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ON FRAMES, DUAL FRAMES, AND THE DUALITY PRINCIPLE

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Dedicated to Professor B. Stanković on the occasion of his 90th birthday

Abstract. Frames in Hilbert spaces generalize orthonormal bases and allow stable representation of all the elements of the space via a given frame and its dual frame. Frames are not only interesting from theoretical point of view, but play significant role in signal and image processing, which leads to many applications in informatics, engineering, medicine, and many other fields. In this paper we give a short survey on the theory for frames in Hilbert spaces, with focus on the duality principle and related open problems. It is the author's hope that this presentation will contribute to the solution of some of the deep problems that remain open despite the intensive development of frame theory during the last decade.

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1. Introduction and Notation

The purpose of this paper is to give a self-contained introduction to some of the central ingredients in frame theory, with particular focus on the parts that are important in the analysis of the so-called R-duals. Frames are sequences of elements in Hilbert spaces that can be used to obtain decompositions of elements in the Hilbert spaces of similar types as the well known ones arising from orthonormal bases; however, in contrast with the ONB case, the corresponding coefficients are not necessarily unique.

R-duals were introduced by Casazza, Kutyniok and Lammers in 2004, in an attempt to generalize the famous duality principle for Gabor frames. Essentially, the duality principle is a result that allows to check the frame condition for so-called Gabor systems in a conceptually simpler way. The R-duals provide a tool with a similar property for arbitrary sequences in general Hilbert spaces; however, at present it is not known whether the theory for R-duals is actually a generalization of the duality principle. We will discuss the known (partial) results about R-duals, as well as a variant defined recently by the author and Christensen (the so-called R-duals of type III). The hope is that the current survey will contribute to a clarification of the remaining open problems.

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The paper is organized as follows. In Section 2, the frame concept is introduced and characterization of frames based on the analysis, synthesis, and frame operators are given. Operators which keep the frame-property are considered. The optimal frame bounds of a frame are related to the operators associated to the frame. Some basic examples of frames are given. Further, the class of frames which are at the same time Schauder bases (the so called Riesz bases) are considered. Characterizations of Riesz bases are given, equivalent conditions for a frame to be a Riesz basis are listed, and operators which preserve the Riesz basis property are considered. The possibility to reconstruct every element in the space based on the frame elements and a dual frame is a very important frame-property. In Section 3 we briefly consider this topic. Characterizations of all the dual frames of a frame are given (based on appropriate operators and based on the use of Bessel sequences). Tight frames and their convenient representation-formulas are discussed. The ranges of the analysis operators of the dual frames are considered. For comprehensive exposition of material on frames and in particular, on Gabor frames, see [4, 9, 17, 18]. Section 4 is devoted to the topic of the duality principle. First, Gabor frames and the duality principle in Gabor analysis are recalled. Then R-duals and the duality principle based on them in general Hilbert spaces are considered. The R-duals cover the duality principle for Gabor tight frames and Gabor Riesz bases; for the rest of the Gabor frames it is still an open question. As a relaxation of R-duals, R-duals of type IV are considered, but one of their properties moves away from the desired duality principle. Finally, we consider the so called R-duals of type III. The class of R-duals III is between the R-duals and R-duals IV, and it covers the duality principle for all Gabor frames.

We end the current section with some notation. The letter $\mathcal{H}$ denotes a separable Hilbert space and I denotes a countable index set. The notion *operator* is used for a linear mapping. For $a \in \mathbb{R}$, T_a denotes the *translation operator* from $L^2(\mathbb{R})$ into $L^2(\mathbb{R})$ given by

$$(T_a f)(x) = f(x-a)$$

and $E_a : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ denotes the *modulation operator* from $L^2(\mathbb{R})$ into $L^2(\mathbb{R})$ given by

$$(E_a f)(x) = e^{2\pi i a x} f(x).$$

The letter G (resp. F) denotes a sequence $\{g_i\}_{i\in I}$ (resp. $\{f_i\}_{i\in I}$) with elements from $\mathcal{H}$. Given operator L, its domain is denoted by $\mathrm{dom}(L)$, its range by $\mathrm{ran}(L)$, and its kernel - by $\ker(L)$. Given sequence G, we will use the following operators:

- the *analysis operator* U_G given by $U_G h := \{\langle h, g_i\rangle\}_{i\in I}$ for $h \in \mathrm{dom}(U_G) = \left\{ f \in \mathcal{H} \ : \ \{\langle f, g_i\rangle\}_{i\in I} \in \ell^2 \right\}$;

- the *synthesis operator* T_G given by $T_G\{c_i\}_{i\in I} := \sum_{i\in I} c_i g_i$ for $\{c_i\}_{i\in I} \in \mathrm{dom}(T_G) = \left\{ \{d_i\}_{i\in I} \in \ell^2 \ : \ \sum_{i\in I} d_i g_i \text{ converges in } \mathcal{H} \right\}$;

- the *'frame' operator* S_G given by $S_G h := \sum_{i\in I} \langle h, g_i\rangle g_i$ for $h \in \mathrm{dom}(S_G) = \left\{ f \in \mathcal{H} \ : \ \sum_{i\in I} \langle f, g_i\rangle g_i \text{ converges in } \mathcal{H} \right\}$.

2. Frames

First recall that a sequence G is a *Schauder basis of* $\mathcal{H}$ if every $h \in \mathcal{H}$ can be represented as $\sum_{i\in I} c_i g_i$ with unique coefficients $\{c_i\}_{i\in I}$; a sequence G is an *orthonormal basis of* $\mathcal{H}$ if it is a Schauder basis of $\mathcal{H}$ and $\langle g_i, g_j\rangle = \delta_{i,j}$, $\forall i, j$. As it is well known, every orthonormal basis $\{e_i\}_{i\in I}$ for $\mathcal{H}$ satisfies the *Parseval equality*:

$$\sum_{i\in I} |\langle h, e_i\rangle|^2 = \|h\|^2, \ \forall h \in \mathcal{H}. \tag{2.1}$$

Extending the concept of an orthonormal basis, the frame concept was introduced by Duffin and Schaeffer [16] in 1952:

Definition 2.1. [16] A (non-zero) sequence G is called a *frame for* $\mathcal{H}$ if there exist positive constants A, B so that

$$A\|h\|^2 \le \sum_{i\in I} |\langle h, g_i\rangle|^2 \le B\|h\|^2, \ \forall h \in \mathcal{H}. \tag{2.2}$$

The constant A (resp. B) is called a *lower* (resp. *upper*) *frame bound of* G. The supremum of all lower frame bounds of G is also a lower frame bound of G and it is called the *optimal lower frame bound of* G; the infimum of all upper frame bounds is also an upper frame bound of G and it is called the *optimal upper frame bound of* G. The frame is called *tight* when its optimal frame bounds coincide.

For validity of the frame condition, it is actually enough to check the inequalities for a dense subset:

Proposition 2.2. [9, Lem. 5.1.7] *Assume that there exist positive constants* A, B *so that*

$$A\|h\|^2 \le \sum_{i\in I} |\langle h, g_i\rangle|^2 \le B\|h\|^2 \text{ for all } h \text{ in a dense subset of } \mathcal{H}.$$

Then G *is a frame for* $\mathcal{H}$ *with frame bounds* A, B.

Frames can be characterized based on the analysis, synthesis, and frame operators:

Proposition 2.3. ([16], [9, Sec. 5], and [8, Theor. 2.1]) *The sequence* G *is a frame for* $\mathcal{H}$ *if and only if any one of the following statements hold.*

(i) *The analysis operator* U_G *is well defined from* $\mathcal{H}$ *into* ℓ^2 *(hence bounded), injective and has a closed range* $\mathrm{ran}(U_G)$. *In this case the inverse operator* $U_G^{-1} : \mathrm{ran}(U_G) \to \mathcal{H}$ *is bounded with* $\|U_G^{-1}\| \le 1/\sqrt{A}$.

(ii) *The synthesis operator* T_G *is well defined from* ℓ^2 *into* $\mathcal{H}$ *(hence bounded) and surjective.*

(iii) *The frame operator S_G is well defined from $\mathcal{H}$ into $\mathcal{H}$ (hence bounded) and surjective. In this case S_G is furthermore bijective, self-adjoint, and positive; if A and B are frame bounds of S_G, then $A\,\mathrm{Id}_{\mathcal{H}} \leq S_G \leq B\,\mathrm{Id}_{\mathcal{H}}$.*

Some of the above properties of the frame-related operators can already be found in the first paper [16].

As a consequence of Proposition 2.3(ii), one can say that the frames for $\mathcal{H}$ are precisely the sequences $\{L\delta_i\}_{i\in I}$ where $L : \ell^2 \to \mathcal{H}$ is a bounded surjective operator and $\{\delta_i\}_{i\in I}$ is the canonical basis of ℓ^2, or equivalently:

Proposition 2.4. [9, Theor. 5.5.5] *Let $\{e_i\}_{i\in I}$ denote an orthonormal basis of $\mathcal{H}$. The frames for $\mathcal{H}$ are precisely the sequences $\{Le_i\}_{i\in I}$ where $L : \mathcal{H} \to \mathcal{H}$ is a bounded surjective operator.*

The above characterization of the class of frames is done via operators acting on a basis. Another approach would be to start from a frame and aim to obtain a frame as well. Such approach is considered by Aldroubi:

Proposition 2.5. [1] *Let G be a frame for $\mathcal{H}$ and let $V : \mathcal{H} \to \mathcal{H}$ be a bounded operator. The sequence $\{Vg_i\}_{i\in I}$ is a frame for $\mathcal{H}$ if and only if there exists a positive constant γ so that*

$$\|V^*v\|^2_{\mathcal{H}} \geq \gamma\|v\|^2_{\mathcal{H}},\ \forall v \in \mathcal{H}. \tag{2.3}$$

As it is well known (see, e.g., [23, Theor. 4.15]), (2.3) is equivalent to surjectivity of V. So, again the surjectivity plays role as in the case of operators acting on orthonormal bases (Prop. 2.4).

The optimal frame bounds of a frame can be expressed via the associated analysis, synthesis, and frame operator:

Proposition 2.6. [9, Prop. 5.4.4 and Theor. 5.4.3] *Let G be a frame for $\mathcal{H}$. Then the optimal frame bounds A^{opt}, B^{opt} of G fulfill*

$$A^{opt} = \|U_G^{-1}\|^{-2} = \|T_G^{\dagger}\|^{-2} = \|S_G^{-1}\|^{-1},\ B^{opt} = \|U_G\|^2 = \|T_G\|^2 = \|S_G\|,$$

where $T_G^{\dagger}$ is the pseudo-inverse of T_G, i.e., the analysis operator of the sequence $\{S_G^{-1}g_i\}_{i\in I}$.

Below we list some simple examples of frames and non-frames [9, 12, 18, 19].

Example 2.7. Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis of $\mathcal{H}$. Then the following holds:

(i) $\{e_i\}_{i=1}^{\infty}$ is a tight frame for $\mathcal{H}$ ($A^{opt} = B^{opt} = 1$);

(ii) $\{e_1, \frac{e_2}{\sqrt{2}}, \frac{e_2}{\sqrt{2}}, \frac{e_3}{\sqrt{3}}, \frac{e_3}{\sqrt{3}}, \frac{e_3}{\sqrt{3}}, ...\}$ is a tight frame for $\mathcal{H}$ ($A^{opt} = B^{opt} = 1$);

(iii) $\{e_1, e_1, e_2, e_3, e_4, ...\}$ is a frame for $\mathcal{H}$ ($A^{opt} = 1, B^{opt} = 2$);

(iv) $\{2e_1, e_2, e_3, e_4, ...\}$ is a frame for $\mathcal{H}$ ($A^{opt} = 1, B^{opt} = 4$);

(v) $\{e_1, \frac{e_2}{2}, \frac{e_3}{3}, ...\}$ is not a frame for $\mathcal{H}$ (it satisfies the upper frame condition, but not the lower one);

(vi) $\{e_1, 2e_2, 3e_3, ...\}$ is not a frame for $\mathcal{H}$ (it satisfies the lower frame condition, but not the upper one).

(vii) $\{\delta_1, -\frac{1}{2}\,\delta_1 + \frac{\sqrt{3}}{2}\,\delta_2, -\frac{1}{2}\,\delta_1 - \frac{\sqrt{3}}{2}\,\delta_2\}$ is a frame for $\mathbb{R}^2$ ($A = B = 3/2$), where $\delta_1 = (1, 0)$ and $\delta_2 = (0, 1)$ is the canonical basis for $\mathbb{R}^2$.

As the above examples show, one may construct simple frames using a Schauder basis (based for example on an orthonormal basis) and adding some elements. However, not all the frames can be constructed in such a way (Schauder basis and some more elements) - for every Hilbert space, there exists a frame such that no subset is a Schauder basis of the space [6]. Note that although a frame does not need to be an orthonormal basis, nor even a Schauder basis, it is always complete in the space (see Prop. 2.3(ii)).

Below we list some more examples of frames which are not in the spirit of Example 2.7, but are based on some appropriate modifications of a function from $L^2(\mathbb{R})$. Such frames are very important for applications. For example, frames consisting of translations and modulations of a function (the so called *Gabor frames*) and frames consisting of translations and dilations of a function (the so called *wavelet frames*) play significant role in signal and image processing, which has broad applications to physics, engineering, medicine, and many other areas.

Example 2.8. (a Gabor frame) Let $g(x) = \pi^{-1/4}e^{-x^2/2}$. For appropriate $a > 0$ and $b > 0$, the Gabor system $\left\{e^{2\pi imbx}g(x - na)\right\}_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$. More information concerning this example can be found in [12].

Example 2.9. (a wavelet frame) Let $\psi(x) = (1 - x^2)e^{-x^2/2}$, the so called *mexican hat*. Then for appropriate $a > 0, b > 0$, the wavelet system $\{a^{-\frac{j}{2}}\psi(a^{-j}x - bk)\}_{j,k\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$. More on this example can be found in [13].

Example 2.10. (a wavelet frame, orthonormal basis) Consider the function

$$\psi(x) = \begin{cases} 1, & x \in [0, \frac{1}{2}), \\ -1, & x \in [\frac{1}{2}, 1), \\ 0, & \text{otherwise}. \end{cases}$$

Then the wavelet system

$$\{2^{\frac{j}{2}}\psi\left(2^j x - k\right)\}_{j,k\in\mathbb{Z}} \tag{2.4}$$

is a frame (actually, an orthonormal basis) for $L^2(\mathbb{R})$, called the *Haar basis*. This is the oldest known example of a function ψ for which the system of the type (2.4) forms an orthonormal basis of $L^2(\mathbb{R})$ (Haar, 1910). For more on the Haar basis, see [13].

2.1. Riesz bases

In the next subsection we consider the class of those frames, which are in addition Schauder bases - the so called Riesz bases.

Definition 2.11. [27, Ch.1 Sec.8] A sequence G is called a *Riesz basis for* $\mathcal{H}$ if it is equivalent to an orthonormal basis, i.e., a sequence of the form $\{Le_i\}_{i\in I}$, where $L : \mathcal{H} \to \mathcal{H}$ is a bounded bijective operator and $\{e_i\}_{i\in I}$ is an orthonormal basis of $\mathcal{H}$.

The next statement gives an equivalent definition for a Riesz basis:

Proposition 2.12. [27, Ch.1 Sec.8, Theor. 9] *A sequence G is a Riesz basis for $\mathcal{H}$ if and only if it is complete in $\mathcal{H}$ and there exist positive constants A, B so that*

$$A\sum_{i\in I}|c_i|^2 \le \|\sum_{i\in I} c_i g_i\|^2 \le B\sum_{i\in I}|c_i|^2, \quad \text{for all finite sequences } \{c_i\} \tag{2.5}$$

(hence, for all $\{c_i\}_{i\in I} \in \ell^2$).

The positive constant A (resp. B) in (2.5) is called a *lower* (resp. *upper*) *Riesz bound of G*. The supremum of all lower Riesz bounds of G is also a lower Riesz bound of G and it is called the *optimal lower Riesz bound of G*; the infimum of all upper Riesz bounds is also an upper Riesz bound of G and it is called the *optimal upper Riesz bound of G*.

Compare the definition of a Riesz basis to Proposition 2.4 which determines the frames. For a Riesz basis one needs a bounded bijective operator, while for frames it is enough to use bounded surjective operators. Thus, all the Riesz bases are actually frames and furthermore, the Riesz bounds correspond to the frame bounds:

Proposition 2.13. [9, Theor. 5.4.1] *Every Riesz basis for $\mathcal{H}$ is a frame for $\mathcal{H}$ and its optimal Riesz bounds coincide with its optimal frame bounds.*

As frames were characterized based on the analysis, synthesis, and frame operators (Prop. 2.3), Riesz bases can also be characterized based on these three operators.

Proposition 2.14. [9, 2] *The sequence G is a Riesz basis for $\mathcal{H}$ if and only if any one of the following statements hold:*

(i) $\mathrm{dom}(U_G) = \mathcal{H}$ *and* U_G *is bijective.*

(ii) $\mathrm{dom}(T_G) = \ell^2$ *and* T_G *is bijective.*

(ii) $\mathrm{dom}(S_G) = \mathcal{H}$, S_G *is bijective, and* $\{S_G^{-1} g_i\}_{i\in I}$ *is biorthogonal to* G.

As a consequence of the above and by [9, Theor. 6.1.1], one can give a characterization of those frames which are Riesz bases:

Proposition 2.15. [9, Theor. 6.1.1] *Let G be a frame for $\mathcal{H}$. The following statements are equivalent:*

(i) *G is a Riesz basis for $\mathcal{H}$.*

(ii) *G is a Schauder basis for $\mathcal{H}$.*

(iii) *U_G is surjective.*

(iv) *T_G is injective.*

(v) *G has a biorthogonal sequence.*

(vi) *G and $\{S_G^{-1} g_i\}_{i \in I}$ are biorthogonal.*

Thus, one can split the class of frames for $\mathcal{H}$ into two subclasses:

- the frames for $\mathcal{H}$ which are at the same time Schauder bases of $\mathcal{H}$ (these are precisely the Riesz bases for $\mathcal{H}$);

- the frames for $\mathcal{H}$ which are not Schauder bases of $\mathcal{H}$ (such frames are called *overcomplete frames*).

Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis of $\mathcal{H}$. As simple examples of Riesz bases of $\mathcal{H}$, see Example 2.7(i),(iv); the rest of the sequences in Example 2.7 are not Riesz bases for $\mathcal{H}$. As a further example, consider the sequence $\{e_i + \frac{1}{i} e_{i+1}\}_{i=1}^{\infty}$, which is a Riesz basis for $\mathcal{H}$ [5].

We end this subsection with an analogue to Proposition 2.5:

Proposition 2.16. [25] *Let G be a Riesz basis for $\mathcal{H}$ and let $V : \mathcal{H} \to \mathcal{H}$ be a bounded operator. The sequence $\{V g_i\}_{i \in I}$ is a Riesz basis for $\mathcal{H}$ if and only if V is bijective. Furthermore, the boundedness of V is also a necessary condition, i.e., if $V : \mathrm{lin}\{g_i\} \to \mathcal{H}$ and $\{V g_i\}_{i \in I}$ is a Riesz basis for $\mathcal{H}$, then V is bounded.*

2.2. Frame sequences and Riesz sequences

While frames and Riesz bases are complete in the Hilbert space under consideration, it is also of interest to consider sequences which are not complete, but satisfy the frame or Riesz basis property just for the closed linear span.

Definition 2.17. A sequence G is called a *frame sequence* (resp. *Riesz sequence*) if it is a frame (resp. Riesz basis) for the closed linear span of its elements.

Note that when we speak about *bounds* of a frame or a frame sequence, we always mean the frame bounds; when we speak about *bounds* of a Riesz basis or a Riesz sequence, we mean the Riesz bounds (which are actually frame bounds, see Prop. 2.13).

Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis of $\mathcal{H}$. As a simple example of a frame sequence (resp. Riesz sequence) consider the sequence $\{e_2, e_2, e_3, e_3, e_4, e_4, \dots\}$ (resp. the sequence $\{e_2, e_3, e_4, \dots\}$).

The next statement gives some characterizations of frame sequences.

Proposition 2.18. [9, Corol. 5.5.2 and 5.5.3] *The following statements are equivalent:*

(i) *G is a frame sequence.*

(ii) $\mathrm{dom}(U_G) = \mathcal{H}$ *and* $\mathrm{ran}(U_G)$ *is closed.*

(iii) $\mathrm{dom}(T_G) = \ell^2$ *and* $\mathrm{ran}(T_G)$ *is closed.*

3. Dual frames

When $\{e_i\}_{i\in I}$ is an orthonormal basis for $\mathcal{H}$, then every element $h \in \mathcal{H}$ can be represented via the basis-elements in the way

$$h = \sum_{i\in I} \langle h, e_i\rangle e_i \tag{3.1}$$

with unique coefficients $\langle h, e_i\rangle$, $i \in I$. Although frames are not necessarily orthonormal bases, they still allow representations of the space-elements via a formula similar to (3.1):

Proposition 3.1. *Let G be a frame for $\mathcal{H}$. Then there exists a frame F for $\mathcal{H}$ so that*

$$h = \sum_{i\in I} \langle h, f_i\rangle g_i = \langle h, g_i\rangle f_i, \ \forall h \in \mathcal{H}; \tag{3.2}$$

such a frame F is called a dual frame of G.

Based on the frame operator, one can determine a specific dual frame:

Proposition 3.2. [16] *Let G be a frame for $\mathcal{H}$ with frame bounds A, B. Then the sequence $\{S_G^{-1} g_i\}_{i\in I}$ is a dual frame of G (called the canonical dual of G) and it has frame bounds $1/B, 1/A$.*

Tight frames are very convenient to use in applications, because they provide representation-formulas with very easy and fast computations: if $A > 0$ is the bound of a tight frame G, then $S_G = A\,\mathrm{Id}_{\mathcal{H}}$, the canonical dual of G is the sequence $\{\frac{1}{A} g_i\}_{i\in I}$, and the following representation-formula holds

$$h = \frac{1}{A} \sum_{i\in I} \langle h, g_i\rangle g_i, \ \forall h \in \mathcal{H}. \tag{3.3}$$

The converse of the above statement also holds:

Proposition 3.3. (see [9, Lemma 5.7.1]) *Let G be a frame for $\mathcal{H}$ and $A > 0$. Then (3.3) holds if and only if G is tight with frame bound A.*

The above statement is also interesting in the sense that it characterizes all the sequences which satisfy the Parseval equality (2.1). It is not only the orthonormal bases, but all the tight frames with bound 1 which satisfy (2.1). As an example of a tight frame with bound 1 which is not an orthonormal basis, see the sequence in Example 2.7(ii).

The next statement determines the frames which have more then one dual frame.

Proposition 3.4. *Let G be a frame for $\mathcal{H}$. The following statements hold.*

(i) [9, Theor. 3.6.3 and 5.4.1] *If G is a Riesz basis for $\mathcal{H}$, then it has a unique dual frame (the canonical dual) which is also a Riesz basis for $\mathcal{H}$ and it is biorthogonal to G.*

(ii) [9, Lemma 5.6.1] *If G is an overcomplete frame for $\mathcal{H}$, then in addition to the canonical dual, G has other dual frames.*

Thus, for an overcomplete frame G for $\mathcal{H}$, one has many representations of the form

$$h = \sum_{i\in I} \langle h, f_i\rangle g_i, \ \forall h \in \mathcal{H}, \tag{3.4}$$

using different possibilities for a dual frame F of G (including the canonical dual). The canonical dual has the ℓ^2-minimal property in the sense that for any $h \in \mathcal{H}$, the coefficients $\{\langle h, f_i\rangle\}_{i\in I}$ in (3.4) have minimal ℓ^2-norm when F is the canonical dual of G compare to F being any other dual frame of G. More general, the following statement holds:

Proposition 3.5. [9, Lemma 5.4.2] *Let G be a frame for $\mathcal{H}$. If $h \in \mathcal{H}$ is written as $h = \sum_{i\in I} c_i g_i$ for some coefficients $\{c_i\}_{i\in I}$, then*

$$\sum_{i\in I} |c_i|^2 = \sum_{i\in I} |\langle h, S_G^{-1} g_i\rangle|^2 + \sum_{i\in I} |c_i - \langle h, S_G^{-1} g_i\rangle|^2.$$

Even though the canonical dual has some special properties, for example the ℓ^2-minimality-property, for some applications other dual frames might be of interest. For example, when frames with specific structure are used, then it might be useful for particular application to use representations based on a dual frame with the same structure. In this sense Gabor frames are nice, the canonical dual of a Gabor frame is always a Gabor frame. In contrary, as determined in [11, 12], the canonical dual of a wavelet frame is not necessarily with the wavelet structure. In [14] one can find an example of a wavelet frame, whose canonical dual is not a wavelet frame, but there are infinitely many other dual frames having the wavelet-structure. This motivates the consideration of dual frames other then the canonical dual. Below we consider characterizations of all the dual frames of a given frame.

3.1. Characterization of the dual frames

We consider two type of characterizations of all the dual frames - one characterization based on the use of appropriate operators (Prop. 3.6) and another characterization done using Bessel sequences (Prop. 3.7). Recall that a sequence G is called a *Bessel sequence in $\mathcal{H}$* if there exists a positive constant B so that $\sum_{i\in I} |\langle h, g_i\rangle|^2 \leq B\|h\|^2$ for all $h \in \mathcal{H}$.

Proposition 3.6. [21] *Let G be a frame for $\mathcal{H}$ and let $\{\delta_i\}_{i\in I}$ denote the canonical basis of ℓ^2. Then the dual frames of G are precisely the sequences $\{V\delta_i\}_{i\in I}$ with V being a bounded left inverse of U_G.*

Proposition 3.7. [21] *Let G be a frame for $\mathcal{H}$. Then the dual frames of G are precisely the sequences*

$$\left\{ S_G^{-1} g_i + h_i - \sum_{j\in I} \langle S_G^{-1} g_i, g_j\rangle h_j \right\}_{i\in I},$$

where $\{h_i\}_{i\in I}$ runs through the Bessel sequences in $\mathcal{H}$.

3.2. On the range of the analysis operators of the dual frames

A Riesz basis G has a unique dual frame, which is also a Riesz basis and the range of its analysis operator is ℓ^2 (see Prop. 2.14(i) and Prop. 3.4). When G is an overcomplete frame, then any dual frame of G is also an overcomplete frame and the range of its analysis operator is a strict subset of ℓ^2. A natural question would be whether one can cover ℓ^2 using the ranges of the analysis operators of the different dual frames and the answer is given in the next statement:

Proposition 3.8. [3] *Let G be a frame for $\mathcal{H}$. Then the closure of the union of all sets* $\operatorname{ran}(U_{G^d})$*, where G^d runs through the dual frames of G, is ℓ^2.*

Furthermore, the set of the dual frames determines a frame in unique way:

Proposition 3.9. [3] *Let G and Ψ be frames for $\mathcal{H}$. If every dual frame of G is also a dual frame of Ψ, then $\Psi = G$.*

In particular, it follows that the set of the dual frames of one frame can not be a strict subset of the set of the dual frames of another frame.

4. Duality principle

The duality principle is one of the strongest results in Gabor analysis. Here we discuss the duality principle and its extension to general Hilbert spaces.

We begin the section recalling Gabor systems, i.e. systems which are defined based on translations and modulations of a fixed function in $L^2(\mathbb{R})$:

Definition 4.1. Let $g \in L^2(\mathbb{R})$ and $a, b > 0$. The sequence $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is called a *Gabor system*. A Gabor system is called a *Gabor frame* (resp. *Gabor Riesz basis*) if it is a frame (resp. Riesz basis) for $L^2(\mathbb{R})$. A Gabor system is called a *Gabor frame sequence* (resp. *Gabor Riesz sequence*) if it is a frame sequence (resp. Riesz sequence).

Given a Gabor system $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$, some values of the product ab are important to determine whether we have a Gabor frame, Gabor Riesz basis, or non-frame:

Theorem 4.2. *[9, Theor. 8.3.1] Let $g \in L^2(\mathbb{R})$ and $a, b > 0$. Then the following statements hold.*

(i) *If $ab > 1$, then $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is not a frame for $L^2(\mathbb{R})$.*

(ii) *If $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$, then*

$$\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}} \text{ is a Riesz basis for } L^2(\mathbb{R}) \text{ if and only if } ab = 1.$$

The above statement gives a sufficient condition for a Gabor system not to be a frame, and distinguishes Riezs bases and overcomplete frames ones it is known that the system is a frame. For sufficient conditions for a Gabor system to form a frame and more on Gabor frames, see, e.g., [9, 17, 19]. Here we will mainly concentrate on the duality principle.

4.1. Duality principle in Gabor analysis

The *duality principle* is one of the strongest results in Gabor analysis. It was discovered at the same time by Janssen [20], Daubechies, Landau, and Landau [15], and Ron and Shen [22]. The duality principle relates a Gabor system with respect to a lattice $(na, mb)_{n,m\in\mathbb{Z}}$ to the Gabor system with respect to the lattice $(n/b, m/a)_{n,m\in\mathbb{Z}}$:

Theorem 4.3. [15, 20, 22] *Let $g \in L^2(\mathbb{R})$ and $a, b > 0$ be given. Then the Gabor system $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$ with bounds A, B if and only if the Gabor system $\{\frac{1}{\sqrt{ab}} E_{m/a}T_{n/b}g\}_{m,n\in\mathbb{Z}}$ is a Riesz sequence with bounds A, B.*

Having in mind Theorem 4.2, one obtains the following known relation:

Proposition 4.4. *Let $g \in L^2(\mathbb{R})$ and $a, b > 0$ be given. Then the Gabor system $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a Riesz basis for $L^2(\mathbb{R})$ with bounds A, B if and only if the Gabor system $\{\frac{1}{\sqrt{ab}} E_{m/a}T_{n/b}g\}_{m,n\in\mathbb{Z}}$ is a Riesz basis for $L^2(\mathbb{R})$ with bounds A, B.*

The above two statements are the central point of our consideration in this section. The aim is to consider extension of these statements to general Hilbert spaces.

4.2. Duality principle in abstract Hilbert spaces

R-duals of type I

In 2014, Casazza, Kutyniok, and Lammers [7] aimed to obtain an analogue of the duality principle to general Hilbert spaces. They introduced the concept of R-duals of a frame as follows:

Definition 4.5. [7] Let $\{e_i\}_{i\in I}$ and $\{h_i\}_{i\in I}$ denote orthonormal bases for $\mathcal{H}$, and let $\{f_i\}_{i\in I}$ be any sequence in $\mathcal{H}$ for which $\sum_{i\in I}|\langle f_i, e_j\rangle|^2 < \infty$, $\forall j \in I$. The *R-dual of* $\{f_i\}_{i\in I}$ with respect to the orthonormal bases $\{e_i\}_{i\in I}$ and $\{h_i\}_{i\in I}$ is defined as the sequence $\{\omega_j\}_{j\in I}$ given by

$$\omega_j = \sum_{i\in I}\langle f_i, e_j\rangle h_i,\ j \in I. \tag{4.1}$$

Since we will also consider other versions of R-duals (called R-duals of type III and R-duals of type IV), from now on we will refer to $\{\omega_j\}_{j\in I}$ in (4.1) as an *R-dual of type I* of $\{f_i\}_{i\in I}$.

The following statements give an analogue of Theorem 4.3 and Proposition 4.4 to general Hilbert spaces.

Theorem 4.6. [7] *Let $\{e_i\}_{i\in I}$ and $\{h_i\}_{i\in I}$ denote orthonormal bases for $\mathcal{H}$, and let $\{f_i\}_{i\in I}$ be any sequence in $\mathcal{H}$ for which $\sum_{i\in I}|\langle f_i, e_j\rangle|^2 < \infty$ for all $j \in I$. Define the R-dual $\{\omega_j\}_{j\in I}$ of type I as in* (4.1)*. Then the following hold:*

(i) *$\{f_i\}_{i\in I}$ is a frame for $\mathcal{H}$ with bounds A, B if and only if $\{\omega_j\}_{j\in I}$ is a Riesz sequence in $\mathcal{H}$ with bounds A, B.*

(ii) *$\{\omega_j\}_{j\in I}$ is a Riesz basis for $\mathcal{H}$ if and only if $\{f_i\}_{i\in I}$ is a Riesz basis for $\mathcal{H}$.*

(iii) *$f_i = \sum_{j\in I}\langle \omega_j, h_i\rangle e_j$ for all $i \in I$, implying that $\{f_i\}_{i\in I}$ is an R-dual of type I of $\{\omega_j\}_{j\in I}$ with respect to the orthonormal bases $\{h_i\}_{i\in I}$ and $\{e_i\}_{i\in I}$.*

The above analogue of the duality principle has naturally led the authors of [7] to the question whether the Gabor system $\{\frac{1}{\sqrt{ab}}\,E_{m/a}T_{n/b}g\}_{m,n\in\mathbb{Z}}$ is an R-dual of type I of $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ and they have answered partially, giving affirmative answer for two classes of frames:

Theorem 4.7. [7] *Assume that $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a tight frame or a Riesz basis for $L^2(\mathbb{R})$. Then $\{\frac{1}{\sqrt{ab}}\,E_{m/a}T_{n/b}g\}_{m,n\in\mathbb{Z}}$ can be realized as an R-dual of type I of $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ with respect to certain choices of orthonormal bases $\{e_{m,n}\}_{m,n\in\mathbb{Z}}$ and $\{h_{m,n}\}_{m,n\in\mathbb{Z}}$.*

It is still an open question whether the above theorem holds for overcomplete frames which are not tight.

In the next subsection we consider a natural relaxation of the R-duals of type I (the so called R-duals of type IV), using Riesz bases instead of orthonormal bases. However, in this way the concept becomes too general. Then we consider another relaxation of the R-duals of type I (called R-duals of type III), give an analogue to the duality principle and show that for any frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ for $L^2(\mathbb{R})$, the Gabor system $\{\frac{1}{\sqrt{ab}}\,E_{m/a}T_{n/b}g\}_{m,n\in\mathbb{Z}}$ can be written as an R-dual of type III of $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$.

Below we discuss characterizations of the R-duals of type I. By Theorem 4.6, a necessary condition for a Riesz sequence to be an R-dual of type I of a frame $\{f_i\}_{i\in I}$ is to have the same optimal bounds as $\{f_i\}_{i\in I}$. In [7], one more necessary condition for a Riesz sequence to be an R-dual of type I is determined:

Lemma 4.8. [7] *If $\{f_i\}_{i\in I}$ is a frame for $\mathcal{H}$ and $\{\omega_j\}_{j\in I}$ is an R-dual of type I of $\{f_i\}_{i\in I}$, then*

$$\dim(\ker T_F) = \dim(\operatorname{span}\{\omega_j\}_{j\in I}^{\perp}). \tag{4.2}$$

The next two examples show that in general the condition (4.2) and same optimal bounds are not sufficient. The first example concerns the Riesz basis case and the second one - the overcomplete case.

Example 4.9. [24] Let $\{z_i\}_{i=1}^{\infty}$ be an orthonormal basis for $\mathcal{H}$. Consider the Riesz bases $\{f_i\}_{i=1}^{\infty} = \{\sqrt{2}z_1, z_2, \sqrt{2}z_3, \sqrt{2}z_4, \sqrt{2}z_5, \sqrt{2}z_6, \ldots\}$ and $\{v_j\}_{j=1}^{\infty} = \{\sqrt{2}z_1, z_2, z_3, z_4, z_5, \ldots\}$. Both $\{f_i\}_{i=1}^{\infty}$ and $\{v_j\}_{j=1}^{\infty}$ have optimal bounds $A = 1$, $B = 2$, and (4.2) holds. However, $\{\nu_j\}_{i=j}^{\infty}$ is not an R-dual of type I of $\{f_i\}_{i=1}^{\infty}$.

Example 4.10. Let $\{z_i\}_{i=1}^{\infty}$ be an orthonormal basis for $\mathcal{H}$. Consider the overcomplete frame $\{f_i\}_{i=1}^{\infty} = \{z_1, z_1, z_2, z_3, z_3, z_4, z_5, z_5, z_6, \ldots\}$ for $\mathcal{H}$ and the Riesz sequence $\{g_j\}_{j=1}^{\infty} = (\sqrt{2}z_1, z_3, z_5, z_7, z_9, \ldots)$. Both $\{f_i\}_{i=1}^{\infty}$ and $\{g_j\}_{j=1}^{\infty}$ have the same optimal bounds $A = 1$, $B = 2$, and (4.2) holds. However, $\{g_j\}_{j=1}^{\infty}$ can not be written as an R-dual of type I of $\{f_i\}_{i=1}^{\infty}$. Indeed, assume that there exist orthonormal bases $\{e_i\}_{i\in I}$ and $\{h_i\}_{i\in I}$ for $\mathcal{H}$ so that $g_j = \sum_{i=1}^{\infty}\langle f_i, e_j\rangle h_i$, $j \in \mathbb{N}$. Then for every $n \in \mathbb{N}$, $z_{2n+1} = g_{n+1} = \sum_{i=1}^{\infty}\langle f_i, e_{n+1}\rangle h_i$, which implies that $\langle z_{2n+1}, h_i\rangle = \langle f_i, e_{n+1}\rangle$ for all $i \in \mathbb{N}$. Then for every $n \in \mathbb{N}$,

$$\begin{aligned} 1 &= \|z_{2n+1}\|^2 = \sum_{i=1}^{\infty}|\langle z_{2n+1}, h_i\rangle|^2 = \sum_{i=1}^{\infty}|\langle f_i, e_{n+1}\rangle|^2 \\ &= \sum_{i=1}^{\infty}|\langle z_i, e_{n+1}\rangle|^2 + \sum_{i=1}^{\infty}|\langle z_{2i-1}, e_{n+1}\rangle|^2 = 1 + \sum_{i=1}^{\infty}|\langle z_{2i-1}, e_{n+1}\rangle|^2. \end{aligned}$$

Therefore, $e_{n+1} \perp z_{2i-1}, \forall i \in \mathbb{N}, \forall n \in \mathbb{N}$. In particular, this implies that $z_1 \perp e_i$, $\forall i \geq 2$, and $z_3 \perp e_i$, $\forall i \geq 2$, which is a contradiction.

As shown in the above examples, for a non-tight frame $\{f_i\}_{i\in I}$ and a Riesz sequence $\{\omega_j\}_{j\in I}$ with the same optimal bounds, the condition (4.2) is not sufficient for $\{\omega_j\}_{j\in I}$ to be an R-dual of type I of $\{f_i\}_{i\in I}$. However, for tight frames, this condition is sufficient:

Proposition 4.11. [24] *Let $\{f_i\}_{i\in I}$ be a tight frame for $\mathcal{H}$ and let $\{\omega_j\}_{j\in I}$ be a tight Riesz sequence in $\mathcal{H}$ with the same bound. Then $\{\omega_j\}_{j\in I}$ is an R-dual of type I of $\{f_i\}_{i\in I}$ if and only if (4.2) holds.*

R-duals of type IV

The R-duals of type I are known to cover the duality principle for the tight Gabor frames and the Gabor Riesz bases. It is still an open question whether they cover the duality principle for the rest of the Gabor frames. A natural approach would be to consider a larger set then the set of the R-duals of type I with the aim to cover the duality principle for all Gabor frames. The simplest way to enlarge the set of the R-duals of type I would be to replace the orthonormal bases in Definition 4.5 by Riesz bases. However, Proposition 4.12(i) below will show that in this way the frame bounds are not necessarily kept, which will move us away from the main aim to obtain an analogue of the duality principle.

The use of Riesz bases have been considered in the more general setting of Banach spaces by Xiao and Zhu [26] and the statement below is a particular case of their general statement in the Banach space setting.

Proposition 4.12. [26] *Let $\{a_i\}_{i\in I}$ and $\{b_i\}_{i\in I}$ be Riesz bases for $\mathcal{H}$ with bounds A_a, B_a and A_b, B_b, respectively, and with canonical duals $\{\widetilde{a}_i\}_{i\in I}$ and $\{\widetilde{b}_i\}_{i\in I}$, respectively. Let $\{f_i\}_{i\in I}$ be a sequence with elements from $\mathcal{H}$ so that $(\langle f_i, a_j\rangle)_{i\in I} \in \ell^2, \forall j \in I$. Consider the sequence $\{\theta_j\}_{j\in I}$ defined by*

$$\theta_j := \sum_{i\in I} \langle f_i, a_j\rangle b_i, j \in I \tag{4.3}$$

(according to the notions in [24], the so called R-dual of type IV of $\{f_i\}_{i\in I}$ with respect to $\{a_i\}_{i\in I}$ and $\{b_i\}_{i\in I}$). Then the following statements hold.

(i) *If $\{f_i\}_{i\in I}$ is a frame for $\mathcal{H}$ with bounds A_F, B_F, then $\{\theta_j\}_{j\in I}$ is a Riesz sequence with bounds $A_aA_bA_F$, $B_aB_bB_F$.*

  If $\{\theta_j\}_{j\in I}$ is a Riesz sequence with bounds A_Θ, B_Θ, then $\{f_i\}_{i\in I}$ is a frame for $\mathcal{H}$ with bounds $B_a^{-1}B_b^{-1}A_\Theta$, $A_a^{-1}A_b^{-1}B_\Theta$.

(ii) *$\{\theta_j\}_{j\in I}$ is a Riesz basis for $\mathcal{H}$ if and only if $\{f_i\}_{i\in I}$ is a Riesz basis for $\mathcal{H}$.*

(iii) *$f_i = \sum_{j\in I}\langle \theta_j, \widetilde{b}_i\rangle \widetilde{a}_j$ for all $i \in I$, implying that $\{f_i\}_{i\in I}$ is an R-dual of type IV of $\{\theta_j\}_{j\in I}$ with respect to the Riesz bases $\{\widetilde{b}_i\}_{i\in I}$ and $\{\widetilde{a}_i\}_{i\in I}$.*

R-duals of type III

As one can see above, some properties of R-duals of type IV (Prop. 4.12(ii)(iii)) are similar to the corresponding properties of the R-duals of type I (Theor. 4.6(ii)(iii)). However, contrary to the fact that R-duals of type I keep the frame bounds (Theor. 4.6(i)), R-duals of type IV do not necessarily keep the frame bounds (Prop. 4.12(i)). This has motivated the authors of [24] to introduce a class of R-duals, which is between the class of the R-duals of type I and the R-duals of type IV, and where the frame bounds are kept.

Before writing the definition of R-duals of type III, recall that given a frame G for $\mathcal{H}$, the frame operator S_G is bijective and positive, and thus S_G and S_G^{-1} have square roots. When G is a just a frame sequence, the operator S_G is considered as a bijection on the closed linear span of the elements of G.

Relaxing the definition of R-duals I and tighten the definition of R-duals IV, we consider the following type of R-duals:

Definition 4.13. [24] Let $F = \{f_i\}_{i\in I}$ be a frame sequence in $\mathcal{H}$. Let $\{e_i\}_{i\in I}$ and $\{h_i\}_{i\in I}$ denote orthonormal bases for $\mathcal{H}$ and $Q : \mathcal{H} \to \mathcal{H}$ be a bounded bijective operator with $\|Q\| \leq \sqrt{||S||}$ and $\|Q^{-1}\| \leq \sqrt{||S^{-1}||}$. The *R-dual of type III of* $\{f_i\}_{i\in I}$ with respect to the triplet $(\{e_i\}_{i\in I}, \{h_i\}_{i\in I}, Q)$, is the sequence $\{\omega_j\}_{j\in I}$ defined by

$$\omega_j := \sum_{i\in I} \langle S_F^{-1/2} f_i, e_j\rangle Q h_i \;\; j \in I. \tag{4.4}$$

Given a frame $\{f_i\}_{i\in I}$ for $\mathcal{H}$, it is clear that an R-dual of type III of $\{f_i\}_{i\in I}$ is also an R-dual of type IV of $\{f_i\}_{i\in I}$. When $\{f_i\}_{i\in I}$ is just assumed to be a frame sequence, at first glance (4.4) does not look like an R-dual of type IV of $\{f_i\}_{i\in I}$, because $S_F^{-1/2}$ is just defined on the closed linear span of $\{f_i\}_{i\in I}$ and not necessarily defined on all e_j, $j \in I$. However, $S_F^{-1/2}$ can be extended to a bounded bijective self-adjoint operator V on $\mathcal{H}$ (see [24, Lemma 1.3]) and thus (4.4) can be written as

$$\omega_j := \sum_{i\in I} \langle f_i, V e_j\rangle Q h_i, \;\; j \in I,$$

which is an R-dual of type IV of $\{f_i\}_{i\in I}$. Thus, for a frame sequence, the class of the R-duals of type III is contained in the class of R-duals of type IV. That R-duals of type I of a frame are contained in the class of R-duals of type III is not an obvious fact either, but it holds true, see Proposition 4.17.

For R-duals of type III, we obtain an analogue to the duality principle and similar properties compare to the properties of R-duals of type I listed in Theorem 4.6:

Theorem 4.14. [24] *Let $\{f_i\}_{i\in I}$ be a frame sequence in $\mathcal{H}$ and let $\{\omega_i\}_{i\in I}$ be an R-dual of $\{f_i\}_{i\in I}$ of type III with respect to the triplet $(\{e_i\}_{i\in I}, \{h_i\}_{i\in I}, Q)$. Then the following statements hold.*

- (i1) *$\{f_i\}_{i\in I}$ is a frame for $\mathcal{H}$ if and only if $\{\omega_j\}_{j\in I}$ is a Riesz sequence; in the affirmative case the bounds for $\{f_i\}_{i\in I}$ are also bounds for $\{\omega_j\}_{j\in I}$.*
- (i2) *$\{f_i\}_{i\in I}$ is a Riesz sequence if and only if $\{\omega_j\}_{j\in I}$ is a frame for $\mathcal{H}$; in the affirmative case the bounds for $\{f_i\}_{i\in I}$ are also bounds for $\{\omega_j\}_{j\in I}$.*
- (ii) *$\{\omega_j\}_{j\in I}$ is a Riesz basis for $\mathcal{H}$ if and only if $\{f_i\}_{i\in I}$ is a Riesz basis for $\mathcal{H}$.*

(iii) *If* $\{f_i\}_{i\in I}$ *is a frame for* $\mathcal{H}$*, then* $f_i = \sum_{j\in I}\langle Q^{-1}\omega_j, h_i\rangle S^{1/2}e_j$ *for all* $i \in I$*, not implying automatically that* $\{f_i\}_{i\in I}$ *is an R-dual of type III of* $\{\omega_j\}_{j\in I}$*.*

(iv) *If* $\{f_i\}_{i\in I}$ *is a frame for* $\mathcal{H}$ *and* $\{\omega_j\}_{j\in I}$ *has the same optimal bounds as* $\{f_i\}_{i\in I}$*, then* $\{f_i\}_{i\in I}$ *is an R-dual of type III of* $\{\omega_j\}_{j\in I}$*.*

Note that the statements in Theorems 4.6(i) and 4.14(i1) are not completely the same. While in Theorem 4.6(i) the optimal bounds of $\{f_i\}_{i\in I}$ and $\{\omega_j\}_{j\in I}$ are the same, in Theorem 4.14(i) it is only stated that the frame bounds for $\{f_i\}_{i\in I}$ are also bounds for $\{\omega_j\}_{j\in I}$, the optimal bounds may differ:

Example 4.15. [24] Let $\{e_i\}_{i=1}^\infty$ denote an orthonormal basis of $\mathcal{H}$. Consider the frame $\{f_i\}_{i=1}^\infty = \{e_1, e_1, e_2, e_3, e_4, \dots\}$ for $\mathcal{H}$ and the Riesz sequence $\{\omega_j\}_{j=1}^\infty = \{e_1 + \frac{1}{\sqrt{2}}\, e_2, e_3, e_4, \dots\}$, which is an R-dual of type III of $\{f_i\}_{i=1}^\infty$. The optimal bounds of $\{f_i\}_{i=1}^\infty$ are 1,2, which are also bounds for $\{\omega_j\}_{j=1}^\infty$, but the optimal bounds of $\{\omega_j\}_{j=1}^\infty$ are $1, 3/2$.

Notice that Theorem 4.14(iii) is not completely the same as Theorem 4.6(iii). While the representation of $\{f_i\}_{i\in I}$ in Theorem 4.6(iii) leads immediately to the conclusion that $\{f_i\}_{i\in I}$ is an R-dual of type I of $\{\omega_j\}_{j\in I}$, similar conclusion does not follow automatically from the representation of $\{f_i\}_{i\in I}$ in Theorem 4.14(iii). However, as written in Theorem 4.14(iv), one can still claim a conclusion that $\{f_i\}_{i\in I}$ is an R-dual of type III of $\{\omega_j\}_{j\in I}$, under the assumption that $\{\omega_j\}_{j\in I}$ has the same optimal bounds as $\{f_i\}_{i\in I}$.

Unlike the case with R-duals of type I, which up to now are known to cover the duality principle for tight Gabor frame and Gabor Riesz bases, the R-duals of type III cover the duality principle for all Gabor frames:

Theorem 4.16. [24] *Let* $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ *be a Gabor frame for* $L^2(\mathbb{R})$*. Then the Gabor Riesz sequence* $\{\frac{1}{\sqrt{ab}}E_{m/a}T_{n/b}g\}_{m,n\in\mathbb{Z}}$ *can be realized as an R-dual of type III of* $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$*.*

It is interesting to consider the relation between R-duals of type I and R-duals of type III. In this spirit, the following holds:

Proposition 4.17. *Let* $\{f_i\}_{i\in I}$ *be a frame for* $\mathcal{H}$*.*

(i) [24] *The class of R-duals of type I of* $\{f_i\}_{i\in I}$ *is contained in the class of R-duals of type III of* $\{f_i\}_{i\in I}$*.*

(ii) [24] *If* $\{f_i\}_{i\in I}$ *is tight, the classes of the R-duals of type I and III of* $\{f_i\}_{i\in I}$ *coincide.*

(iii) [10] *If* $\{f_i\}_{i\in I}$ *is not tight, then the set of its R-duals of type I is a proper subset of the set of its R-duals of type III.*

In the tight case, the R-duals of type I are enough to cover the duality principle and thus, for tight frames, one does not need to enlarge the class of

R-duals of type I. From this point of view, Prop. 4.17(ii) shows a good property of the R-duals of type III.

At the end we provide a characterization of the R-duals of type III:

Proposition 4.18. [24] *Let $\{f_i\}_{i\in I}$ be a frame for $\mathcal{H}$, let $\{\omega_j\}_{j\in I}$ be a Riesz sequence in $\mathcal{H}$ and assume that the bounds of $\{f_i\}_{i\in I}$ are also bounds for $\{\omega_j\}_{j\in I}$. Then*

$$\{\omega_j\}_{j\in I} \text{ is an R-dual of type III of } \{f_i\}_{i\in I} \text{ if and only if } (4.2) \text{ holds.}$$

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