W_{t_0} belongs to the Wiener chaos space of order one. Hence the solution of (5.5) belongs to the Wiener chaos space of order zero. From (5.2) we obtain the chaos expansion form of the solution

$$\begin{aligned} u(t) &= \sum_{k\in\mathbb{N}} u_{(0,0,\dots),k}\, \xi_k(t)\, H_0(\omega) \\ &= \sum_{k\in\mathbb{N}} \xi_k(t_0)\, \xi_k(t) \;=\; d_{t_0}(t), \end{aligned}$$

which is the Dirac delta function concentrated at t_0.

3. Let $\delta u = \sum_{j=1}^{\infty} H_{2\varepsilon^{(j)}}(\omega)$. The solution belongs to the Wiener chaos space of order one. From (5.2) we obtain the form of the coefficients

$$u_{\varepsilon^{(j)},k} = \begin{cases} 0, & j \neq k \\ f_{2\varepsilon^{(j)}}, & j = k \end{cases} = \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases}.$$

Thus the solution is obtained in the form

$$u(t,\omega) = \sum_{j\in\mathbb{N}} u_{\varepsilon^{(j)},j}\ \xi_j(t)\ H_{\varepsilon^{(j)}} = \sum_{j\in\mathbb{N}} \xi_j(t)\ H_{\varepsilon^{(j)}} = W_t(\omega)$$

and represents singular white noise.

4. Consider the equation

$$\delta u = \frac{1}{2}\, B_{t_0}^{\lozenge 2}(\omega),$$

with right hand side $\frac{1}{2}\, B_{t_0}^{\lozenge 2}(\omega) = \frac{1}{2}\, (B_{t_0}^2(\omega) - t_0)$ in the Wiener chaos space of order two. The solution will belong to the chaos space of order one, i.e. it will be a Gaussian process. Since

$$\frac{1}{2}\, B_{t_0}^{\lozenge 2} = \frac{1}{2} \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \left(\int_0^{t_0} \xi_k(t)dt\right) \left(\int_0^{t_0} \xi_l(s)ds\right) H_{\varepsilon^{(k)}+\varepsilon^{(l)}},$$

by symmetry of the coefficients it follows that

$$f_{\varepsilon^{(l)}+\varepsilon^{(k)}} = f_{\varepsilon^{(k)}+\varepsilon^{(l)}} = \frac{1}{2}\left(\int_0^{t_0} \xi_k(t)dt\right)\left(\int_0^{t_0} \xi_l(s)ds\right).$$

By partial integration we obtain

$$\begin{aligned}\int_0^{t_0}\left(\int_0^t \xi_k(s)ds\right)\xi_l(t)dt &= \left(\int_0^{t_0} \xi_k(s)ds\right)\left(\int_0^{t_0} \xi_l(s)ds\right)\\ &\quad - \int_0^{t_0}\left(\int_0^t \xi_l(s)ds\right)\xi_k(t)dt,\end{aligned}$$

i.e. by symmetry of k and l:

$$\begin{aligned}\int_0^{t_0}\left(\int_0^t \xi_k(s)ds\right)\xi_l(t)dt &= \int_0^{t_0}\left(\int_0^t \xi_l(s)ds\right)\xi_k(t)dt\\ &= \frac{1}{2}\left(\int_0^{t_0} \xi_k(s)ds\right)\left(\int_0^{t_0} \xi_l(s)ds\right).\end{aligned}$$

Now, for each $j \in \mathbb{N}$, by (5.4) we obtain

$$\begin{aligned}u_{\varepsilon^{(j)},k} = \frac{1}{2}(f_{\varepsilon^{(j)}+\varepsilon^{(k)}} + f_{\varepsilon^{(k)}+\varepsilon^{(j)}}) &= \frac{1}{2}\left(\int_0^{t_0} \xi_k(t)dt\right)\left(\int_0^{t_0} \xi_j(s)ds\right)\\ &= \int_0^{t_0}\left(\int_0^t \xi_j(s)ds\right)\xi_j(t)dt.\end{aligned}$$

Thus,

$$\begin{aligned}u(t,\omega) &= \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\left(\int_0^{t_0}\left(\int_0^t \xi_j(s)ds\right)\xi_k(t)dt\right)\otimes \xi_k(t)\otimes H_{\varepsilon^{(j)}}(\omega)\\ &= \sum_{j=1}^{\infty}\left(\int_0^t \xi_j(s)ds\right)\kappa_{[0,t_0]}(t)\otimes H_{\varepsilon^{(j)}}(\omega)\\ &= B_t(\omega)\kappa_{[0,t_0]}(t).\end{aligned}$$

Note that the Skorokhod integral coincides with the Itô integral for which it is well-known that $\int_0^{t_0} B_t\, dB_t = \frac{1}{2}\,(B_{t_0}^2(\omega) - t_0)$.

5. Similarly to the previous case, the equation

$$\delta u = \frac{1}{2}\, W_{t_0}^{\lozenge 2}(\omega)$$

has the solution

$$u(t,\omega) = W_t(\omega)\delta_{t_0}(t).$$

Remark 5.7. Note that the operators $\mathbb{D}$ and δ are not inverse operators. From the previous examples we have seen, e.g. that for $Z = \sum_{k=1}^{\infty} H_{2\varepsilon^{(k)}}$ we have $\mathbb{D}(\frac{1}{2}Z) = W_t$, while $\delta(W_t) = Z$. Also, $\mathbb{D}(\frac{1}{2}B_{t_0}^{\lozenge 2}) = B_{t_0}\kappa_{[0,t_0]}$, while $\delta(B_{t_0}\kappa_{[0,t_0]}) = B_{t_0}\delta(\kappa_{[0,t_0]}) = B_{t_0}^2 = B_{t_0}^{\lozenge 2} + t_0$. The "disturbing" factor $\frac{1}{2}$ is a consequence of the fact that Z and $B_t^{\lozenge 2}$ belong to the Wiener chaos space $\mathcal{H}_2$.

It is also clear that $\mathcal{R}(\frac{1}{2}Z) = \delta(\mathbb{D}(\frac{1}{2}Z) = \delta(W_t) = Z$ and $\mathcal{R}(\frac{1}{2}B_{t_0}^{\lozenge 2}) = \delta(\mathbb{D}(\frac{1}{2}B_{t_0}^{\lozenge 2})) = \delta(B_{t_0}\kappa_{[0,t_0]}) = B_{t_0}^2 = B_{t_0}^{\lozenge 2} + t_0$, which are both in compliance with $\mathcal{R}(H_\alpha) = |\alpha| H_\alpha$ and Theorem 3.1.

The operators $\mathbb{D}$ and δ do not commute, which can easily be seen from $\mathbb{D}(\delta(W_t)) = \mathbb{D}(Z) = 2W_t$ and $\delta(\mathbb{D}(W_t)) = \delta(d_t) = W_t$.

Theorem 5.8. *Let $u \in X \otimes S'(\mathbb{R}) \otimes (S)_{-1}$. If $u \in Dom_-(\mathbb{D})$, then $\delta(u) \in Dom_-(\mathbb{D})$ and the following relation holds:*

$$\mathbb{D}(\delta u) = u + \delta(\mathbb{D}u). \tag{5.6}$$

Proof. Let u be of the form $u = \sum_{\alpha\in\mathcal{I}} \sum_{k=1}^{\infty} u_{\alpha,k} \otimes \xi_k \otimes H_\alpha$. Then, $\delta(u) = \sum_{\alpha\in\mathcal{I}} \sum_{k=1}^{\infty} u_{\alpha,k} \otimes H_{\alpha+\varepsilon^{(k)}}$ and consequently

$$\begin{aligned}
\mathbb{D}(\delta(u)) &= \sum_{\alpha\in\mathcal{I}} \sum_{k=1}^{\infty} u_{\alpha,k} \sum_{i=1}^{\infty} (\alpha+\varepsilon^{(k)})_i \otimes \xi_i \otimes H_{\alpha+\varepsilon^{(k)}-\varepsilon^{(i)}} \\
&= \sum_{\alpha\in\mathcal{I}} \sum_{k=1}^{\infty} u_{\alpha,k} \left((\alpha_k+1) \otimes \xi_k \otimes H_\alpha + \sum_{i\neq k} \alpha_i \otimes \xi_i \otimes H_{\alpha+\varepsilon^{(k)}-\varepsilon^{(i)}} \right) \\
&= \sum_{\alpha\in\mathcal{I}} \sum_{k=1}^{\infty} u_{\alpha,k} \otimes \xi_k \otimes H_\alpha + \sum_{\alpha\in\mathcal{I}} \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \alpha_i u_{\alpha,k} \otimes \xi_i \otimes H_{\alpha+\varepsilon^{(k)}-\varepsilon^{(i)}} \\
&= u + \delta(\mathbb{D}(u)).
\end{aligned}$$

The latter equality follows from $\mathbb{D}(u) = \sum_{\alpha\in\mathcal{I}} \sum_{i=1}^{\infty} \alpha_i \left(\sum_{k=1}^{\infty} u_{\alpha,k} \otimes \xi_k \right) \otimes \xi_i \otimes H_{\alpha-\varepsilon^{(i)}} \in X \otimes S'(\mathbb{R}) \otimes S'(\mathbb{R}) \otimes (S)_{-1}$ which implies

$$\delta(\mathbb{D}(u)) = \sum_{\alpha\in\mathcal{I}} \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \alpha_i u_{\alpha,k} \otimes \xi_i \otimes H_{\alpha-\varepsilon^{(i)}+\varepsilon^{(k)}}.$$

Since $u \in Dom_-(\mathbb{D})$, from Theorem 2.19 it follows that $\mathbb{D}u \in Dom_-(\delta)$. Theorem 2.22 ensures that the result remains in $X \otimes S'(\mathbb{R}) \otimes (S)_{-1}$, thus the right hand side of (5.6) is well defined and belongs to $X \otimes S'(\mathbb{R}) \otimes (S)_{-1}$. This means that the left hand is also an element in $X \otimes S'(\mathbb{R}) \otimes (S)_{-1}$, thus $\delta(u)$ must be in the domain of $\mathbb{D}$. □

Remark 5.9. Note that if $u \in X \otimes L^2(\mathbb{R}) \otimes (S)_{-1}$, then

$$\delta(u) = \int_{\mathbb{R}} u \lozenge W_t \, dt,$$

where the right hand side is interpreted as the X-valued Bochner integral in the Riemann sense. This is in accordance with the known fact that Itô-Skorokhod integration with the rules of Itô integration (Itô's calculus) generates the same results as integration interpreted in the classical Riemann sense following the rules of ordinary calculus, if the integrand is interpreted as the Wick product with white noise. For example,

$$\begin{aligned}\int_{[0,t_0]} B_t dB_t &= \delta(\kappa_{[0,t_0]}(t)B_t) = \int_{[0,t_0]} B_t \lozenge W_t dt = \int_{[0,t_0]} B_t \lozenge B_t' dt \\ &= \frac{1}{2} B_{t_0}^{\lozenge 2} = \frac{1}{2}(B_{t_0}^2 - t_0).\end{aligned}$$

The general case follows easily from the definition of the Skorokhod integral. If $u = \sum_{\alpha \in \mathcal{I}} u_\alpha \otimes H_\alpha = \sum_{\alpha \in \mathcal{I}} \sum_{k=1}^{\infty} u_{\alpha,k} \otimes \xi_k \otimes H_\alpha$ is in $X \otimes L^2(\mathbb{R}) \otimes (S)_{-1}$ then $u_{\alpha,k} = \int_{\mathbb{R}} u_\alpha(t) \xi_k(t) dt$ for all $\alpha \in \mathcal{I}$, $k \in \mathbb{N}$. Thus,

$$\begin{aligned}\delta(u) &= \sum_{\alpha \in \mathcal{I}} \sum_{k=1}^{\infty} u_{\alpha,k} \otimes H_{\alpha+\varepsilon^{(k)}} = \sum_{\alpha \in \mathcal{I}} \sum_{k=1}^{\infty} \int_{\mathbb{R}} u_\alpha(t) \xi_k(t) dt \otimes H_{\alpha+\varepsilon^{(k)}} \\ &= \int_{\mathbb{R}} \left(\sum_{\alpha \in \mathcal{I}} \sum_{k=1}^{\infty} u_\alpha(t) \xi_k(t) \otimes H_{\alpha+\varepsilon^{(k)}} \right) dt \\ &= \int_{\mathbb{R}} \left(\sum_{\alpha \in \mathcal{I}} u_\alpha(t) \otimes H_\alpha \right) \lozenge \left(\sum_{k=1}^{\infty} \xi_k(t) \otimes H_{\varepsilon^{(k)}} \right) dt \\ &= \int_{\mathbb{R}} u \lozenge W_t \, dt.\end{aligned}$$

The following theorem extends the result of Theorem 4.6 and reflects a nice connection between the Wick product and the ordinary product if one of the multiplicands is a Gaussian process from $\mathcal{H}_1$.

Theorem 5.10. ([21])

(a) *Let $f \in X \otimes (S)_{-1}$ be an element of $\mathcal{H}_0 \bigoplus \mathcal{H}_1$ of the form $f = \sum_{k=0}^{\infty} f_k H_{\varepsilon^{(k)}}$. Then, for any $h \in X \otimes (S)_{-1}$ of the form $h = \sum_{\alpha \in \mathcal{I}} h_\alpha H_\alpha$,*

$$h \cdot f - h \lozenge f = \sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} h_{\alpha+\varepsilon^{(k)}} f_k (\alpha_k + 1) H_\alpha \tag{5.7}$$

holds, where the right hand side is an element in $X \otimes (S)_1$ if only finitely many of its coefficients are nonzero, otherwise it is understood as a formal (not necessarily convergent) expansion.Some special cases under which it is a convergent expansion in $X \otimes (S)_{-1}$ are provided below:

(b) *Especially, if* $g \in X \otimes S(\mathbb{R})$*, where* g *denotes the unique solution to* $\delta(g) = f$*, then*

$$h \cdot \delta(g) - h \Diamond \delta(g) = \langle \mathbb{D}(h), g \rangle$$

holds in $X \otimes (S)_{-1}$.

(c) *Especially, if* $h \in X \otimes (S)_1$ *and* $g \in X \otimes S'(\mathbb{R})$*, where* g *denotes the unique solution to* $\delta(g) = f$*, then*

$$h \cdot \delta(g) - h \Diamond \delta(g) = \langle g, \mathbb{D}(h) \rangle$$

holds in $X \otimes (S)_{-1}$.

(d) *In case* $g \in X \otimes S(\mathbb{R})$ *and* $\mathbb{D}(h) \in X \otimes L^2(\mathbb{R}) \otimes (S)_{-1}$*, as well as in the case* $g \in X \otimes L^2(\mathbb{R})$ *and* $\mathbb{D}(h) \in X \otimes L^2(\mathbb{R}) \otimes (S)_1$*, formula* (5.7) *reduces to*

$$h \cdot \delta(g) - h \Diamond \delta(g) = \int_{\mathbb{R}} g(t) \cdot \mathbb{D}(h)(t)\, dt.$$

Proof. (a) Assume $E(f) = f_0 = 0$. Then, according to Theorem 5.1 there exists a unique g such that $\delta(g) = f$ and moreover this g is given by $g = \sum_{k=1}^{\infty} f_k \xi_k$ as an element of $X \otimes S'(\mathbb{R})$. Thus,

$$h \Diamond f = h \Diamond \delta(g) = \sum_{\gamma \in \mathcal{I}} \sum_{n=1}^{\infty} h_{\gamma - \varepsilon^{(n)}} f_n H_\gamma$$

and

$$\begin{aligned} h \cdot \delta(g) &= \sum_{\alpha \in \mathcal{I}} \sum_{n=1}^{\infty} h_{\alpha - \varepsilon^{(n)}} f_n H_{\alpha - \varepsilon^{(n)}} H_{\varepsilon^{(n)}} \\ &= \sum_{\alpha \in \mathcal{I}} \sum_{n=1}^{\infty} h_{\alpha - \varepsilon^{(n)}} f_n (H_\alpha + (\alpha_n - 1) H_{\alpha - 2\varepsilon^{(n)}}). \end{aligned}$$

This implies

$$\begin{aligned} h \cdot \delta(g) - h \Diamond \delta(g) &= \sum_{\alpha \in \mathcal{I}} \sum_{n=1}^{\infty} h_{\alpha - \varepsilon^{(n)}} f_n (\alpha_n - 1) H_{\alpha - 2\varepsilon^{(n)}} \\ &= \sum_{\alpha \in \mathcal{I}} \sum_{n=1}^{\infty} h_{\alpha + \varepsilon^{(n)}} f_n (\alpha_n + 1) H_\alpha. \end{aligned}$$

Now, for arbitrary f let $\tilde{f} = f - E(f)$ and $\tilde{g}$ such that $f = E(f) + \delta(\tilde{g})$. Since for constants the Wick product and the ordinary product coincide, we have

$$\begin{aligned} h \cdot f - h \Diamond f &= h \cdot E(f) + h \cdot \delta(\tilde{g}) - h \Diamond E(f) - h \Diamond \delta(\tilde{g}) = h \cdot \delta(\tilde{g}) - h \Diamond \delta(\tilde{g}) \\ &= \sum_{\alpha \in \mathcal{I}} \sum_{n=1}^{\infty} h_{\alpha + \varepsilon^{(n)}} f_n (\alpha_n + 1) H_\alpha. \end{aligned}$$

Convergence of the series on the right hand side of (5.7) can be proven only in the special cases (b), (c) and (d). For example, if (b) holds, then $g = \sum_{k=1}^{\infty} f_k \xi_k$ and $f_k = \langle \xi_k, g \rangle$, $k \in \mathbb{N}$, which reduces to $f_k = \int_{\mathbb{R}} g(t)\xi_k(t)dt$ in case of $g \in L^2(\mathbb{R})$ and since $\mathbb{D}(h) = \sum_{\alpha\in\mathcal{I}} \sum_{n=1}^{\infty} h_{\alpha+\varepsilon^{(n)}}(\alpha_n+1)\xi_n H_\alpha$, we may write the right hand side of (5.7) as

$$\begin{aligned} \sum_{\alpha\in\mathcal{I}} \sum_{n=1}^{\infty} h_{\alpha+\varepsilon^{(n)}}(\alpha_n+1)\langle \xi_n, g\rangle H_\alpha &= \langle \sum_{\alpha\in\mathcal{I}} \sum_{n=1}^{\infty} h_{\alpha+\varepsilon^{(n)}}(\alpha_n+1)\xi_n H_\alpha, g\rangle \\ &= \langle \mathbb{D}(h), g\rangle. \end{aligned}$$

Assume that $h \in X \otimes (S)_{-1,-p}$ for some $p > 0$ and that $g \in X \otimes S_l(\mathbb{R})$ for all $l > 0$. Then $h \cdot \delta(g) - h\lozenge\delta(g) = \sum_{\alpha\in\mathcal{I}} \sum_{n=1}^{\infty} h_\alpha f_n \alpha_n H_{\alpha-\varepsilon^{(n)}}$ is well defined in $X \otimes (S)_{-1,-q}$ for $q \geq p+2$. This follows from the fact that $|\alpha| \leq (2\mathbb{N})^\alpha$ and thus

$$\begin{aligned} &\sum_{\alpha\in\mathcal{I}} \sum_{n=1}^{\infty} \|h_\alpha\|_X^2 \|f_n\|_X^2 |\alpha_n|^2 (2\mathbb{N})^{-q(\alpha-\varepsilon^{(n)})} \\ &= \sum_{\alpha\in\mathcal{I}} \sum_{n=1}^{\infty} \|h_\alpha\|_X^2 \|f_n\|_X^2 |\alpha_n|^2 (2\mathbb{N})^{-q\alpha}(2n)^q \\ &\leq \sum_{\alpha\in\mathcal{I}} \|h_\alpha\|_X^2 (2\mathbb{N})^{-(q-2)\alpha} \sum_{n=1}^{\infty} \|f_n\|_X^2 (2n)^q \\ &\leq \sum_{\alpha\in\mathcal{I}} \|h_\alpha\|_X^2 (2\mathbb{N})^{-p\alpha} \sum_{n=1}^{\infty} \|f_n\|_X^2 (2n)^l < \infty \end{aligned}$$

for $q-2 \geq p$ and $q \leq l$. Since l is arbitrary this holds for all $q \geq p+2$.

The proof of (c) and (d) is similar. □

Remark 5.11. Especially, if f_1, f_2 are both Gaussian processes such that $f_i = \delta(g_i)$, $g_i \in X \otimes L^2(\mathbb{R})$, $i = 1, 2$, then

$$\delta(g_1) \cdot \delta(g_2) - \delta(g_1)\lozenge\delta(g_2) = \int_{\mathbb{R}} g_1(t) g_2(t) dt.$$

This is in compliance with the $(L)^2$-result from [13].

Example 5.12. For example if $g = d_t$ (the Dirac delta distribution) we have $f = \delta(d_t) = W_t$, $\langle d_t, \mathbb{D}(h)\rangle = \mathbb{D}(h)(t)$ and thus retrieve the result of Theorem 4.6.

From (5.7) it follows that

$$B_t^2 - B_t^{\lozenge 2} = \int_{\mathbb{R}} \kappa_{[0,t]}(s) \mathbb{D}(B_t)(s) ds = \int_{\mathbb{R}} \kappa_{[0,t]}(s) \kappa_{[0,t]}(s) ds = t.$$

Remark 5.13. One might define a new type of "scalarized" Wick product containing in itself an integral operator, i.e. the scalar product in $L^2(\mathbb{R})$ or the dual pairing $\langle\cdot,\cdot\rangle$ of a distribution in $\mathcal{S}'(\mathbb{R})$ and a test function in $\mathcal{S}(\mathbb{R})$. Thus, if $a = \sum_{\alpha\in\mathcal{I}} a_\alpha H_\alpha \in L^2(\mathbb{R})\otimes(S)_{-1}$, $b = \sum_{\beta\in\mathcal{I}} b_\beta H_\beta \in L^2(\mathbb{R})\otimes(S)_{-1}$, then $a\blacklozenge b \in (S)_{-1}$ is defined by

$$a\blacklozenge b = \sum_{\gamma\in\mathcal{I}}\sum_{\alpha+\beta=\gamma}\langle a_\alpha, b_\beta\rangle H_\gamma.$$

Similarly, if $a\in\mathcal{S}'(\mathbb{R})\otimes(S)_{-1}$, $b\in\mathcal{S}(\mathbb{R})\otimes(S)_{-1}$, the result will be $a\blacklozenge b\in(S)_{-1}$.

Now, the right hand side of (5.7) can be rewritten as $\mathbb{D}(h)\blacklozenge\mathbb{D}(f)$. Clearly,

$$\begin{aligned}\mathbb{D}(h)\blacklozenge\mathbb{D}(f) &= \sum_{\alpha\in\mathcal{I}}\sum_{k\in\mathbb{N}} h_{\alpha+\varepsilon^{(k)}}(\alpha_k+1)\xi_k H_\alpha \blacklozenge \sum_{k\in\mathbb{N}} f_k\xi_k H_{(0,0,0,\dots)}\\ &= \sum_{\gamma\in\mathcal{I}}\langle\sum_{k\in\mathbb{N}} h_{\alpha+\varepsilon^{(k)}}(\alpha_k+1)\xi_k, \sum_{l\in\mathbb{N}} f_l\xi_l\rangle H_\gamma\\ &= \sum_{\gamma\in\mathcal{I}}\sum_{k\in\mathbb{N}} h_{\alpha+\varepsilon^{(k)}}(\alpha_k+1) f_k H_\gamma,\end{aligned}$$

since $\langle\xi_k,\xi_l\rangle = 1$ only for $k=l$ and $\langle\xi_k,\xi_l\rangle=0$ for $k\neq l$.

Thus, Theorem 5.10 b) - d) state that

$$h\cdot f = h\lozenge f + \mathbb{D}(h)\blacklozenge\mathbb{D}(f).$$

In [15, 32] a more general formula appears in the $f,g\in X\otimes(L)^2$ case, where the Wick product scalarizes through the n-fold integral:

$$h\cdot f = h\lozenge f + \sum_{n\in\mathbb{N}}\frac{1}{n!}(\mathbb{D}^{(n)}(h)\blacklozenge\mathbb{D}^{(n)}(f)) = \sum_{n\in\mathbb{N}_0}\frac{1}{n!}(\mathbb{D}^{(n)}(h)\blacklozenge\mathbb{D}^{(n)}(f)), \tag{5.8}$$

under suitable conditions that ensure the convergence of the latter sum.

In a very similar manner to (5.8) it is possible to express the Wick product through the ordinary product. This has been proved in [15] and used also in [38]. For $f,g\in X\otimes(L)^2$ it holds that

$$h\lozenge f = \sum_{n\in\mathbb{N}_0}\frac{(-1)^n}{n!}\langle\mathbb{D}^{(n)}(h),\mathbb{D}^{(n)}(f)\rangle, \tag{5.9}$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $L^2(\mathbb{R})^{\otimes n}$.

Both identities: (5.8) and (5.9) can be generalized to the case when $f\in Dom_+(\mathbb{D})$ and $h\in Dom_-(\mathbb{D})$ or vice versa. In this case we interpret $\langle\cdot,\cdot\rangle$ as the dual pairing between $S'(\mathbb{R})^{\otimes n}$ and $S(\mathbb{R})^{\otimes n}$.

6. Properties of the Malliavin operators

The following theorem states the duality between the Malliavin derivative and the Skorokhod integral in form of (6.1), which is also called the integration by parts formula.

Theorem 6.1. *(Duality) Assume that either of the following holds:*

(a) $F \in Dom_-(\mathbb{D})$ *and* $u \in Dom_+(\delta)$

(b) $F \in Dom_+(\mathbb{D})$ *and* $u \in Dom_-(\delta)$

(c) $F \in Dom_0(\mathbb{D})$ *and* $u \in Dom_0(\delta)$.

Then the following duality relationship between the operators $\mathbb{D}$ *and* δ *holds:*

$$E\left(F \cdot \delta(u)\right) \;=\; E\left(\langle \mathbb{D}F,\, u\rangle\right)\,, \tag{6.1}$$

where (6.1) *denotes the equality of the generalized expectations of two objects in* $X \otimes (S)_{-1}$ *and* $\langle\cdot,\cdot\rangle$ *denotes the dual paring of* $S'(\mathbb{R})$ *and* $S(\mathbb{R})$.

Proof. First we note that Theorem 2.16 implies that in all three cases (a), (b) and (c), the product on the left hand side of (6.1) is well defined and $F \cdot \delta(u)$ is an element in $X \otimes (S)_{-1}$. Also, the application of the dual pairing in $S'(\mathbb{R})$ will make $\langle \mathbb{D}, u\rangle$ also an element in $X \otimes (S)_{-1}$. Now we prove that both objects have the same expectation.

Let $u \in Dom(\delta)$ be given in its chaos expansion form $u = \sum_{\beta\in\mathcal{I}}\sum_{j\in\mathbb{N}} u_{\beta,j} \otimes \xi_j \otimes H_\beta$. Then $\delta(u) = \sum_{\beta\in\mathcal{I}}\sum_{j\in\mathbb{N}} u_{\beta,j} \otimes H_{\beta+\varepsilon^{(j)}}$. Let $F \in Dom(\mathbb{D})$ be given as $F = \sum_{\alpha\in\mathcal{I}} f_\alpha \otimes H_\alpha$. Then $\mathbb{D}(F) = \sum_{\alpha\in\mathcal{I}}\sum_{k\in\mathbb{N}} (\alpha_k+1)\, f_{\alpha+\varepsilon^{(k)}} \otimes \xi_k \otimes H_\alpha$. Therefore we obtain

$$\begin{aligned}
F \cdot \delta(u) &= \sum_{\alpha\in\mathcal{I}}\sum_{\beta\in\mathcal{I}}\sum_{j\in\mathbb{N}} f_\alpha u_{\beta,j} \otimes H_\alpha \cdot H_{\beta+\varepsilon^{(j)}} \\
&= \sum_{\alpha\in\mathcal{I}}\sum_{\beta\in\mathcal{I}}\sum_{j\in\mathbb{N}} f_\alpha u_{\beta,j} \otimes \sum_{\gamma\le\min\{\alpha,\beta+\varepsilon^{(j)}\}} \gamma!\binom{\alpha}{\gamma}\binom{\beta+\varepsilon^{(j)}}{\gamma} H_{\alpha+\beta+\varepsilon^{(j)}-2\gamma}.
\end{aligned}$$

The generalized expectation of $F \cdot \delta(u)$ is the zeroth coefficient in the previous sum, which is obtained when $\alpha+\beta+\varepsilon^{(j)} = 2\gamma$ and $\gamma \le \min\{\alpha, \beta+\varepsilon^{(j)}\}$, i.e. only for the choice $\beta = \alpha - \varepsilon^{(j)}$ and $\gamma = \alpha$, $j \in \mathbb{N}$. Thus,

$$E\left(F \cdot \delta(u)\right) = \sum_{\alpha\in\mathcal{I},|\alpha|>0}\sum_{j\in\mathbb{N}} f_\alpha u_{\alpha-\varepsilon^{(j)},\,j} \cdot \alpha! = \sum_{\alpha\in\mathcal{I}}\sum_{j\in\mathbb{N}} f_{\alpha+\varepsilon^{(j)}} u_{\alpha,j} \cdot (\alpha+\varepsilon^{(j)})!\,.$$

On the other hand,

$$\begin{aligned}
\langle \mathbb{D}(F), u\rangle &= \sum_{\alpha\in\mathcal{I}}\sum_{\beta\in\mathcal{I}}\sum_{k\in\mathbb{N}}\sum_{j\in\mathbb{N}} (\alpha_k+1)\, f_{\alpha+\varepsilon^{(k)}}\, u_{\beta,j}\, \langle \xi_k, \xi_j\rangle H_\alpha \cdot H_\beta \\
&= \sum_{\alpha\in\mathcal{I}}\sum_{\beta\in\mathcal{I}}\sum_{j\in\mathbb{N}} (\alpha_j+1)\, f_{\alpha+\varepsilon^{(j)}}\, u_{\beta,j} \sum_{\gamma\le\min\{\alpha,\beta\}} \gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma} H_{\alpha+\beta-2\gamma}
\end{aligned}$$

and its generalized expectation is obtained for $\alpha = \beta = \gamma$. Thus

$$\begin{aligned} E\left(\langle \mathbb{D}(F), u\rangle\right) &= \sum_{\alpha\in\mathcal{I}}\sum_{j\in\mathbb{N}} (\alpha_j+1)\, f_{\alpha+\varepsilon^{(j)}} u_{\alpha,j} \cdot \alpha! \\ &= \sum_{\alpha\in\mathcal{I}}\sum_{j\in\mathbb{N}} f_{\alpha+\varepsilon^{(j)}} u_{\alpha,j} \cdot (\alpha+\varepsilon^{(j)})! \\ &= E\left(F\cdot\delta(u)\right). \end{aligned}$$

□

The next theorem states a higher order duality formula, which connects the kth order iterated Skorokhod integral and the Malliavin derivative operator of kth order, $k \in \mathbb{N}$.

Theorem 6.2. *Let $f \in Dom_+(\mathbb{D}^{(k)})$ and $u \in Dom_-(\delta^{(k)})$, or alternatively let $f \in Dom_-(\mathbb{D}^{(k)})$ and $u \in Dom_+(\delta^{(k)})$, $k \in \mathbb{N}$. Then the duality formula*

$$E\left(f\cdot\delta^{(k)}(u)\right) = E\left(\langle \mathbb{D}^{(k)}(f), u\rangle\right)$$

holds, where $\langle\cdot,\cdot\rangle$ denotes the duality pairing of $S'(\mathbb{R})^{\otimes k}$ and $S(\mathbb{R})^{\otimes k}$.

Remark 6.3. The previous theorems are special cases of a more general identity. It can be proven, that under suitable assumptions that make all the products well defined the following formulae hold:

$$F\,\delta(u) = \delta(Fu) + \langle \mathbb{D}(F), u\rangle, \tag{6.2}$$

$$F\,\delta^{(k)}(u) = \sum_{i=0}^{k}\binom{k}{i}\,\delta^{(k-i)}(\langle \mathbb{D}^{(i)}F, u\rangle), \quad k\in\mathbb{N}. \tag{6.3}$$

The special case of (6.3) when $u \in Dom_0(\delta)$ i.e. when u is square integrable has been proven in [34]. Taking the expectation in (6.2) and using the fact that $\delta(Fu) = 0$, the duality formula (6.1) follows.

Example 6.4. Let $\psi \in L^2(\mathbb{R})$. In Remark 4.10 we have shown that the stochastic exponentials $\exp^{\Diamond}\{\delta(\psi)\}$ are eigenvalues of the Malliavin derivative i.e. $\mathbb{D}(\exp^{\Diamond}\{\delta(\psi)\}) = \psi\cdot\exp^{\Diamond}\{\delta(\psi)\}$. We will prove that they are also eigenvalues of the Ornstein-Uhlenbeck operator. Indeed, using (6.2) we obtain

$$\begin{aligned} \mathcal{R}(\exp^{\Diamond}\{\delta(\psi)\}) &= \delta(\psi\cdot\exp^{\Diamond}\{\delta(\psi)\}) = \delta(\psi)\exp^{\Diamond}\{\delta(\psi)\} - \langle \mathbb{D}(\exp^{\Diamond}\{\delta(\psi)\}), \psi\rangle \\ &= \delta(\psi)\exp^{\Diamond}\{\delta(\psi)\} - \langle \psi\cdot\exp^{\Diamond}\{\delta(\psi)\}, \psi\rangle \\ &= (\delta(\psi) - \|\psi\|^2_{L^2(\mathbb{R})})\exp^{\Diamond}\{\delta(\psi)\}. \end{aligned}$$

In the next theorem we prove a weaker type of duality instead of (6.1) which holds if $F \in Dom_-(\mathbb{D})$ and $u \in Dom_-(\delta)$ are both generalized processes. Recall that $\ll,\cdot,\cdot\gg_r$ denotes the scalar product in $(S)_{0,r}$.

Theorem 6.5. *(Weak duality, [25]) Let $F \in Dom_{-p}(\mathbb{D})$ and $u \in X \otimes (S)_{-1,-q}$ for $p, q \in \mathbb{N}$. For any $\varphi \in S_{-n}(\mathbb{R})$, $n \in \mathbb{N}$, it holds that*

$$\ll \langle \mathbb{D}F, \varphi \rangle_{-r}, u \gg_{-r} = \ll F, \delta(\varphi u) \gg_{-r},$$

for $r > 1 + \max\{q, p+1, n+1\}$.

Proof. Let $F = \sum_{\alpha \in \mathcal{I}} f_\alpha H_\alpha \in X \otimes (S)_{-1,-p}$, $u = \sum_{\alpha \in \mathcal{I}} u_\alpha H_\alpha \in X \otimes (S)_{-1,-q}$ and $\varphi = \sum_{k \in \mathbb{N}} \varphi_k \xi_k \in S_{-n}(\mathbb{R})$. Let $r > 1 + \max\{q, p+1, n+1\}$. Then, for $k > p+1$, $\mathbb{D}F \in X \otimes S_{-k}(\mathbb{R}) \otimes (S)_{-1,-p} \subseteq X \otimes S_{-(r-1)}(\mathbb{R}) \otimes (S)_{-1,-(r-1)} \subseteq X \otimes S_{-r}(\mathbb{R}) \otimes (S)_{0,-r}$. Also, $\varphi u \in X \otimes S_{-n}(\mathbb{R}) \otimes (S)_{-1,-q}$ implies that for $w > \max\{q, n+1\}$, $\delta(\varphi u) \in X \otimes (S)_{-1,-w} \subseteq X \otimes (S)_{-1,-(r-1)} \subseteq X \otimes (S)_{0,-r}$. Clearly, $\varphi \in S_{-n}(\mathbb{R}) \subseteq S_{-r}(\mathbb{R})$. Thus,

$$\begin{aligned} \langle \mathbb{D}F, \varphi \rangle_{-r} &= \langle \sum_{k \in \mathbb{N}} \sum_{\alpha \in \mathcal{I}} (\alpha_k + 1) f_{\alpha + \varepsilon^{(k)}} H_\alpha \otimes \xi_k, \sum_{k \in \mathbb{N}} \varphi_k \xi_k \rangle_{-r} \\ &= \sum_{k \in \mathbb{N}} \varphi_k \sum_{\alpha \in \mathcal{I}} (\alpha_k + 1) f_{\alpha + \varepsilon^{(k)}} H_\alpha \, (2k)^{-r}, \end{aligned}$$

and consequently

$$\begin{aligned} &\ll \langle \mathbb{D}F, \varphi \rangle_{-r}, u \gg_{-r} \\ &\quad = \ll \sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} \varphi_k (\alpha_k + 1) f_{\alpha + \varepsilon^{(k)}} (2k)^{-r} H_\alpha, \sum_{\alpha \in \mathcal{I}} u_\alpha H_\alpha \gg_{-r} \\ &\quad = \sum_{\alpha \in \mathcal{I}} \alpha! u_\alpha \sum_{k \in \mathbb{N}} \varphi_k (\alpha_k + 1) f_{\alpha + \varepsilon^{(k)}} (2k)^{-r} (2\mathbb{N})^{-r\alpha}. \end{aligned}$$

On the other hand,

$$\varphi u = \sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} u_\alpha \varphi_k \xi_k \otimes H_\alpha$$

and

$$\delta(\varphi u) = \sum_{\alpha > \mathbf{0}} \sum_{k \in \mathbb{N}} u_{\alpha - \varepsilon^{(k)}} \varphi_k H_\alpha.$$

Thus,

$$\begin{aligned} \ll F, \delta(\varphi u) \gg_{-r} &= \ll \sum_{\alpha \in \mathcal{I}} f_\alpha H_\alpha, \sum_{\alpha > \mathbf{0}} \sum_{k \in \mathbb{N}} u_{\alpha - \varepsilon^{(k)}} \varphi_k H_\alpha \gg_{-r} \\ &= \sum_{\alpha > \mathbf{0}} \alpha! f_\alpha \sum_{k \in \mathbb{N}} u_{\alpha - \varepsilon^{(k)}} \varphi_k (2\mathbb{N})^{-r\alpha} \\ &= \sum_{\beta \in \mathcal{I}} \sum_{k \in \mathbb{N}} (\beta + \varepsilon^{(k)})! f_{\beta + \varepsilon^{(k)}} u_\beta \varphi_k (2\mathbb{N})^{-r(\beta + \varepsilon^{(k)})} \\ &= \sum_{\beta \in \mathcal{I}} \sum_{k \in \mathbb{N}} \beta! (\beta_k + 1) f_{\beta + \varepsilon^{(k)}} u_\beta \varphi_k (2k)^{-r} (2\mathbb{N})^{-r\beta}, \end{aligned}$$

which completes the proof.

□

The following theorem states the product rule for the Ornstein-Uhlenbeck operator. Its special case for $F, G \in Dom_0(\mathcal{R})$ states that $F \cdot G$ is also in $Dom_0(\mathcal{R})$ and (6.4) holds; the proof can be found e.g. in [17].

Theorem 6.6. *(Product rule for $\mathcal{R}$)*

a) Let $F \in Dom_+(\mathcal{R})$ and $G \in Dom_-(\mathcal{R})$, or vice versa. Then $F \cdot G \in Dom_-(\mathcal{R})$ and

$$\mathcal{R}(F \cdot G) = F \cdot \mathcal{R}(G) + G \cdot \mathcal{R}(F) - 2 \cdot \langle \mathbb{D}F, \mathbb{D}G \rangle, \tag{6.4}$$

holds, where $\langle \cdot, \cdot \rangle$ is the dual paring between $S'(\mathbb{R})$ and $S(\mathbb{R})$.

b) Let $F, G \in Dom_-(\mathcal{R})$. Then $F \cdot G \in Dom_-(\mathcal{R})$ and

$$\mathcal{R}(F \lozenge G) = F \lozenge \mathcal{R}(G) + \mathcal{R}(F) \lozenge G. \tag{6.5}$$

Proof. *a*) First let us note that according to Theorem 2.16, $F \cdot \mathcal{R}(G)$ and $G \cdot \mathcal{R}(F)$ are both well defined and belong to $X \otimes (S)_{-1}$. Similarly, $\langle \mathbb{D}(F), \mathbb{D}(G) \rangle$ belongs to $X \otimes (S)_{-1}$, thus the right hand side of (6.4) is in $X \otimes (S)_{-1}$, which means that $F \cdot G \in Dom_-(\mathcal{R})$ according to Theorem 3.1.

Now let $F = \sum_{\alpha \in \mathcal{I}} f_\alpha \otimes H_\alpha \in Dom_+(\mathcal{R})$ and $G = \sum_{\beta \in \mathcal{I}} g_\beta \otimes H_\beta \in Dom_-(\mathcal{R})$. Then, $\mathcal{R}(F) = \sum_{\alpha \in \mathcal{I}} |\alpha| \, f_\alpha \otimes H_\alpha$ and $\mathcal{R}(G) = \sum_{\beta \in \mathcal{I}} |\beta| \, g_\beta \otimes H_\beta$.

The left hand side of (6.4) can be written in the form

$$\begin{aligned} \mathcal{R}(F \cdot G) &= \mathcal{R}\left(\sum_{\alpha \in \mathcal{I}} \sum_{\beta \in \mathcal{I}} f_\alpha \, g_\beta \sum_{\gamma \leq \min\{\alpha,\beta\}} \gamma! \binom{\alpha}{\gamma} \binom{\beta}{\gamma} H_{\alpha+\beta-2\gamma} \right) \\ &= \sum_{\alpha \in \mathcal{I}} \sum_{\beta \in \mathcal{I}} f_\alpha \, g_\beta \sum_{\gamma \leq \min\{\alpha,\beta\}} \gamma! \binom{\alpha}{\gamma} \binom{\beta}{\gamma} |\alpha + \beta - 2\gamma| \, H_{\alpha+\beta-2\gamma} \\ &= \sum_{\alpha \in \mathcal{I}} \sum_{\beta \in \mathcal{I}} f_\alpha \, g_\beta \sum_{\gamma \leq \min\{\alpha,\beta\}} \gamma! \binom{\alpha}{\gamma} \binom{\beta}{\gamma} (|\alpha| + |\beta| - 2|\gamma|) \, H_{\alpha+\beta-2\gamma}. \end{aligned}$$

On the other hand, the first two terms on the right hand side of (6.4) are

$$\mathcal{R}(F) \cdot G = \sum_{\alpha \in \mathcal{I}} \sum_{\beta \in \mathcal{I}} f_\alpha \, g_\beta \otimes \sum_{\gamma \leq \min\{\alpha,\beta\}} \gamma! \binom{\alpha}{\gamma} \binom{\beta}{\gamma} |\alpha| \, H_{\alpha+\beta-2\gamma} \tag{6.6}$$

and

$$F \cdot \mathcal{R}(G) = \sum_{\alpha \in \mathcal{I}} \sum_{\beta \in \mathcal{I}} f_\alpha \, g_\beta \otimes \sum_{\gamma \leq \min\{\alpha,\beta\}} \gamma! \binom{\alpha}{\gamma} \binom{\beta}{\gamma} |\beta| H_{\alpha+\beta-2\gamma}. \tag{6.7}$$

Since $F \in Dom_+(\mathcal{R}) \subset Dom_+(\mathbb{D})$ and $G \in Dom_-(\mathcal{R}) = Dom_-(\mathbb{D})$ we have $\mathbb{D}(F) = \sum_{\alpha \in \mathcal{I}} \sum_{k \in \mathbb{N}} \alpha_k \, f_\alpha \otimes \xi_k \otimes H_{\alpha - \varepsilon^{(k)}}$ and $\mathbb{D}(G) = \sum_{\beta \in \mathcal{I}} \sum_{j \in \mathbb{N}} \beta_j \, g_\beta \otimes \xi_j \otimes$

$H_{\beta-\varepsilon^{(k)}}$. Thus, the third term on the right hand side of (6.4) is

$$\begin{aligned}\langle \mathbb{D}(F), \mathbb{D}(G)\rangle &= \langle \sum_{|\alpha|>0}\sum_{k\in\mathbb{N}} \alpha_k\, f_\alpha \otimes \xi_k \otimes H_{\alpha-\varepsilon^{(k)}}, \sum_{|\beta|>0}\sum_{j\in\mathbb{N}} \beta_j\, g_\beta \otimes \xi_j \otimes H_{\beta-\varepsilon^{(j)}}\rangle \\ &= \sum_{|\alpha|>0}\sum_{|\beta|>0}\sum_{k\in\mathbb{N}}\sum_{j\in\mathbb{N}} \alpha_k\ \beta_j\ f_\alpha\, g_\beta\ \langle \xi_k, \xi_j\rangle \otimes\ H_{\alpha-\varepsilon^{(k)}} \cdot H_{\beta-\varepsilon^{(j)}} \\ &= \sum_{|\alpha|>0}\sum_{|\beta|>0}\sum_{k\in\mathbb{N}} \alpha_k\beta_k f_\alpha g_\beta \otimes \sum_{\gamma\le\min\{\alpha-\varepsilon^{(k)},\beta-\varepsilon^{(k)}\}} \gamma! \binom{\alpha-\varepsilon^{(k)}}{\gamma}\binom{\beta-\varepsilon^{(k)}}{\gamma} H_{\alpha+\beta-2\varepsilon^{(k)}-2\gamma},\end{aligned}$$

where we used the fact that $\langle \xi_k, \xi_j\rangle = 0$ for $k \neq j$ and $\langle \xi_k, \xi_j\rangle = 1$ for $k = j$. Now we put $\theta = \gamma + \varepsilon^{(k)}$ and use the identities

$$\alpha_k \cdot \binom{\alpha-\varepsilon^{(k)}}{\gamma} = \alpha_k \cdot \binom{\alpha-\varepsilon^{(k)}}{\theta-\varepsilon^{(k)}} = \theta_k \cdot \binom{\alpha}{\theta}, \quad k \in \mathbb{N},$$

and $\theta_k \cdot (\theta - \varepsilon^{(k)})! = \theta!$. Thus we obtain

$$\begin{aligned}\langle \mathbb{D}(F), \mathbb{D}(G)\rangle &= \sum_{\alpha\in\mathcal{I}}\sum_{\beta\in\mathcal{I}}\sum_{k\in\mathbb{N}} f_\alpha g_\beta \sum_{\theta\le\min\{\alpha,\beta\}} \theta_k^2\, (\theta-\varepsilon^{(k)})! \binom{\alpha}{\theta}\binom{\beta}{\theta} H_{\alpha+\beta-2\theta} \\ &= \sum_{\alpha\in\mathcal{I}}\sum_{\beta\in\mathcal{I}}\sum_{k\in\mathbb{N}} f_\alpha\, g_\beta \sum_{\theta\le\min\{\alpha,\beta\}} \theta_k \theta! \binom{\alpha}{\theta}\binom{\beta}{\theta} H_{\alpha+\beta-2\theta} \\ &= \sum_{\alpha\in\mathcal{I}}\sum_{\beta\in\mathcal{I}} f_\alpha\, g_\beta \sum_{\theta\le\min\{\alpha,\beta\}} \left(\sum_{k\in\mathbb{N}} \theta_k\right) \theta! \binom{\alpha}{\theta}\binom{\beta}{\theta} H_{\alpha+\beta-2\theta} \\ &= \sum_{\alpha\in\mathcal{I}}\sum_{\beta\in\mathcal{I}} f_\alpha\, g_\beta \sum_{\theta\le\min\{\alpha,\beta\}} |\theta|\, \theta! \binom{\alpha}{\theta}\binom{\beta}{\theta} H_{\alpha+\beta-2\theta}.\end{aligned}$$

Combining all previously obtained results we now have

$$\begin{aligned}\mathcal{R}(F\cdot G) &= \sum_{\alpha\in\mathcal{I}}\sum_{\beta\in\mathcal{I}} f_\alpha g_\beta \sum_{\gamma\le\min\{\alpha,\beta\}} \gamma! \binom{\alpha}{\gamma}\binom{\beta}{\gamma} (|\alpha|+|\beta|-2|\gamma|)\, H_{\alpha+\beta-2\gamma} \\ &= \sum_{\alpha\in\mathcal{I}}\sum_{\beta\in\mathcal{I}} f_\alpha\, g_\beta \sum_{\gamma\le\min\{\alpha,\beta\}} \gamma! \binom{\alpha}{\gamma}\binom{\beta}{\gamma} |\alpha|\ H_{\alpha+\beta-2\gamma} \\ &+ \sum_{\alpha\in\mathcal{I}}\sum_{\beta\in\mathcal{I}} f_\alpha\, g_\beta \sum_{\gamma\le\min\{\alpha,\beta\}} \gamma! \binom{\alpha}{\gamma}\binom{\beta}{\gamma} |\beta|\ H_{\alpha+\beta-2\gamma} \\ &- 2\sum_{\alpha\in\mathcal{I}}\sum_{\beta\in\mathcal{I}} f_\alpha\, g_\beta \sum_{\gamma\le\min\{\alpha,\beta\}} |\gamma|\, \gamma! \binom{\alpha}{\gamma}\binom{\beta}{\gamma} H_{\alpha+\beta-2\gamma} \\ &= \mathcal{R}(F)\cdot G + F\cdot\mathcal{R}(G) - 2\cdot\langle \mathbb{D}(F), \mathbb{D}(G)\rangle\end{aligned}$$

and thus (6.4) holds.

b) If $F, G \in Dom_-(\mathcal{R})$, then $\mathcal{R}(F), \mathcal{R}(G) \in X \otimes (S)_{-1}$. From Theorem 2.15 it follows that $\mathcal{R}(F)\lozenge G, \mathcal{R}(G)\lozenge F \in X \otimes (S)_{-1}$. Thus, the right hand side of (6.5) is in $X \otimes (S)_{-1}$ i.e. $F\lozenge G \in Dom_-(\mathcal{R})$.

From

$$G\lozenge\mathcal{R}(F) = \sum_{\gamma\in\mathcal{I}}\sum_{\alpha+\beta=\gamma} |\alpha| f_\alpha g_\beta \, H_\gamma,$$

$$F\lozenge\mathcal{R}(G) = \sum_{\gamma\in\mathcal{I}}\sum_{\alpha+\beta=\gamma} f_\alpha |\beta| g_\beta \, H_\gamma,$$

it follows that

$$G\lozenge\mathcal{R}(F) + F\lozenge\mathcal{R}(G) = \sum_{\gamma\in\mathcal{I}} |\gamma| \sum_{\alpha+\beta=\gamma} f_\alpha g_\beta \, H_\gamma = \mathcal{R}(F\lozenge G).$$

□

Corollary 6.7. *Let $F \in Dom_+(\mathcal{R})$ and $G \in Dom_-(\mathcal{R})$, or vice versa (including also the possibility $F, G \in Dom_0(\mathcal{R})$). Then the following property holds:*

$$E(F \cdot \mathcal{R}(G)) = E\left(\langle \mathbb{D}F, \mathbb{D}G\rangle\right).$$

Proof. From the chaos expansion form of $\mathcal{R}(F{\cdot}G)$ it follows that $E\mathcal{R}(F{\cdot}G) = 0$. Moreover, taking the expectations on both sides of (6.6) and (6.7) we obtain

$$E\left(\mathcal{R}(F) \cdot G\right) = E\left(F \cdot \mathcal{R}(G)\right).$$

Now, from Theorem 6.6 it follows that

$$0 = 2E\left(F \cdot \mathcal{R}(G)\right) - 2E(\langle \mathbb{D}F, \mathbb{D}G\rangle),$$

and the assertion follows. □

In the classical literature ([30, 36]) it is proven that the Malliavin derivative satisfies the product rule (with respect to ordinary multiplication) i.e. if $F, G \in Dom_0(\mathbb{D})$, then $F \cdot G \in Dom_0(\mathbb{D})$ and (6.8) holds. The following theorem recapitulates this result and extends it for generalized and test processes, and extends it also for Wick multiplication [1].

Theorem 6.8. *(Product rule for $\mathbb{D}$)*

a) Let $F \in Dom_-(\mathbb{D})$ and $G \in Dom_+(\mathbb{D})$ or vice versa. Then $F \cdot G \in Dom_-(\mathbb{D})$ and

$$\mathbb{D}(F \cdot G) = F \cdot \mathbb{D}G + \mathbb{D}F \cdot G. \tag{6.8}$$

b) Let $F, G \in Dom_-(\mathbb{D})$. Then $F\lozenge G \in Dom_-(\mathbb{D})$ and

$$\mathbb{D}(F\lozenge G) = F\lozenge\mathbb{D}G + \mathbb{D}F\lozenge G.$$

Proof. a)

$$\mathbb{D}(F\cdot G) = \mathbb{D}(\sum_{\alpha\in\mathcal{I}} f_\alpha H_\alpha \cdot \sum_{\beta\in\mathcal{I}} g_\beta H_\beta) =$$

$$\mathbb{D}\left(\sum_{\alpha\in\mathcal{I}}\sum_{\beta\in\mathcal{I}} f_\alpha\, g_\beta \sum_{\gamma\le\min\{\alpha,\beta\}} \gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma} H_{\alpha+\beta-2\gamma}\right) =$$

$$\sum_{\alpha\in\mathcal{I}}\sum_{\beta\in\mathcal{I}}\sum_{k\in\mathbb{N}} f_\alpha g_\beta \sum_{\gamma\le\min\{\alpha,\beta\}} \gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma}(\alpha_k+\beta_k-2\gamma_k)\,\xi_k H_{\alpha+\beta-2\gamma-\varepsilon^{(k)}}$$

On the other side we have

$$F\cdot\mathbb{D}(G) = \sum_{\alpha\in\mathcal{I}} f_\alpha H_\alpha \cdot \sum_{\beta\in\mathcal{I}}\sum_{k\in\mathbb{N}} \beta_k\, g_\beta \xi_k\, H_{\beta-\varepsilon^{(k)}} =$$

$$\sum_{\alpha\in\mathcal{I}}\sum_{\beta\in\mathcal{I}}\sum_{k\in\mathbb{N}} f_\alpha g_\beta \sum_{\gamma\le\min\{\alpha,\beta-\varepsilon^{(k)}\}} \gamma!\binom{\alpha}{\gamma}\binom{\beta-\varepsilon^{(k)}}{\gamma}\beta_k\,\xi_k H_{\alpha+\beta-2\gamma-\varepsilon^{(k)}}$$

and

$$G\cdot\mathbb{D}(F) = \sum_{\alpha\in\mathcal{I}}\sum_{\beta\in\mathcal{I}}\sum_{k\in\mathbb{N}} f_\alpha g_\beta \sum_{\gamma\le\min\{\alpha-\varepsilon^{(k)},\beta\}} \gamma!\binom{\alpha-\varepsilon^{(k)}}{\gamma}\binom{\beta}{\gamma}\alpha_k\,\xi_k H_{\alpha+\beta-2\gamma-\varepsilon^{(k)}}.$$

Summing up chaos expansions for $F\cdot\mathbb{D}(G)$ and $G\cdot\mathbb{D}(F)$ and applying the identities

$$\begin{aligned}\alpha_k\binom{\alpha-\varepsilon^{(k)}}{\gamma} &= \alpha_k\cdot\frac{(\alpha-\varepsilon^{(k)})!}{\gamma!\,(\alpha-\varepsilon^{(k)}-\gamma)!} = \frac{\alpha!}{\gamma!\,(\alpha-\gamma)!}\cdot(\alpha_k-\gamma_k)\\ &= \binom{\alpha}{\gamma}(\alpha_k-\gamma_k)\end{aligned}$$

and

$$\beta_k\binom{\beta-\varepsilon^{(k)}}{\gamma} = \binom{\beta}{\gamma}(\beta_k-\gamma_k),$$

for all $\alpha,\beta\in\mathcal{I}$, $k\in\mathbb{N}$ and $\gamma\in\mathcal{I}$ such that $\gamma\le\min\{\alpha,\beta\}$ and the expression $(\alpha_k-\gamma_k)+(\beta_k-\gamma_k)=\alpha_k+\beta_k-2\gamma_k$ we obtain (6.8).

From Theorem 2.16 it follows that all products on the right hand side of (6.8) are well defined, thus the right hand sifde of (6.8) is an element of $X\otimes S'(\mathbb{R})\otimes(S)_{-1}$. Thus, $F\cdot G\in Dom_-(\mathbb{D})$.

b) By definition of the Malliavin derivative and the Wick product it can be easily verified that

$$\begin{aligned}\mathbb{D}(F)\lozenge G + F\lozenge\mathbb{D}(G) &= \sum_{\gamma\in\mathcal{I}}\sum_{k=1}^{\infty}\sum_{\alpha+\beta-\varepsilon^{(k)}=\gamma}\alpha_k f_\alpha g_\beta H_\gamma + \sum_{\gamma\in\mathcal{I}}\sum_{k=1}^{\infty}\sum_{\alpha+\beta-\varepsilon^{(k)}=\gamma}\beta_k f_\alpha g_\beta H_\gamma\\ &= \sum_{\gamma\in\mathcal{I}}\sum_{k=1}^{\infty}\sum_{\alpha+\beta=\gamma}\gamma_k f_\alpha g_\beta H_{\gamma-\varepsilon^{(k)}} = \mathbb{D}(F\lozenge G).\end{aligned}$$

If $F,G \in Dom_-(\mathbb{D})$, then $\mathbb{D}(F),\mathbb{D}(G) \in X \otimes S'(\mathbb{R}) \otimes (S)_{-1}$. From Theorem 2.15 it follows that $\mathbb{D}(F)\lozenge G$ and $F\lozenge\mathbb{D}(G)$ both belong to $X \otimes S'(\mathbb{R}) \otimes (S)_{-1}$. Thus, $F\lozenge G \in Dom_-(\mathbb{D})$.

□

A generalization of Theorem 6.8 for higher order derivatives, i.e. the Leibnitz formula is given in the next theorem.

Theorem 6.9. *Let $F,G \in Dom_-(\mathbb{D}^{(k)})$, $k \in \mathbb{N}$, then $F\lozenge G \in Dom_-(\mathbb{D}^{(k)})$ and the Leibnitz rule holds:*

$$\mathbb{D}^{(k)}\left(F\lozenge G\right) = \sum_{i=0}^{k} \binom{k}{i} \mathbb{D}^{(i)}(F)\lozenge\mathbb{D}^{(k-i)}(G),$$

where $\mathbb{D}^{(0)}(F) = F$ and $\mathbb{D}^{(0)}(G) = G$.

Moreover, if $G \in Dom_+(\mathbb{D}^{(k)})$, then $F \cdot G \in Dom_-(\mathbb{D}^{(k)})$ and

$$\mathbb{D}^{(k)}\left(FG\right) = \sum_{i=0}^{k} \binom{k}{i} \mathbb{D}^{(i)}(F)\,\mathbb{D}^{(k-i)}(G). \tag{6.9}$$

Proof. The Leibnitz rule (6.9) follows by induction and applying Theorem 6.8. Clearly, (6.9) holds also if $F,G \in Dom_0(\mathbb{D}^{(k)})$. □

Theorem 6.10. *Assume that either of the following hold:*

- *$F \in Dom_-(\mathbb{D})$, $G \in Dom_+(\mathbb{D})$ and $u \in Dom_+(\delta)$,*
- *$F,G \in Dom_+(\mathbb{D})$ and $u \in Dom_-(\delta)$,*
- *$F,G \in Dom_0(\mathbb{D})$ and $u \in Dom_0(\delta)$.*

Then the second integration by parts formula holds:

$$E(F\langle\mathbb{D}G,u\rangle) + E(G\langle\mathbb{D}F,u\rangle) = E(F\,G\,\delta(u)). \tag{6.10}$$

Proof. The assertion (6.10) follows directly from the duality formula (6.1) and the product rule (6.8). Assume the first case holds when $F \in Dom_-(\mathbb{D})$, $G \in Dom_+(\mathbb{D})$ and $u \in Dom_+(\delta)$. Then $F \cdot G \in Dom_-(\mathbb{D})$, too, and we have

$$\begin{aligned} E(F\,G\,\delta(u)) &= E(\langle\mathbb{D}(F\cdot G),u\rangle) = E(\langle F\cdot\mathbb{D}(G) + G\cdot\mathbb{D}(F),u\rangle) \\ &= E(F\,\langle\mathbb{D}(G),u\rangle) + E(G\,\langle\mathbb{D}(F),u\rangle). \end{aligned}$$

The second and third case can be proven in an analogous way. □

The next theorem states the chain rule for the Malliavin derivative. The classical $(L)^2$-case has been known throughout the literature and its Wick-version was introduced in [1].

Theorem 6.11. *(Chain rule) Let ϕ be a twice continuously differentiable function with bounded derivatives.*

1. *If $F \in Dom_+(\mathbb{D})$ (resp. $F \in Dom_0(\mathbb{D})$), then $\phi(F) \in Dom_+(\mathbb{D})$ (resp. $\phi(F) \in Dom_0(\mathbb{D})$) and the chain rule holds:*

$$\mathbb{D}\left(\phi(F)\right) = \phi'(F) \cdot \mathbb{D}(F). \tag{6.11}$$

2. *If $F \in Dom_-(\mathbb{D})$ and ϕ is analytic, then $\phi^\Diamond(F) \in Dom_-(\mathbb{D})$ and*

$$\mathbb{D}\left(\phi^\Diamond(F)\right) = \phi'^\Diamond(F)\Diamond\mathbb{D}(F). \tag{6.12}$$

Proof. First we prove that (6.11) holds true when ϕ is a polynomial of degree n, $n \in \mathbb{N}$. Then we use the Stone-Weierstrass theorem and approximate a continuously differentiable function ϕ by a polynomial $\widetilde{p}_n$ of degree n, and since we assumed that ϕ is regular enough, $\widetilde{p}'_n$ will also approximate ϕ'.
(i) Denote by $q_n(x) = x^n$, $n \in \mathbb{N}$ and let $p(x) = \sum_{k=0}^{n} a_k q_k(x) = \sum_{k=0}^{n} a_k\, x^k$ be a polynomial of degree n with real coefficients a_0, a_1, ... , a_n, and $a_n \neq 0$. By induction on n, we prove the chain rule for q_n, i.e. we prove

$$\mathbb{D}\left(p_n(F)\right) = p'_n(F) \cdot \mathbb{D}(F), \quad n \in \mathbb{N}. \tag{6.13}$$

For $n = 1$, $q_1(x) = x$ and (6.13) holds since

$$\mathbb{D}(q_1(F)) = \mathbb{D}(F) = 1 \cdot \mathbb{D}(F) = q'_1(F) \cdot \mathbb{D}(F).$$

Assume (6.13) holds for $k \in \mathbb{N}$. Then, for $q_{k+1} = x^{k+1}$ by Theorem 6.8 we have

$$\begin{aligned}\mathbb{D}(q_{k+1}(F)) &= \mathbb{D}(F^{k+1}) = \mathbb{D}(F \cdot F^k)\\ &= \mathbb{D}(F) \cdot F^k + F \cdot \mathbb{D}(F^k) = \mathbb{D}(F) \cdot F^k + F \cdot kF^{k-1} \cdot \mathbb{D}(F)\\ &= (k+1)F^k \cdot \mathbb{D}(F) = q'_{k+1}(F) \cdot \mathbb{D}(F).\end{aligned}$$

Thus, (6.13) holds for every $n \in \mathbb{N}$.
Since $\mathbb{D}$ is a linear operator, (6.13) holds for any polynomial p_n, i.e.

$$\mathbb{D}(p_n(F)) = \sum_{k=0}^{n} a_k \mathbb{D}(q_k(F)) = \sum_{k=0}^{n} a_k q'_k(F) \cdot \mathbb{D}(F) = p'_n(F) \cdot \mathbb{D}(F).$$

(ii) Let $\phi \in C^2(\mathbb{R})$ and $F \in Dom_p(\mathbb{D})$, $p \in \mathbb{N}$. Then, by the Stone–Weierstrass theorem, there exists a polynomial $\widetilde{p_n}$ such that

$$\|\phi(F) - \widetilde{p_n}(F)\|_{X\otimes(S)_{1,p}} = \|\phi(F) - \sum_{k=0}^{n} a_k F^k\|_{X\otimes(S)_{1,p}} \to 0$$

and

$$\|\phi'(F) - \widetilde{p_n}'(F)\|_{X\otimes(S)_{1,p}} = \|\phi'(F) - \sum_{k=1}^{n} a_k k F^{k-1}\|_{X\otimes(S)_{1,p}} \to 0$$

as $n \to \infty$.

We denote by $\mathcal{X}_{lp} = X \otimes S_l(\mathbb{R}) \otimes (S)_{1,p}$. From (6.13) and the fact that $\mathbb{D}$ is a bounded operator (Theorem 2.19) we obtain (for $l < p-1$)

$$\begin{aligned}
&\|\mathbb{D}(\phi(F)) - \phi'(F) \cdot \mathbb{D}(F)\|_{X\otimes S_l(\mathbb{R})\otimes(S)_{1,p}} = \|\mathbb{D}(\phi(F)) - \phi'(F) \cdot \mathbb{D}(F)\|_{\mathcal{X}_{lp}} \\
&= \|\mathbb{D}(\phi(F)) - \mathbb{D}(\widetilde{p_n}(F)) + \mathbb{D}(\widetilde{p_n}(F)) - \phi'(F) \cdot \mathbb{D}(F)\|_{\mathcal{X}_{lp}} \\
&\leq \|\mathbb{D}(\phi(F)) - \mathbb{D}(\widetilde{p_n}(F))\|_{\mathcal{X}_{lp}} + \|\mathbb{D}(\widetilde{p_n}(F)) - \phi'(F)\mathbb{D}(F)\|_{\mathcal{X}_{lp}} \\
&= \|\mathbb{D}(\phi(F) - \widetilde{p_n}(F))\|_{\mathcal{X}_{lp}} + \|\widetilde{p_n}'(F)\mathbb{D}(F) - \phi'(F)\mathbb{D}(F)\|_{\mathcal{X}_{lp}} \\
&\leq \|\mathbb{D}\| \cdot \|(\phi(F) - \widetilde{p_n}(F))\|_{X\otimes(S)_{1,p}} + \|\widetilde{p_n}'(F) - \phi'(F)\| \cdot \|\mathbb{D}(F)\|_{X\otimes(S)_{1,p}} \to 0,
\end{aligned}$$

as $n \to \infty$. From this follows (6.11) as well as the estimate

$$\|\mathbb{D}(\phi(F))\|_{X\otimes S_l(\mathbb{R})\otimes(S)_{1,p}} \leq \|\phi'(F)\|_{X\otimes(S)_{1,p}} \cdot \|\mathbb{D}(F)\|_{X\otimes S_l(\mathbb{R})\otimes(S)_{1,p}} < \infty,$$

and thus $\phi(F) \in Dom_p(\mathbb{D})$.

(iii) The proof of (6.12) for the Wick version can be conducted in a similar manner. According to Theorem 6.8 we have

$$\mathbb{D}(F^{\lozenge k}) = k\ F^{\lozenge(k-1)} \lozenge \mathbb{D}(F).$$

If ϕ is an analytic function given by $\phi(x) = \sum_{k=0}^{\infty} a_k x^k$, then $\phi'(x) = \sum_{k=1}^{\infty} a_k k x^{k-1}$, thus

$$\phi^{\lozenge}(F) = \sum_{k=0}^{\infty} a_k F^{\lozenge k}, \quad \phi'^{\lozenge}(F) = \sum_{k=1}^{\infty} a_k k F^{\lozenge(k-1)}.$$

Thus,

$$\mathbb{D}(\phi^{\lozenge}(F)) = \sum_{k=0}^{\infty} a_k \mathbb{D}(F^{\lozenge k}) = \sum_{k=0}^{\infty} a_k k F^{\lozenge(k-1)} \lozenge \mathbb{D}(F) = \phi'^{\lozenge}(F) \lozenge \mathbb{D}(F).$$

□

Example 6.12. For example, $\mathbb{D}(B_{t_0}^2) = 2B_{t_0} \cdot \mathbb{D}(B_{t_0}) = 2B_{t_0} \cdot \kappa_{[0,t_0]}(t)$, $\mathbb{D}(B_{t_0}^{\lozenge 2}) = 2B_{t_0} \cdot \kappa_{[0,t_0]}(t)$ and $\mathbb{D}(W_{t_0}^{\lozenge 2}) = 2W_{t_0} \lozenge \mathbb{D}(W_{t_0}) = 2W_{t_0} \cdot d_{t_0}(t)$, since the Wick product reduces to the ordinary product if one of the multiplicands is deterministic. This is in compliance with Example 4.4 and Example 5.6.

Also, $\mathbb{D}(\exp^{\lozenge}(W_{t_0})) = \exp^{\lozenge}(W_{t_0}) \cdot d_{t_0}(t)$, or more general $\mathbb{D}(\exp^{\lozenge} \delta(h)) = \exp^{\lozenge} \delta(h) \cdot h$ for any $h \in S'(\mathbb{R})$, which verifies once again that the stochastic exponentials are eigenvectors of the Malliavin derivative (see Remark 4.10).

Example 6.13. Geometric Brownian motion is defined by

$$G_{t_0} = G_0 \cdot e^{(\mu - \frac{1}{2}\sigma^2)t_0 + \sigma B_{t_0}},$$

for some constants $\mu, \sigma > 0$. Then,

$$\begin{aligned}
\mathbb{D}G_{t_0} &= G_0 \cdot e^{(\mu-\frac{1}{2}\sigma^2)t_0} \cdot \mathbb{D}(e^{\sigma B_{t_0}}) = G_0 \cdot e^{(\mu-\frac{1}{2}\sigma^2)t_0} \cdot \sigma \cdot e^{\sigma B_{t_0}} \cdot \mathbb{D}(B_{t_0}) \\
&= G_0 \cdot e^{(\mu-\frac{1}{2}\sigma^2)t_0} \cdot \sigma \cdot e^{\sigma B_{t_0}} \cdot \kappa_{[0,t_0]}(t) = \sigma \cdot G_{t_0} \cdot \kappa_{[0,t_0]}(t) \\
&= \begin{cases} \sigma \cdot G_{t_0}, & t \in [0, t_0] \\ 0, & t \notin [0, t_0] \end{cases}.
\end{aligned}$$

7. Applications of the Malliavin calculus

One of the first and most important applications of the Malliavin calculus concerns the existence and smoothness of a density for the probability law of random variables. Other, more recent applications in finance ([2, 30, 37]) have been developed for option pricing and computing greeks (greeks measure the stability of the option price under variations of the parameters) via the Clark-Ocone formula. A few years ago it was also discovered that Malliavin calculus is in a close relationship with Stein's method and can be used for estimating the distance of a random variable from Gaussian variables.

In this section we provide an overwiev of some applications and capabilities of the Malliavin calculus.

For simplicity we will assume that $X = \mathbb{R}$.

7.1. Measurability and densities

Let $A \in \mathcal{B}$ be a Borel set in $S'(\mathbb{R})$. Denote by κ_A its indicator function i.e. the random variable $\kappa_A(\omega) = 1$ for $\omega \in A$ and $\kappa_A(\omega) = 0$ for $\omega \in A^c$. Then $\kappa_A = \sum_{\alpha\in\mathcal{I}} a_\alpha H_\alpha$, where $a_\alpha = E(\kappa_A \cdot H_\alpha)$, $\alpha \in \mathcal{I}$. Especially, $a_{\mathbf{0}} = E(\kappa_A) = P(A)$.

Proposition 7.1. ([36]) *$\kappa_A \in Dom_0(\mathbb{D})$ if and only if $P(A) = 0$ or $P(A) = 1$.*

Proof. Since $E(\kappa_A) = P(A)$, the chaos expanasion of the indicator function is $\kappa_A = P(A) + \sum_{\alpha>\mathbf{0}} a_\alpha H_\alpha$, $a_\alpha = E(H_\alpha \kappa_A)$

Assume first that $P(A) \in \{0, 1\}$. Then $\kappa_A = const$ a.e. (it is either 0 or 1 a.e.), thus $a_\alpha = 0$ for all $\alpha > \mathbf{0}$. Clearly, (2.8) is satisfied and $\kappa_A \in Dom_0(\mathbb{D})$.

It remains to prove the other direction, that $\sum_{\alpha>\mathbf{0}} |\alpha|\alpha!|a_\alpha|^2$ cannot be finite unless $a_\alpha = 0$ for all $\alpha > \mathbf{0}$.

Assume $\kappa_A \in Dom_0(\mathbb{D})$. Let $\phi \in C_0^\infty(\mathbb{R})$ be such that $\phi(t) = t^2$ for $t \in [-1, 1]$. According to Theorem 6.11 we have

$$\mathbb{D}(\phi(\kappa_A)) = \phi'(\kappa_A)\mathbb{D}(\kappa_A).$$

Since $\phi(\kappa_A) = \kappa_A^2 = \kappa_A$, it follows that

$$\mathbb{D}(\kappa_A) = 2 \cdot \kappa_A \cdot \mathbb{D}(\kappa_A).$$

Thus both for $\omega \in A$ and for $\omega \in A^c$ we obtain $\mathbb{D}(\kappa_A) = 0$. Now, from Corollary 4.2 it follows that $\kappa_A(\omega) = const$ for a.e. $\omega \in \Omega$. For the chaos expansion of κ_A this means that $\kappa_A = E(\kappa_A) = P(A)$ a.e. and $a_\alpha = 0$ for all $\alpha > \mathbf{0}$ and $const = P(A)$. This implies that $P(A)$ is either zero or one. □

Remark 7.2. If $P(A) \in (0, 1)$, then $\kappa_A \notin Dom_0(\mathbb{D})$. For example, $f(\omega) = \kappa_{\{B_t(\omega)>0\}} \notin Dom_0(\mathbb{D})$ since $P\{B_t > 0\} \in (0, 1)$.

On the other hand, $\kappa_A \in Dom_-(\mathbb{D})$ regardless of the value of $P(A)$. This follows from $a_\alpha = E(\kappa_A\, H_\alpha) \le E(H_\alpha) \le 1$, thus

$$\|\kappa_A\|^2_{Dom_{-p}(\mathbb{D})} \le \sum_{\alpha>\mathbf{0}} |\alpha|^2 (2\mathbb{N})^{-p\alpha} \le \sum_{\alpha>\mathbf{0}} (2\mathbb{N})^{-(p-2)\alpha} < \infty, \quad p > 3.$$

Remark 7.3. Let A be a closed subspace of $S'(\mathbb{R})$. Denote by $\sigma[A]$ the sub-σ-algebra of $\mathcal{B}$ generated by A. A random variable f is measurable with respect to $\sigma[A]$ if and only if $\mathbb{D}(f) = 0$ a.e. on A^c.

In particular it can be proven ([3, 18, 36]), that if a stochastic process f_t is adapted to the Brownian filtration $A_t = \sigma[B_s : s \leq t]$, then *supp* $\mathbb{D}(f_t) = [0, t]$ i.e. $\mathbb{D}f_t = \sum_{\alpha\in\mathcal{I}} \sum_{k\in\mathbb{N}} \alpha_k f_\alpha(t) \otimes \xi_k(s) \otimes H_{\alpha-\varepsilon^{(k)}} = 0$ for $s > t$.

Remark 7.4. Let $h \in L^2(\mathbb{R})$ and

$$M(s) = \exp^{\Diamond} \delta(h\kappa_{[0,s]}) = \exp\left(\int_0^s h(t)dB_t - \frac{1}{2}\int_0^s h^2(t)dt\right), \quad s \geq 0,$$

be the stochastic exponential of $h\kappa_{[0,s]}$. According to Remark 4.10 it is an eigenvector of the Malliavin derivative, thus $\mathbb{D}(M(s)) = h(t)\kappa_{[0,s]}(t)M(s)$, i.e.

$$\mathbb{D}(M(s)) = h(t)M(s), \qquad \text{for} \quad t \in [0, s].$$

It is known ([2, 36]) that $M(s)$ is a martingale with respect to the Brownian filtration, thus for $0 \leq t \leq s$ we have

$$E(\mathbb{D}M(s)|A_t) = E(h(t)M(s)|A_t) = h(t)E(M(s)|A_t) = h(t)M(t).$$

On the other hand, from Corollary 5.3 it follows that $M(s) = E(M) + \delta(u)$ for $u = \mathbb{D}(\mathcal{R}^{-1}(M - EM))$. Since $\delta(h(t)\kappa_{[0,s]}M(s)) = \delta(\mathbb{D}(M(s))) = \mathcal{R}(M(s))$, it follows that $u = h(t)\kappa_{[0,s]}M(s)$, i.e.

$$\begin{aligned} M(s) &= E(M) + \int_0^s h(t)M(t)dB_t \\ &= E(M) + \int_0^s E(\mathbb{D}M(s)|A_t)\, dB_t. \end{aligned}$$

Since the stochastic exponentials are dense in $(L)^2$ it follows that the latter formula can be extended to all $M \in Dom_0(\mathbb{D})$. This result is known as the Clark-Ocone formula.

Theorem 7.5. *(Clark-Ocone formula) Let $F \in Dom_0(\mathbb{D})$ be adapted to the Brownian filtration. Then,*

$$F(s) = E(F) + \int_0^s E(\mathbb{D}F(s)|A_t)\, dB_t.$$

Example 7.6.

$$B_T^2 = T + \int_0^T E(\mathbb{D}B_T^2|A_t)dB_t = T + \int_0^T E(2B_T\kappa_{[0,T]}|A_t)dB_t = T + 2\int_0^T B_t dB_t,$$

by the martingale property of Brownian motion.

Remark 7.7. For the stochastic exponential M it also holds that

$$\begin{aligned}\mathbb{D}(E(M(s)|A_t)) &= \mathbb{D}(M(t)) = h(x)\kappa_{[0,t]}M(t)\\ &= h(x)\kappa_{[0,t]}E(M(s)|A_t) = \kappa_{[0,t]}E(h(x)\kappa_{[0,s]}M(s)|A_t)\\ &= \kappa_{[0,t]}E(\mathbb{D}M(s)|A_t),\end{aligned}$$

for $0 \le t \le s$. This result extends to all adapted processes: if $F \in Dom_0(\mathbb{D})$ is adapted, then $E(F|A_t) \in Dom_0(\mathbb{D})$ and

$$\mathbb{D}(E(F(s)|A_t)) = \kappa_{[0,t]}E(\mathbb{D}F(s)|A_t).$$

In the sequel we are going to show that absolutely continuous distributions can be characterized via the Malliavin derivative and there exists an explicit formula for the density of the distribution. For this purpose we note that $\|\mathbb{D}F\|^2_{L^2(\mathbb{R})} = \langle \mathbb{D}F, \mathbb{D}F\rangle_{L^2(\mathbb{R})}$ is an element in $(L)^2$. If F is of the form $F = \sum_{\alpha\in\mathcal{I}} f_\alpha H_\alpha$, then $\|\mathbb{D}F\|^2_{L^2(\mathbb{R})} = \sum_{k\in\mathbb{N}}\left(\sum_{\alpha\in\mathcal{I}} f_{\alpha+\varepsilon^{(k)}}(\alpha_k+1)H_\alpha\right)^2$.

Theorem 7.8. ([18]) *Let $F \in Dom_0(\mathbb{D})$ be such that $\|\mathbb{D}F\|_{L^2(\mathbb{R})} \neq 0$ a.e. and $\frac{\mathbb{D}F}{\|\mathbb{D}F\|^2} \in Dom_0(\delta)$. Then for every $\phi \in C_0^2(\mathbb{R})$,*

$$E(\phi'(F)) = E\left(\phi(F)\cdot\delta\left(\frac{\mathbb{D}F}{\|\mathbb{D}F\|^2_{L^2(\mathbb{R})}}\right)\right). \tag{7.1}$$

Moreover, F is an absolutely continuous random variable and its density φ is given by

$$\varphi(t) = E\left(\kappa_{\{F>t\}}\cdot\delta\left(\frac{\mathbb{D}F}{\|\mathbb{D}F\|^2_{L^2(\mathbb{R})}}\right)\right). \tag{7.2}$$

Proof. Using the chain rule (Theorem 6.11) and the duality relationship (Theorem 6.1) we obtain

$$\begin{aligned}E(\phi'(F)) &= E\left(\frac{\phi'(F)}{\langle u,\mathbb{D}F\rangle}\cdot\langle u,\mathbb{D}F\rangle\right) = E\left(\langle\frac{u}{\langle u,\mathbb{D}F\rangle},\phi'(F)\mathbb{D}F\rangle\right)\\ &= E\left(\langle\frac{u}{\langle u,\mathbb{D}F\rangle},\mathbb{D}(\phi(F))\rangle\right) = E\left(\delta\left(\frac{u}{\langle u,\mathbb{D}F\rangle}\right)\cdot\phi(F)\right)\end{aligned}$$

holds for any $u \in Dom_0(\delta)$. Especially, for $u = \mathbb{D}F$ we obtain (7.1).

Putting $\phi(x) = \int_{-\infty}^x \kappa_{(a,b)}(s)ds$, $\phi'(x) = \kappa_{(a,b)}(x)$ into (7.1) (in fact we approximate $\kappa_{(a,b)}$ with a sequence of smooth functions) we obtain by Fubini's theorem that

$$\begin{aligned}P\{a<F<b\} &= E\left(\int_{-\infty}^F \kappa_{(a,b)}(s)ds\cdot\delta\left(\frac{\mathbb{D}F}{\|\mathbb{D}F\|^2_{L^2(\mathbb{R})}}\right)\right)\\ &= \int_a^b\left(\kappa_{\{F>s\}}\cdot\delta\left(\frac{\mathbb{D}F}{\|\mathbb{D}F\|^2_{L^2(\mathbb{R})}}\right)\right)ds,\end{aligned}$$

which proves (7.2).

□

Example 7.9. Let $F \in (L)^2$ be a standardized Gaussian random variable in $\mathcal{H}_1$ with chaos expansion $F = \sum_{j=1}^{\infty} f_j H_{\varepsilon^{(j)}}$, $\sum_{j=1}^{\infty} |f_j|^2 = 1$. Then $\mathbb{D}F = \sum_{j=1}^{\infty} f_j \xi_j \in \mathcal{H}_0$ and $\|\mathbb{D}F\|^2_{L^2(\mathbb{R})} = 1$. Also, $\delta(\mathbb{D}F) = \mathcal{R}(F) = F$ since it is a fixed point of the Ornstein-Uhlenbeck operator. Thus, by (7.2) the density is given by $\varphi(t) = E(\kappa_{\{F>t\}}F)$. Indeed, it is easy to verify that $\int_t^{\infty} x \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}$.

7.2. Gaussian approximations

In this section we present some results obtained by combining the Malliavin calculus with Stein's method as recently investigated in [35]. It is well-known that a random variable N has $\mathcal{N}(0,1)$ distribution if and only if

$$E\left(N \cdot F(N) - F'(N)\right) = 0,$$

for every smooth function F. Thus, according to Stein's lemma [4], one can measure the distance to $N \sim \mathcal{N}(0,1)$, for an arbitrary random variable Z by measuring the expectation of $Z \cdot F(Z) - F'(Z)$. We will show using Malliavin calculus that

$$E\left(Z \cdot F(Z)\right) = E\left(F'(Z)\,\langle \mathbb{D}Z, \mathbb{D}\mathcal{R}^{-1} Z\rangle\right)$$

holds for every $F \in C^2(\mathbb{R})$. Thus, in order to measure the distance to $N \sim \mathcal{N}(0,1)$, one needs to estimate

$$E|1 - \langle \mathbb{D}Z, \mathbb{D}\mathcal{R}^{-1} Z\rangle|, \tag{7.3}$$

where $E|1 - \langle \mathbb{D}Z, \mathbb{D}\mathcal{R}^{-1} Z\rangle| = 0$ if and only if $Z \sim \mathcal{N}(0,1)$.

Theorem 7.10. *Let $f \in Dom_+(\mathbb{D})$ or $f \in Dom_0(\mathbb{D})$ such that $E(f) = 0$ and let $F \in C^2(\mathbb{R})$. Then*

$$E\left(f \cdot F(f)\right) = E\left(F'(f) \cdot \langle \mathbb{D}f, \mathbb{D}\mathcal{R}^{-1} f\rangle\right).$$

Proof. Since $Ef = 0$ from (3.1) it follows that $\mathcal{R}\mathcal{R}^{-1}f = f$. Therefore, by the duality formula (6.1) and Theorem 6.11 we have

$$\begin{aligned} E\left(f \cdot F(f)\right) &= E\left(\mathcal{R}\mathcal{R}^{-1}(f) \cdot F(f)\right) = E\left(\delta\mathbb{D}\mathcal{R}^{-1}(f) \cdot F(f)\right) \\ &= E\left(\langle \mathbb{D}F(f), \mathbb{D}\mathcal{R}^{-1} f\rangle\right) = E\left(F'(f) \cdot \langle \mathbb{D}(f), \mathbb{D}\mathcal{R}^{-1}(f)\rangle\right). \end{aligned}$$

□

An immediate consequence of Theorem 7.10 and Stein's lemma is the fact that $\langle \mathbb{D}f, \mathbb{D}\mathcal{R}^{-1}f\rangle = 1$ implies that $f \sim \mathcal{N}(0,1)$. Moreover, we can prove that in this case f belongs to the first order chaos space.

Theorem 7.11. *Let $f \in Dom_+(\mathbb{D})$ or $f \in Dom_0(\mathbb{D})$ such that $E(f) = 0$ and $\langle \mathbb{D}f, \mathbb{D}\mathcal{R}^{-1}f\rangle = 1$. Then $f \in \mathcal{H}_1$, $\|f\|^2_{(L)^2} = 1$ and $f \sim \mathcal{N}(0,1)$.*

Proof. Let $\langle \mathbb{D}f, \mathbb{D}\mathcal{R}^{-1}f\rangle = 1$. Assume that f has chaos expansion representation $f = \sum_{\alpha\in\mathcal{I}} f_\alpha H_\alpha$.

From Theorem 5.1 follows that the equation $\delta(u) = f$, for $Ef = 0$ has a unique solution $u = \mathbb{D}\mathcal{R}^{-1} f$ and it is of the form (5.2).

Thus,

$$\begin{aligned}
1 &= \langle \mathbb{D}f, \mathbb{D}\,\mathcal{R}^{-1}\, f\rangle \\
&= \langle \sum_{\alpha\in\mathcal{I}}\sum_{k\in\mathbb{N}}(\alpha_k+1)\, f_{\alpha+\varepsilon^{(k)}}\ \xi_k\otimes H_\alpha, \sum_{\beta\in\mathcal{I}}\sum_{j\in\mathbb{N}}(\beta_j+1)\,\frac{f_{\beta+\varepsilon^{(j)}}}{|\beta+\varepsilon^{(j)}|}\ \xi_j\otimes H_\beta\rangle \\
&= \sum_{\alpha\in\mathcal{I}}\sum_{\beta\in\mathcal{I}}\sum_{k\in\mathbb{N}}(\alpha_k+1)\, f_{\alpha+\varepsilon^{(k)}}\cdot\frac{f_{\beta+\varepsilon^{(k)}}}{|\beta+\varepsilon^{(k)}|}(\beta_k+1)\sum_{\gamma\le\min\{\alpha,\beta\}}\gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma}H_{\alpha+\beta-2\gamma}.
\end{aligned}$$

The latter expression can be equal to one if and only if its expectation is equal to one, and all higher order coefficients in the chaos expansion are equal to zero.

Thus, $E(\langle \mathbb{D}f, \mathbb{D}\,\mathcal{R}^{-1}\, f\rangle) = 1$ implies (for $\alpha=\beta=\gamma$) that

$$\begin{aligned}
\sum_{\alpha\in\mathcal{I}}\sum_{k\in\mathbb{N}}\frac{(\alpha_k+1)^2}{|\alpha+\varepsilon^{(k)}|}\cdot f^2_{\alpha+\varepsilon^{(k)}}\cdot\alpha! &= \sum_{\alpha\in\mathcal{I}}\sum_{k\in\mathbb{N}}\frac{(\alpha_k+1)}{|\alpha|+1}\cdot(\alpha+\varepsilon^{(k)})!f^2_{\alpha+\varepsilon^{(k)}} \\
&= \sum_{\alpha\in\mathcal{I}}\sum_{k\in\mathbb{N}}\frac{\alpha_k}{|\alpha|}\cdot\alpha!f_\alpha^2 \\
&= \sum_{\alpha\in\mathcal{I}}\frac{\alpha!}{|\alpha|}\left(\sum_{k\in\mathbb{N}}\alpha_k\right)f_\alpha^2 = \sum_{\alpha\in\mathcal{I}}\alpha!f_\alpha^2 \\
&= \|f\|^2_{(L)^2} = 1.
\end{aligned}$$

On the other hand, all higher order coefficients have to be equal to zero, which leaves only the possibility that

$$f_{\alpha+\varepsilon^{(k)}} = 0, \quad \text{for all } |\alpha| > 0,$$

i.e. $f_\alpha = 0$ for all $|\alpha|\ge 2$. Thus, $f\in\mathcal{H}_1$. According to Theorem 2.2 this implies that f is Gaussian. □

Corollary 7.12. *Let $f\in Dom_+(\mathbb{D})$ or $f\in Dom_0(\mathbb{D})$ such that $E(f)=0$ and $\langle\mathbb{D}f,\mathbb{D}\mathcal{R}^{-1}f\rangle = \sigma^2$. Then $f\in\mathcal{H}_1$, $\|f\|^2_{(L)^2} = \sigma^2$ and $f\sim\mathcal{N}(0,\sigma^2)$.*

We extend the previous theorem also for generalized random variables (e.g. the white noise process at a fixed time point). These processes have an infinite variance (infinite $(L)^2$ norm) and they can be regarded as elements of the Kondratiev spaces. Recall that $\langle\cdot,\cdot\rangle_{-p}$ denotes the scalar product in the Schwartz space $S_{-p}(\mathbb{R})$.

Theorem 7.13. *Let $f\in Dom_{-p}(\mathbb{D})$ and $E(f)=0$. Assume that*

$$\langle\mathbb{D}f,\mathbb{D}\mathcal{R}^{-1}f\rangle_{-p} = C(p)$$

for some constant $C(p)$. Then $f \in \mathcal{H}_1$, $C(p) = \|f\|^2_{(S)_{-1,-p}}$ and f has a generalized Gaussian distribution.

Proof. Similarly as in the proof of Theorem 7.11 we assume that f is of the form $f = \sum_{\alpha\in\mathcal{I}} f_\alpha H_\alpha$. From

$$\begin{aligned} C(p) &= \langle \mathbb{D}f, \mathbb{D}\,\mathcal{R}^{-1} f\rangle_{-p} \\ &= \sum_{\alpha\in\mathcal{I}}\sum_{k\in\mathbb{N}}(\alpha_k+1)\, f_{\alpha+\varepsilon^{(k)}} H_\alpha \sum_{\beta\in\mathcal{I}}\sum_{j\in\mathbb{N}}(\beta_j+1)\,\frac{f_{\beta+\varepsilon^{(j)}}}{|\beta+\varepsilon^{(j)}|} H_\beta\ \langle \xi_k,\xi_j\rangle_{-p} \\ &= \sum_{\alpha\in\mathcal{I}}\sum_{\beta\in\mathcal{I}}\sum_{k\in\mathbb{N}}(\alpha_k+1) f_{\alpha+\varepsilon^{(k)}} \frac{f_{\beta+\varepsilon^{(k)}}}{|\beta+\varepsilon^{(k)}|}(\beta_k+1)(2k)^{-p}\sum_{\gamma\le\min\{\alpha,\beta\}}\gamma!\binom{\alpha}{\gamma}\binom{\beta}{\gamma}H_{\alpha+\beta-2\gamma} \end{aligned}$$

follows that

$$\begin{aligned} C(p) &= \sum_{\alpha\in\mathcal{I}}\sum_{k\in\mathbb{N}}\frac{(\alpha_k+1)^2}{|\alpha+\varepsilon^{(k)}|} f^2_{\alpha+\varepsilon^{(k)}}\alpha!(2k)^{-p} \\ &= \sum_{\alpha\in\mathcal{I}}\frac{\alpha!}{|\alpha|} f^2_\alpha \sum_{k=1}^{\infty}\alpha_k(2k)^{-p} \end{aligned}$$

and $f_\alpha = 0$ for all $|\alpha| \ge 2$ i.e. $f \in \mathcal{H}_1$. Thus,

$$\begin{aligned} C(p) &= \sum_{j=1}^{\infty}\frac{\varepsilon^{(j)}!}{|\varepsilon^{(j)}|} f^2_{\varepsilon^{(j)}}\sum_{k=1}^{\infty}\delta_{kj}(2k)^{-p} = \sum_{j=1}^{\infty} f^2_{\varepsilon^{(j)}}(2j)^{-p} \\ &= \sum_{j=1}^{\infty} f^2_{\varepsilon^{(j)}}(2\mathbb{N})^{-p\varepsilon^{(j)}} = \|f\|^2_{(S)_{-1,-p}}, \end{aligned}$$

where $\delta_{kj} = 0$, $k \ne j$ and $\delta_{kj} = 1$, $k = j$ is the Kronecker symbol. Thus, $f \in \mathcal{H}_1$ and by Theorem 2.2 it follows that f is Gaussian.

□

Example 7.14. White noise is a generalized Gaussian process. For each fixed time point t_0 we have $\|W_{t_0}\|^2_{(S)_{-1,-p}} = \sum_{j=1}^{\infty}|\xi_j(t_0)|^2(2j)^{-p} < \infty$, for $p \ge 1$ by boundedness of the Hermite functions: $\sup_{t\in\mathbb{R}}|\xi_n(t)| \le Cn^{-\frac{1}{12}}$, $n \in \mathbb{N}$.

Remark 7.15. Although all elements in $\mathcal{H}_0 \oplus \mathcal{H}_1$ are Gaussian, the converse is in general not true. Gaussian random variables or processes can have a chaos expansion involving higher order chaotic coefficients as shown in the next example. This is in compliance with the central limit theorem that ensures that partial sums of various distributions may converge to a Gaussian distribution. Therefore it is of great importance to measure the distance to Gaussianity which is successfully done by combining the Stein method and the Malliavin calculus.

Example 7.16. The process $X_t = \int_0^t \mathrm{sign}(B_s)dB_s$ is Gaussian by Lévy's characterization of Brownian motion, but it has a chaos expansion involving infinitely many components from $\mathcal{H}_j$, $j \geq 2$. This was proven in [14].

Theorem 7.17. ([33]) *Let* $Z \in Dom_+(\mathbb{D})$ *or* $Z \in Dom_0(\mathbb{D})$ *be such that* $E(Z) = 0$ *and* $Var(Z) = 1$*. Then the expectation* (7.3) *satisfies*

$$E\left(|1 - \langle \mathbb{D}Z, \mathbb{D}\mathcal{R}^{-1} Z\rangle|\right) \leq \sqrt{Var\left(\langle \mathbb{D}Z, \mathbb{D}\mathcal{R}^{-1} Z\rangle\right)}.$$

Proof. The assertion follows directly from $E(Y)^2 \leq E(Y^2)$, i.e. $E(Y) \leq \sqrt{Var(Y)}$ and from $Var(1-U) = Var(U)$. □

Thus, in order to measure how close is Z to being normally distributed, one has to estimate how close is $Var\left(\langle \mathbb{D}Z, \mathbb{D}\mathcal{R}^{-1} Z\rangle\right)$ to zero. This quantity is larger than the Kolmogorov distance, but nevertheless still a good approximation.

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