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A NOTE ON CUT PROPERTIES OF SEMILATTICE VALUED FUZZY SETS[1]

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Abstract. Semilattice valued fuzzy sets are investigated in the framework of cuts. A theorem of synthesis for such fuzzy sets is proved. Families of cuts for meet(join)-fuzzy sets are proved to be special semi-closure systems. Conversely, for every such semi-closure system, we prove existence of a semilattice and a semilattice-valued fuzzy set whose collection of cuts is the given semi-closure system. We also show that for an arbitrary collection of subsets of a nonempty set, there is a semilattice valued fuzzy set whose collection of cuts contains these subsets. Using meet-irreducible elements in a finite meet-semilattice, we give conditions under which all the cuts of a meet-semilattice-valued fuzzy set are different and we describe a representation of the semilattice by the collection of cuts ordered dually by inclusion.

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1. Introduction

In this paper, a fuzzy set on an arbitrary nonempty set X is a function from X into a meet (join)-semilattice S. A fuzzy set as a function can be characterized by the particular collection of subsets of its domain. As it is known, these subsets are called cut sets of the function (for details, see [2], [3]). This collection is considered as a partially ordered set with respect to the set inclusion. If the co-domain of a fuzzy set is a lattice, then the collection of cuts of this fuzzy set formes a closure system on X.

Šešelja and Tepavčević ([2]) introduced poset-valued fuzzy sets for which the co-domain is a poset. They gave necessary and sufficient conditions under which a family of subsets of an arbitrary nonempty set represents a collection of cuts of a fuzzy set whose co-domain is a partially ordered set. In [3], they investigated the analogue representation for fuzzy sets whose co-domain is a

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meet (join)-semilattice. Such representations are an alternative way to find the function from an arbitrary nonempty set to a poset, which may also be a lattice or a semilattice.

In this paper, in connection to the above investigations, semilattice-valued fuzzy sets are investigated from the point of view of cut sets. The theorem of synthesis for semilattice-valued fuzzy sets by means of cuts is proved. Families of cuts for meet(join)-fuzzy sets turn out to be special (dual) semi-closure systems. Conversely, for every (dual) semi-closure system over a nonempty domain, we prove that there is a semilattice and a semilattice-valued fuzzy set, whose collection of cuts is the given semi-closure system. We also show that for an arbitrary nonempty collection of subsets of a nonempty set, there is a semilattice valued fuzzy set whose collection of cuts contains these subsets. Finally, dealing with finite semilattices and using meet-irreducible elements, we give conditions under which all the cuts of a meet-semilattice valued fuzzy sets are different and we present a representation of a semilattice by the collection of cuts of a semilattice valued fuzzy set, ordered dually to inclusion.

Notations and some basic facts about fuzzy sets with the co-domain which is a meet (join)-semilattice are adopted from [3] and notions related to partially ordered sets from [1].

2. Preliminaries

2.1. Ordered structures

We list some known notions related to order structures (including their basic properties), mostly intending to introduce notation used in the text.

A partially ordered set – a **poset** $(S, \leq)$ is a nonempty set S equipped with an ordering relation $\leq$. For every $p \in S$, **the principal filter generated by** p is denoted by $\uparrow p$:

$$\uparrow p = \{x \in S \mid p \leq x\}.$$

Let $(P, \leq)$ be a poset. For arbitrary $M \subseteq S$, **the set of upper bounds of** M is given by $M^u := \{p \in P \mid x \leq p, \text{ for every } x \in M\}$, and **the set of lower bounds of** M is $M^l := \{p \in P \mid x \geq p, \text{ for every } x \in M\}$.

A poset is **bounded** if it has the smallest element, the **bottom**, denoted by 0, and the greatest, the **top**, denoted by 1.

A **meet-semilattice** is a poset in which for every two-element subset $\{x, y\}$ there is the greatest lower bound (glb, meet, infimum), denoted by $x \wedge y$. A meet-semilattice is **complete** if the greatest lower bound, infimum, exists for every nonempty subset M of S. Infimum of M is denoted by $\bigwedge M$. A **join-semilattice** is a poset in which for every two-element subset $\{x, y\}$ there is the least upper bound (lub, join, supremum), denoted by $x \vee y$. It is **complete** if for every nonempty $M \subseteq S$, supremum, $\bigvee M$, exists. Meet and join are binary operations on S, hence a meet and join-semilattices are algebras, denoted by $(S, \wedge)$ and $(S, \vee)$ respectively.

A (**complete**) **lattice** is a poset which is a (complete) meet-semilattice and a (complete) join-semilattice. Every complete lattice is bounded.

If $(S, \leq)$ and $(T, \leq)$ are posets, then a map $f : S \longrightarrow T$ is an **order isomorphism** if it is a bijection compatible with the order in both directions, i.e., if for $x, y \in S$,

$$x \leq y \text{ if and only if } f(x) \leq f(y). \tag{2.1}$$

If S and T are semilattices, then a function from S to T which preserve meets is a **meet-homomorphism** and if it preserves joins, it is a **join-homomorphism**. Namely these functions are given respectively by:

$$f(x \wedge y) = f(x) \wedge f(y) \text{ and } f(x \vee y) = f(x) \vee f(y), \quad \text{for all } x, y \in S. \tag{2.2}$$

If S and T are lattices, then a function f from S to T is a **homomorphism** if it preserves meets and joins, namely if it satisfies both equalities in (2.2).

Two semilattices are isomorphic as algebras if and only if they are order isomorphic in the sense of (2.1).

As usual in the set theory, a **family** of subsets of a set X is a function from an index set I into the power set $P(X)$. The co-domain of a family is the corresponding collection of subsets of X. A family is empty if $I = \emptyset$.

Let $\mathcal{C}$ be a collection of subsets of a nonempty set X (i.e., let $\mathcal{C} \subseteq \mathcal{P}(X)$, where $\mathcal{P}(X)$ is the power set of X). As it is known, this collection is called a **closure system** on X if it closed under set intersections of arbitrary (including empty) subcollections. A closure system is a complete lattice under inclusion.

For the rest of this paper, a complete meet-semilattice contains the bottom element, 0, and a complete join-semilattice contains the top, 1.

3. Main Results

Let $(S, \leq)$ be a meet (join)-semilattice and $X \neq \emptyset$. A mapping $\mu : X \to S$ is called a **meet** (**join**)-**fuzzy set** on X, or a **fuzzy set** on X. The collection

$$S^X = \{\mu \mid \mu : X \to S\},$$

is called the **meet** (**join**) **fuzzy power set** of X. The collection S^X is ordered with respect to the ordering relation in S:

$$\mu \leq \nu \text{ if and only if for every } x \in X, \ \mu(x) \leq \nu(x).$$

Under this order, S^X is a semilattice. The **image** of X under μ is denoted as usual, by $\mu(X)$:

$$\mu(X) = \{p \in P \mid p = \mu(x), \text{ for some } x \in X\}.$$

Let $\mu \in S^X$ and $p \in S$. Then a **cut set** (**cut**) of μ is a subset of X defined by

$$\mu_p = \{x \in X \mid \mu(x) \geq p\}.$$

In other words, a cut set of μ is the **inverse image** of the principal filter generated by p under μ:

$$\mu_p = \mu^{-1}(\uparrow p).$$

It is obvious that for $p, q \in S$,

$$p \leq q \rightarrow \mu_q \subseteq \mu_p. \tag{3.1}$$

The **collection of all cuts of** μ is denoted by μ_S:

$$\mu_S = \{\mu_p \mid p \in S\}.$$

Here we investigate order theoretic properties of μ_S under the set inclusion.

In the following we use the known fact that if a particular subset of any complete meet-semilattice S has an upper bound, then join of such subset exists.

Proposition 3.1. *If S is a complete meet-semilattice and $\mu : X \rightarrow S$ is a function in S^X, then for every $x \in X$,*

$$\mu(x) = \bigvee \{p \in S \mid x \in \mu_p\},$$

which means that the join on the right hand side exists and is equal to $\mu(x)$.

Proof. Let $M = \{p \in S \mid x \in \mu_p\}$. If $s \in M$, then for every $x \in X$ we have

$$x \in \mu_s, \text{ if and only if } s \leq \mu(x).$$

It means that $\mu(x) \in M^u$, i.e., $M^u \neq \emptyset$ and thus the join of M exists, since it is the meet of all upper bounds.

Since $x \in \mu_{\mu(x)}$, $\mu(x)$ is an element of M. Due to the fact that it is an upper bound of M, $\mu(x) = \bigvee \{p \in S \mid x \in \mu_p\}$. □

An analogous proposition concerning join-fuzzy sets is also valid.

In the following, we deal with the converse problem. Starting with a particular family of subsets of a domain, we describe a construction of a fuzzy set whose cuts are members of the family.

Using the existence of the join of a nonempty subset M of a semilattice S, we get the following lemma.

Lemma 3.2. *Let $(S, \leq)$ be a complete meet-semilattice and $\mu \in S^X$. If $M^u \neq \emptyset$, for a subset M of S, then*

$$\bigcap \{\mu_p \mid p \in M\} = \mu_{\bigvee \{p \mid p \in M\}}. \tag{3.2}$$

Proof. If $M^u \neq \emptyset$, then $\bigvee \{p \mid p \in M\}$ exists. Since $p \leq \bigvee \{p \mid p \in M\}$ for every $p \in M$, $\mu_{\bigvee \{p \mid p \in M\}} \subseteq \bigcap \{\mu_p \mid p \in M\}$. On the other hand, if $x \in \bigcap \{\mu_p \mid p \in M\}$, then $\mu(x) \geq p$ for every $p \in M$, hence $\mu(x) \geq \bigvee \{p \mid p \in M\}$ and $x \in \mu_{\bigvee \{p \mid p \in M\}}$. This means that the opposite inclusion holds, hence also the equality. □

The analogous statement for join-fuzzy sets follows.

Lemma 3.3. *Let $(S, \leq)$ be a complete join-semilattice and $\mu \in S^X$. If M is a nonempty subset of S, then*

$$\bigcap\{\mu_p \mid p \in M\} = \mu_{\bigvee\{p \mid p \in M\}}.$$

The proof is similar as in Lemma 3.2; upper bounds in a complete join-semilattice S exist for every nonempty subset.

The next lemma is a straightforward application of properties of meet (join)-semilattices to the collection of cuts of μ on X.

Lemma 3.4. *Let μ be a meet (join)-fuzzy set on X and μ_S be a collection of cuts of μ. Then the following conditions are satisfied.*

1. *If $(S, \leq)$ is a meet-semilattice, then $(\mu_S, \subseteq)$ is a join-semilattice.*
2. *If $(S, \leq)$ is a join-semilattice, then $(\mu_S, \subseteq)$ is a meet-semilattice.*

The following property could be called the Theorem of synthesis for a fuzzy set over a meet (join)-semilattice.

Theorem 3.5. *Let $\{M_i \mid i \in S\}$ be a family of subsets of a nonempty set X, indexed by elements of meet-semilattice S in the following way:*

For every $x \in X$, $\bigvee\{p \mid x \in M_p\}$ exists in S and

$$\bigcap\{M_p \mid x \in M_p\} = M_{\bigvee\{p \mid x \in M_p\}}. \tag{3.3}$$

Then, $\{M_i \mid i \in S\}$ is the family of cut sets of fuzzy set $\mu : X \to S$, defined by

$$\mu(x) = \bigvee\{p \in S \mid x \in M_p\}. \tag{3.4}$$

Proof. We note that from the condition (3.3) it follows that for all $p, q \in S$, if $p \leq q$ then $M_q \subseteq M_p$.

Function μ is well defined. Now we prove that $\mu_S = \{M_i \mid i \in S\}$, i.e., for every $p \in S, \mu_p = M_p$.

Let $p \in S$ and let $x \in \mu_p$. This means that

$$\bigvee\{p \in S \mid x \in M_p\} = \mu(x) \geq p.$$

From $\bigcap\{M_p \mid x \in M_p\} = M_{\bigvee\{p \in S \mid x \in M_p\}} \subseteq M_p$, we have $x \in M_p$.

Now, suppose that $x \in M_p$ for some $p \in S$. By the assumption $\bigvee\{p \in S \mid x \in M_p\}$ exist and $\mu(x) = \bigvee\{p \in S \mid x \in M_p\} \geq p$. Therefore, $x \in \mu_p$.

We proved that for every $p \in S$, $M_p = \mu_p$. □

Remark 3.6. The theorem analogous to the previous one is valid also for join-semilattices, in which case the condition that for every $x \in X$, $\bigvee\{p \mid x \in M_p\}$ exists in S is superfluous.

An illustration of the theorem is given in the following example.

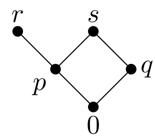

Figure 1: Meet-semilattice S

Example 3.7. Let $X = \{a, b, c\}$ be an abstract finite set and let the meet-semilattice S be as in Figure 1.

Let $M_S = \{\{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$ be the family indexed by elements from S by the mapping $i : S \to M_S \subseteq P(X)$ defined by $i(s) = M_s$ for each $s \in S$ as follows:
$M_p = \{b, c\},\ M_q = \{a, c\},\ M_r = \{b\},\ M_s = \{c\}\ \ M_0 = \{a, b, c\}$.

x	M_t	t	N	$\vee N$
a	M_0, M_q	$0, q$	$\{0, q\}$	q
b	M_0, M_p, M_r	$0, p, r$	$\{0, p, r\}$	r
c	M_0, M_p, M_q, M_s	$0, p, q, s$	$\{0, p, q, s\}$	s

Table 1: join of particular subset N of S

Using the formula (3.4) we formulate the definition of function μ as follows.

$$\mu = \begin{pmatrix} a & b & c \\ q & r & s \end{pmatrix}$$

As a consequence, the collection of cuts of μ coincides with the starting family M_S of subsets of X.

The collection of cuts can be ordered by the inclusion as a join-semilattice (Figure 2).

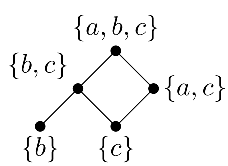

Figure 2: Join-semilattice of collection μ_S of cuts

In the case of fuzzy sets with the lattice co-domain, it is proved in [4] that the collection of cuts is a closure system. In the present discussion, by the construction of a meet (join)-fuzzy set μ, the collection of cuts of μ is not a closure system in general. The following is a counter example in the case of meet-semilattices.

Example 3.8. Consider The abstract finite set $X = \{x, y, z\}$ and the meet-semilattice S given by the diagram in Figure 3.

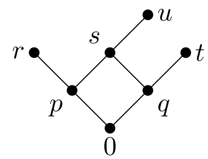

Figure 3: Meet-semilattice S

Let μ be the meet-fuzzy set on X defined as follows:

$$\mu = \begin{pmatrix} x & y & z \\ r & u & t \end{pmatrix}.$$

All cut sets of function μ are:

$$\mu_0 = \{x, y, z\}, \mu_p = \{x, y\}, \mu_q = \{y, z\}, \mu_r = \{x\}, \mu_s = \mu_u = \{y\}, \mu_t = \{z\}.$$

The collection of cuts μ_S is a join-semilattice but it is not closed under intersections.

Hence, in general the collection of cuts of a meet (join)-fuzzy set is not a closure system.

However, such collection of cuts satisfies a similar property, but it is weaker than the mentioned one, as defined in the sequel.

Definition 3.9. Let X be a nonempty set and $\mathcal{F}$ a collection of subsets in X. Then $\mathcal{F}$ is called

a **semi-closure system** if it is closed under nonempty intersection of arbitrary subcollection, i.e., if $\mathcal{F}_1 \subseteq \mathcal{F}$, and $\cap\mathcal{F}_1 \neq \emptyset$, then $\cap\mathcal{F}_1 \in \mathcal{F}$.

a **dual semi-closure system** if it closed under intersection of arbitrary nonempty subcollection, i.e., if $\mathcal{F}_1 \subseteq \mathcal{F}$ and $\mathcal{F}_1 \neq \emptyset$, then $\cap\mathcal{F}_1 \in \mathcal{F}$.

Since the intersection of an empty family of subsets of X is X, we have that a semi-closure system is a complete join-semilattice under inclusion.

Analogously, if $\mathcal{F}$ is a dual semi-closure system, then $\bigcap\mathcal{F}$ (possibly empty) is the smallest set in $\mathcal{F}$ under inclusion. Hence, a dual semi-closure system is a complete meet-semilattice under inclusion.

Theorem 3.10. *Let X be a nonempty set.*

(i) If S is a complete meet-semilattice and μ is a fuzzy set in S^X, then the collection μ_S of cut sets of μ, ordered by inclusion, is a semi-closure system on X.

(ii) If S is a complete join-semilattice and μ is a fuzzy set in S^X, then the collection μ_S of cut sets of μ, ordered by inclusion, is a dual semi-closure system on X.

Proof. (i) By Proposition 3.1 there is $R = \{s \in S | x \in \mu_s\} \subseteq S$ such that $\mu(x) = \bigvee R = q$ for some $q \in S$. Let $\mathcal{F} \subseteq \mu_S$, it means that $\mathcal{F} = \{\mu_p | p \in M\}$ for some $M \subseteq S$. Let $M = R$. Then the smallest cut $\bigcap \mathcal{F} = \bigcap\{\mu_p | p \in M\}$ is not empty as there is the greatest element $q = \bigvee R$ in S. We obtain directly by Lemma 3.2 $\bigcap \mathcal{F} = \bigcap\{\mu_p | p \in R\} = \mu_{\bigvee R} = \mu_q$ for some $q \in S$. It means that μ_S closed under intersection. Therefore μ_S is semi-closure system on X.

(ii) Let $\mathcal{F} = \{\mu_p | p \in M\} \subseteq \mu_S$ for some $M = R$ and $\mathcal{F} \neq \emptyset$ immediately. It is proved by Lemma 3.3. □

The following theorem is the converse of the Theorem 3.10 (ii). By a fuzzy set construction, we identify the collection of cuts of join-fuzzy sets with semi-closure systems.

Theorem 3.11. *Let $\mathcal{F}$ be a semi-closure system over X such that $\bigcup \mathcal{F} = X$. Then there is a complete meet-semilattice S and a meet-fuzzy set $\mu : X \to S$ such that $\mu_S = \mathcal{F}$.*

Proof. From the assumption that $\mathcal{F}$ is a semi-closure system over X, $(\mathcal{F}, \subseteq)$ is a complete join-semilattice. By dual inclusion $\supseteq_{\mathcal{F}}$, we have that $(S, \supseteq_{\mathcal{F}})$ is a complete meet-semilattice.

Let $\mu : X \to S$ be defined by: $\mu(x) = \bigcap\{f \in \mathcal{F} \mid x \in f\}$ (as in [3] for poset-valued fuzzy sets). The mapping μ is well defined, since the set $\bigcap\{f \in \mathcal{F} \mid x \in f\}$ is not empty (contains at least x) and $\mathcal{F}$ is closed under intersections (if not empty).

Now, we can prove that for every $f \in \mathcal{F}$, $\mu_f = f$, similarly as in [3]. □

Example 3.12. Let $X = \{a, b, c\}$ be an abstract finite set and let $\mathcal{F}$ be the following collection of subsets of X:

$$F = \{\{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}\},$$

which is a semi-closure system. $\mathcal{F}$ is a join-semilattice under inclusion (Figure 4). Under the dual inclusion, we found meet-semilattice in Figure 5.

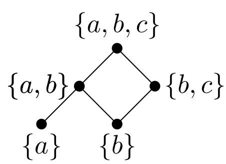

Figure 4: Join-semilattice $\mathcal{F}$

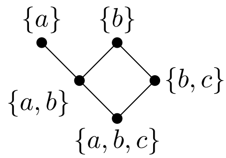

Figure 5: Meet-semilattice S

We calculated intersection of all subsets in $\mathcal{F}$ containing x for every $x \in X$ and we obtain a meet-fuzzy set μ:

$$\mu = \begin{pmatrix} a & b & c \\ \{a\} & \{b\} & \{c\} \end{pmatrix}$$

with the following cuts of the meet-fuzzy set μ.

$$\mu_{\{a\}} = \{a\}, \mu_{\{b\}} = \{b\}, \mu_{\{a,b\}} = \{a,b\}, \mu_{\{b,c\}} = \{b,c\}, \nu_{\{a,b,c\}} = \{a,b,c\}.$$

Therefore, we have for every $f \in \mathcal{F}, \mu_f = f$, i.e., $\mu_{\mathcal{F}} = \mathcal{F}$.

The following theorem is analogous to Theorem 3.11, dealing with fuzzy sets obtained from dual semi-closure systems.

Theorem 3.13. *Let $\mathcal{F}$ be a dual semi-closure system over a nonempty set X such that $\bigcup \mathcal{F} = X$. Then there is a complete join-semilattice S and a join fuzzy set $\mu : X \to S$, such that $\mu_S = \mathcal{F}$.*

Proof. Analogous to the proof of Theorem 3.11. □

In the following theorem we generalize Theorems 3.11 and 3.13. Namely, we prove that *every* nonempty collection of subsets on a nonempty set can consist of some cuts of a suitable meet(join)-fuzzy set.

Theorem 3.14. *Let $\mathcal{F}$ be an arbitrary nonempty collection of subsets of a nonempty set X. Then there is a complete meet (join)-semilattice S and a fuzzy set $\mu : X \to S$, such that $\mathcal{F} \subseteq \mu_S$, and for every $f \in \mathcal{F}$, $\mu_f = f$.*

Proof. Let $\mathcal{F} \subseteq \mathcal{P}(X)$ and $\mathcal{F} \neq \emptyset$, for $X \neq \emptyset$. We construct a collection $\mathcal{S}$ of subsets of X as follows. $\mathcal{S} = \mathcal{F} \cup \mathcal{G}$, where $\mathcal{G}$ consists of all nonempty missing set-intersections of arbitrary subcollections of $\mathcal{F}$, including X which is the intersection of the empty set as a subcollection. Obviously, $(\mathcal{S}, \leq)$ is a complete meet-semilattice, where the order $\leq$ is the dual of the set inclusion. Define $\mu : X \to \mathcal{S}$ by

$$\mu(x) = \bigcap (s \in \mathcal{S} \mid x \in s),$$

which is defined for every $x \in X$, since the intersection is not $\emptyset$, because it contains at least x.

Now, if $f \in \mathcal{F}$, then μ_f is a cut of μ, and we have:

$x \in \mu_f$ if and only if $\mu(x) \geq f$ if and only if $\bigcap(s \in \mathcal{S} \mid x \in s) \subseteq f$ if and only if $x \in f$.

Hence, $\mu_f = f$ for every $f \in \mathcal{F}$.

Similarly, starting with $\mathcal{F}$, we can construct S to be a dual semi-closure system. Namely, we add to $\mathcal{F}$ all missing set intersections of nonempty subcollections of $\mathcal{F}$. Then obviously, $(\mathcal{S}, \leq)$ is a complete join-semilattice, where, as above, the order $\leq$ is the dual of the set inclusion. Now the construction of the fuzzy set μ is analogous to the one described in the first part of the proof. Again, as above, we have $\mu_f = f$ for every $f \in \mathcal{F}$. □

In the final part we represent a finite meet-semilattice as a family of cuts of a semilattice valued fuzzy set. Recall that a is a meet irreducible element in a meet-semilattice $(S, \wedge)$, if from $a = b \wedge c$ it follows that $a = b$ or $a = c$ (see [5]).

Proposition 3.15. *Let S be a finite meet-semilattice, M a set of all meet-irreducible elements of S and $\mu : X \to S$ a meet-fuzzy set. If $M \subseteq \mu(X)$, then all the cuts are different and $(\mu(X), \supseteq)$ is a meet-semilattice isomorphic to $(S, \leq)$.*

Proof. For $p \in S$ let $M_p \subseteq M$ be the set of all meet-irreducible elements such that $p_i \in M_p$ if and only if $p \leq p_i$. It is easy to prove that every element in a meet semilattice is the meet of all meet-irreducible elements above it (the proof goes by induction on the height of the semilattice). Let $p \neq q$. By $p = \bigwedge M_p$ and $q = \bigwedge M_q$, we have that $M_p \neq M_q$. This means that e.g., there is an element $x \in M_p$ such that $x \notin M_q$, hence $q \not\leq x$. Now, since $\mu_p = \uparrow p \cap \mu(X)$, $\mu_q = \uparrow q \cap \mu(X)$, and $x \in \mu_p$, we have that $x \notin \mu_q$ and $\mu_p \neq \mu_q$. Hence, all the cuts are different.

Now, consider the mapping $p \mapsto \mu_p$, which is a bijection by previous considerations. By 3.1, if $p \leq q$, then $\mu_q \subseteq \mu_p$. Now, suppose that $\mu_q \subseteq \mu_p$, i.e., $\uparrow q \cap \mu(X) \subseteq \uparrow p \cap \mu(X)$. Since $M \subseteq \mu(X)$, we have that $M_q \subseteq \uparrow p \cap \mu(X)$, hence $M_q \subseteq M_p$ and hence $p = \bigwedge M_p \leq \bigwedge M_q = q$. Thus, $(\mu(X), \supseteq)$ is order isomorphic to $(S, \leq)$ and thus it is also a meet-semilattice.

□

The following example is a consequence stating that each meet-semilattice can be represented by a semilattice of cuts of a meet-fuzzy set.

Corollary 3.16. *Let S be a finite meet-semilattice, then there is a meet-fuzzy set such that the family of cuts under inclusion is anti-isomorphic to S.*

Conclusion and Future Work

Our investigation of semilattice-valued fuzzy sets is mostly oriented to special collections of subsets of the domain, being cutsof these fuzzy sets. It turned out that these are weaker than closure systems, and we were using them to give representation theorems not only for semilattice-valued fuzzy sets, but also for semilattices.

As future work, we are planing to investigate under which condition the equivalence relation induced by equality of cuts is a congruence relation on the semilattice.

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