Novi Sad J. Math.
Vol. 46, No. 1, 2016, 1-14

ON APPROXIMATION TO FUNCTIONS IN THE $W(L^p, \xi(t))$ CLASS BY A NEW MATRIX MEAN

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Abstract. In this paper we shall present results related to trigonometric approximation of functions belonging to the weighted generalized Lipschitz class by the $(C^1 \cdot T)$ matrix means of their Fourier series. The results of Lal in [8] will be extended to a more general summability method. Moreover, we present the results on degree of approximation to conjugates of functions belonging to a weighted generalized Lipschitz class by the $(C^1 \cdot T)$ matrix means of their conjugate Fourier series.

AMS Mathematics Subject Classification (2010): 41A25; 42A05; 42A10; 42A24; 42A50

Key words and phrases: degree of approximation; trigonometric approximation; Fourier series; weighted generalized Lipschitz class; matrix means

1. Introduction and Notations

Summability methods have been used in various fields of mathematics. For example, summability methods are applied in function theory in connection with the analytic continuation of holomorphic functions and the boundary behaviour of a power series, in applied analysis for generation of iteration methods for the solution of a system of linear equations, and for acceleration of convergence in approximation theory, in the theory of Fourier series both for creation and acceleration of convergence of a Fourier series, and in other fields of mathematics like probability theory (Markov chains) and number theory (prime number theorem) [1]. In this work we are interested in a summability method in the theory of Fourier series. For this aim, we shall give the following notations to be used in this paper.

Let $L := L(0, 2\pi)$ denote the space of functions that are $2\pi-$ periodic and Lebesque integrable on $[0, 2\pi]$ and let

$$S[f] = \frac{a_o}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx) \equiv \sum_{k=0}^{\infty} A_k(f; x) \tag{1.1}$$

be the Fourier series of a function $f \in L$; i.e., for any $k = 0, 1, 2, \cdots$

$$a_k = a_k(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos ktdt \ , \ \ b_k = b_k(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin ktdt.$$

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Let

$$s_n(f;x)=\frac{1}{2}a_0+\sum_{k=1}^{n}(a_k\cos kx+b_k\sin kx)\equiv\sum_{k=1}^{n}A_k(f;x)$$

denote the partial sum of the first $(n+1)$ terms of the Fourier series of $f\in L$ at a point x. The conjugate series of (1.1) is given by

$$\tilde{S}[f]=\sum_{k=1}^{\infty}(a_k\sin kx-b_k\cos kx)\equiv\sum_{k=1}^{\infty}\tilde{A}_k(f;x).$$

Note that there is no free term in $\tilde{S}[f]$. Therefore, the series conjugate to the series $\tilde{S}[f]$ is the series $S[f]$ without free term.

The function $\tilde{f}\in L$ for which $S[\tilde{f}]=\tilde{S}[f]$ is called trigonometrically conjugate, or simply conjugate, to $f(\cdot)$. It can be shown that the functions $f(\cdot)$ and $\tilde{f}(\cdot)$ are connected by the equality

$$\tilde{f}(x)=-\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x+t)cot\frac{t}{2}dt$$

$$=-\frac{1}{2\pi}\int_{0}^{\pi}\eta(t)cot\frac{t}{2}dt=-\frac{1}{2\pi}\lim_{\varepsilon\to0}\int_{\varepsilon}^{\pi}\eta(t)cot\frac{t}{2}dt \tag{1.2}$$

where $\eta(t):=\eta(x,t)=f(x+t)-f(x-t)$. If $f\in L$, then equality (1.2) exists for almost all x [25].

Let

$$\tau_n(f;x)=\tau_n(f,T;x):=\sum_{k=0}^{n}a_{n,k}s_k(f;x),\quad\forall n\geq0$$

where $T\equiv(a_{n,k})$ is a lower triangular infinite matrix satisfying the Silverman-Toeplitz[24] condition of regularity such that:

$$a_{n,k}=\begin{cases}\geq0, & k\leq n;\\ 0, & k>n\end{cases}\qquad(k,n=0,1,2,\ldots)$$

and

$$\sum_{k=0}^{n}a_{n,k}=1,\quad(n=0,1,2,\ldots). \tag{1.3}$$

The Fourier series of a function f is said to be T-summable to s, if $\tau_n(f;x)\to s(x)$ as $n\to\infty$. The Fourier series of f is called Cesàro-T $(C^1\cdot T)$ summable to $s(x)$ if

$$t_n^{CT}:=\frac{1}{n+1}\sum_{m=0}^{n}\tau_m(f;x)=\frac{1}{n+1}\sum_{m=0}^{n}\sum_{k=0}^{m}a_{m,k}s_k(f;x)\to s(x),$$

as $n\to\infty$.

The Cesàro-T $(C^1\cdot T)$ means give us the following means for some important cases:

- Cesàro- Nörlund $(C^1 \cdot N_p)$ means, with

$$a_{m,k} = \begin{cases} \frac{p_{m-k}}{P_m}, & k \leq m; \\ 0, & k > m \end{cases} \qquad (k, m = 0, 1, 2, \ldots), P_m = \sum_{k=0}^{m} p_k \neq 0;$$

- $(C, 1)(E, 1)$ Product means, with $a_{m,k} = \frac{1}{2^m}\binom{m}{k}$;
- $(C, 1)(E, q)$ Product means, with $a_{m,k} = \frac{1}{(1+q)^m}\binom{m}{k} q^{m-k}$;
- Generalized Nörlund means, with $a_{m,k} = \dfrac{p_{m-k} q_k}{r_m}$ where $r_m = \sum_{k=0}^{m} p_{m-k} q_k$.

The degree of approximation of a function $f : \mathbb{R} \to \mathbb{R}$ by a trigonometric polynomial T_n of degree n is defined by

$$\|T_n - f\|_\infty = \sup\{|T_n(x) - f(x)|, x \in \mathbb{R}\}$$

with respect to the supremum norm [25]. The degree of approximation of a function $f \in L_p$ $(p \geq 1)$ is given by

$$E_n(f) = \min_n \|T_n - f\|_p$$

where $\|.\|_p$ denotes the L_p-norm with respect to x and will be defined by

$$\|f\|_p := \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(x)|^p \, dx \right\}^{\frac{1}{p}}.$$

This method of approximation is called the trigonometric Fourier approximation.

We recall the following definitions:

1. A function f is said to belong to the $Lip\alpha$ class if $|f(x+t) - f(x)| = O(|t^\alpha|)$, $0 < \alpha \leq 1$;
2. A function f is said to belong to the $Lip(\alpha, p)$ class if $\omega_p(\delta, f) = O(\delta^\alpha)$, where

$$\omega_p(\delta, f) = \sup_{|t| \leq \delta} \{\frac{1}{2\pi} \int_0^{2\pi} |f(x+t) - f(x)|^p dx\}^{\frac{1}{p}}, \quad 0 < \alpha \leq 1; \quad p \geq 1;$$

3. A function f is said to belong to the $Lip(\xi(t), p)$ class if $\omega_p(\delta, f) = O(\xi(t))$ where $\xi(t)$ is a positive increasing function and $p \geq 1$;
4. We write $f \in W(L^p, \xi(t))$ if $\|(f(x+t) - f(x)) sin^\beta(x/2)\|_p = O(\xi(t))$, $\beta \geq 0$ and $p \geq 1$.

If $\beta = 0$, then $W(L^p, \xi(t))$ reduces to $Lip(\xi(t), p)$; and, if $\xi(t) = t^\alpha$, the $Lip(\xi(t), p)$ class reduces to the $Lip(\alpha, p)$ class. If $p \to \infty$ then the $Lip(\alpha, p)$ class coincides with the $Lip\alpha$ class. Accordingly, we have the following inclusions:

$$Lip\alpha \subset Lip(\alpha, p) \subset Lip(\xi(t), p) \subset W(L^p, \xi(t))$$

for all $0 < \alpha \leq 1$ and $p \geq 1$.

2. Approximation by matrix means of a Fourier series

The degree of approximation, using the various summability methods in the $Lip\alpha$ class, has been determined by many mathematicians such as Bernstein [25], de la Valle-Poussin [25], Jackson [25], Mcfadden [4]. Similar problems for the $Lip(\alpha, p)$ class have been studied by researchers like Quade [18], Khan [5], Qureshi [20], Chandra[2], Leindler [11]. Other research related to the $Lip(\alpha, p)$ class can also be found in [3], [12], [13], [14] and [15].

The weighted $W(L^p, \xi(t))$ class is a generalization of the classes $Lip\alpha$, $Lip(\alpha, p)$ and $Lip(\xi(t), p)$. The degree of approximation of a function belonging to the weighted $W(L^p, \xi(t))$ class has been studied by Qureshi in [21]. In [6] and [8] Lal has considered the degree of approximation of functions belonging to the weighted $W(L^p, \xi(t))$ class by the $(C, 1)(E, 1)$ means and $(C^1 \cdot N_p)$ means, respectively. Nigam has studied the same problem for the $(C, 1)(E, q)$ means, which are much more general than the $(C, 1)(E, 1)$ means in [16]. Sing, Mittal and Sonker have generalized the results of Lal[8] in [23]. Therefore, taking into account this generalization of the function classes, we shall give two theorems on degree of approximation to functions belonging to the classes $W(L^p, \xi(t))$ and $Lip\alpha$ by the $(C^1 \cdot T)$ matrix means, being more general than $(C, 1)(E, 1)$, $(C^1 \cdot N_p)$ and $(C, 1)(E, q)$ means given in [6], [8, 23] and [16], respectively.

Also, throughout this section, we shall use the following notations:

$$\Psi(x, t) := \Psi(t) = f(x + t) + f(x - t) - 2f(x)$$

and

$$K_T(n, t) := \frac{1}{2\pi(n + 1)} \sum_{m=0}^{n} \sum_{k=0}^{m} a_{m,k} \frac{sin(k + \frac{1}{2})t}{sin(\frac{t}{2})}.$$

Before stating the theorems, we develop the following auxiliary results needed in the proofs of both of them.

Lemma 2.1. *For $0 < t \leq \pi/n$, we have $K_T(n, t) = O(n)$.*

Proof. For $0 < t \leq \pi/n$, from $(sin(t/2))^{-1} \leq \pi/t$ and $sin(n + 1)t \leq (n + 1)t$, we have

$$|K_T(n, t)| \leq \frac{1}{2\pi(n + 1)} \sum_{m=0}^{n} \sum_{k=0}^{m} a_{m,k} \left| \frac{sin(k + \frac{1}{2})t}{sin(\frac{t}{2})} \right|$$

$$\leq \frac{2n + 1}{2\pi(n + 1)} \sum_{m=0}^{n} \sum_{k=0}^{m} a_{m,k} = O(n)$$

by considering (1.3). □

Lemma 2.2. *For $\pi/n < t \leq \pi$ and any n, we have*

$$K_T(n,t) = O(\frac{t^{-2}}{n+1}) + O(t^{-1}).$$

Proof.

$$\begin{aligned} K_T(n,t) &= \frac{1}{2\pi(n+1)sin(\frac{t}{2})} \sum_{m=0}^{n}\sum_{k=0}^{m} a_{m,k} sin(k+\frac{1}{2})t \\ &= \frac{1}{2\pi(n+1)sin(\frac{t}{2})} \left\{ \sum_{m=0}^{\tau} + \sum_{m=\tau+1}^{n} \right\} \sum_{k=0}^{m} a_{m,k} sin(k+\frac{1}{2})t \\ &= : I_1 + I_2, \end{aligned}$$

where τ denotes the integer part of $1/t$. Owing to (1.3) and Jordan's inequality, $(sin(t/2))^{-1} \leq \pi/t$, for $0 < t \leq \pi$, we obtain

$$|I_1| = O\left(\frac{1}{(n+1)t}\right) \sum_{m=0}^{\tau}\sum_{k=0}^{m} a_{m,k} = O(\frac{\tau t^{-1}}{n+1}) = O(\frac{t^{-2}}{n+1}). \tag{2.1}$$

We now estimate I_2. By using (1.3) again and the Jordan inequality $(sin(\frac{t}{2}))^{-1} \leq \pi/t$, for $0 < t \leq \pi$, we get

$$|I_2| = O\left(\frac{1}{(n+1)t}\right) \sum_{m=\tau+1}^{n}\sum_{k=0}^{m} a_{m,k} = O\left(\frac{(n-\tau)t^{-1}}{n+1}\right) = O(1/t). \tag{2.2}$$

Combining (2.1) and (2.2), we have

$$K_T(n,t) = O(\frac{t^{-2}}{n+1}) + O(t^{-1}).$$

□

Theorem 2.3. *Let $f \in L$ and let $T \equiv (a_{n,k})$ be a lower triangular regular matrix with nonnegative entries and row sums 1. If $f \in Lip\alpha$ $(0 < \alpha \leq 1)$, then the degree of approximation by the $(C^1 \cdot T)$ means of its Fourier series is given by*

$$\|t_n^{CT}(f) - f(x)\|_\infty = \begin{cases} O(n^{-\alpha}), & 0 < \alpha < 1; \\ O(\frac{logn}{n}), & \alpha = 1. \end{cases}$$

Proof. We know that

$$s_n(f,x) - f(x) = \frac{1}{2\pi} \int_0^{\pi} \Psi(t) \left(\frac{sin(n+\frac{1}{2})t}{sin(\frac{t}{2})}\right) dt. \tag{2.3}$$

Taking into account (2.3) and the definitions of $t_n^{CT}(f)$ and the $(C^1 \cdot T)$ means of $s_n(f)$, we write

$$\begin{aligned} |t_n^{CT}(f) - f(x)| &= \frac{1}{n+1}\left|\sum_{m=0}^{n}\sum_{k=0}^{m} a_{m,k}(s_k(f;x) - f(x))\right| \\ &= \frac{1}{2\pi(n+1)}\left|\int_0^{\pi} \Psi(t)\sum_{m=0}^{n}\sum_{k=0}^{m} a_{m,k}\left(\frac{sin(k+\frac{1}{2})t}{sin(\frac{t}{2})}\right) dt\right| \\ &\leq \int_0^{\pi} |\Psi(t)K_T(n,t)|dt = \left[\int_0^{\pi/n} + \int_{\pi/n}^{\pi}\right] |\Psi(t)K_T(n,t)|dt \\ &= : J_1 + J_2. \end{aligned}$$

Since $f \in Lip\alpha$, $\Psi(t)$ belongs to the $Lip\alpha$ class. Therefore, from Lemma 2.1, we obtain

$$J_1 = \int_0^{\pi/n} |\Psi(t)K_T(n,t)|dt = O(n)\int_0^{\pi/n} t^{\alpha}dt = O(n^{-\alpha}) \tag{2.4}$$

for $0 < \alpha \leq 1$.
By using Lemma 2.2, then we have

$$\begin{aligned} J_2 &= \int_{\pi/n}^{\pi} |\Psi(t)K_T(n,t)|dt = O\left\{\int_{\pi/n}^{\pi} t^{\alpha}\left(\frac{t^{-2}}{n+1} + t^{-1}\right) dt\right\} \\ &= O\left\{\int_{\pi/n}^{\pi} \frac{t^{\alpha-2}}{n+1}dt\right\} + O\left\{\int_{\pi/n}^{\pi} t^{\alpha-1}dt\right\} =: J_2^1 + J_2^2. \end{aligned}$$

Accordingly,

$$J_2^1 = O\left\{\int_{\pi/n}^{\pi} \frac{t^{\alpha-2}}{n+1}dt\right\} = \begin{cases} O(n^{-\alpha}), & 0 < \alpha < 1; \\ O(\frac{logn}{n}), & \alpha = 1. \end{cases} \tag{2.5}$$

and

$$J_2^2 = O\left\{\int_{\pi/n}^{\pi} t^{\alpha-1}dt\right\} = O\left\{n^{-\alpha}\right\}. \tag{2.6}$$

Taking into account (2.4), (2.5) and (2.6), we obtain

$$\|t_n^{CT}(f) - f(x)\|_{\infty} = \sup_{x\in[0,2\pi]} |t_n^{CT}(f) - f(x)| = \begin{cases} O(n^{-\alpha}), & 0 < \alpha < 1; \\ O(\frac{logn}{n}), & \alpha = 1. \end{cases}$$

by using $1/n \leq logn/n$, for large values of n. Therefore the proof of Theorem 2.3 is completed. □

Theorem 2.4. *Let $f \in L$ and $\xi(t)$ be a positive increasing function. If $f \in W(L^p, \xi(t))$ with $0 \leq \beta \leq 1 - 1/p$, the degree of approximation by $(C^1 \cdot T)$ means of its Fourier series is given by*

$$\|t_n^{CT}(f) - f(x)\|_p = O(n^{\beta+1/p}\xi(\frac{1}{n})),$$

provided that the function $\xi(t)$ satisfies the following conditions:

$$\left\{\frac{\xi(t)}{t}\right\}$$

is a decreasing function and

$$\left\{\int_0^{\pi/n}\left(\frac{|\Psi(t)|sin^{\beta}(t/2)}{\xi(t)}\right)^p dt\right\}^{1/p} = O(1) \tag{2.7}$$

$$\left\{\int_{\pi/n}^{\pi}\left(\frac{|\Psi(t)|t^{-\delta}}{\xi(t)}\right)^p dt\right\}^{1/p} = O(n^{\delta}), \tag{2.8}$$

where δ is an arbitrary number such that $q(\beta - \delta) - 1 > 0$, $p^{-1} + q^{-1} = 1$, $p \geq 1$, and (2.7) and (2.8) hold uniformly in x.

Proof. Proceeding as above, we have

$$\begin{aligned}
|t_n^{CT}(f) - f(x)| &= \frac{1}{n+1}\left|\sum_{m=0}^{n}\sum_{k=0}^{m} a_{m,k}(s_k(f;x) - f(x))\right| \\
&= \frac{1}{2\pi(n+1)}\left|\int_0^{\pi}\Psi(t)\sum_{m=0}^{n}\sum_{k=0}^{m} a_{m,k}\left(\frac{sin(k+\frac{1}{2})t}{sin(\frac{t}{2})}\right)dt\right| \\
&\leq \left|\int_0^{\pi/n}\Psi(t)K_T(n,t)dt\right| + \left|\int_{\pi/n}^{\pi}\Psi(t)K_T(n,t)dt\right| \\
&=: J_3 + J_4. \qquad (2.9)
\end{aligned}$$

By considering Hölder's inequality, condition (2.7), Lemma 2.1, Jordan's inequality, and $\Psi(t)\in W(L^p,\xi(t))$, we get

$$\begin{aligned}
J_3 &= \left|\int_0^{\pi/n} \frac{\Psi(t)sin^\beta(t/2)}{\xi(t)}\frac{\xi(t)K_T(n,t)}{sin^\beta(t/2)}dt\right| \\
&\leq \left(\int_0^{\pi/n}\left|\frac{\Psi(t)sin^\beta(t/2)}{\xi(t)}\right|^p dt\right)^{1/p}\left(\lim_{\varepsilon\to 0}\int_\varepsilon^{\pi/n}\left|\frac{\xi(t)K_T(n,t)}{sin^\beta(t/2)}\right|^q dt\right)^{1/q} \\
&= O(1)\left(\lim_{\varepsilon\to 0}\int_\varepsilon^{\pi/n}\left(\frac{\xi(t)n}{sin^\beta(t/2)}\right)^q dt\right)^{1/q} = O(n\xi(\pi/n))\left(\lim_{\varepsilon\to 0}\int_\varepsilon^{\pi/n} t^{-\beta q}dt\right)^{1/q}
\end{aligned}$$

$$= O\left(\xi(\frac{1}{n})n^{1+\beta-1/q}\right) = O\left(n^{\beta+1/p}\xi(\frac{1}{n})\right), \tag{2.10}$$

in view of $p^{-1}+q^{-1}=1$ and $\xi(\pi/n)/(\pi/n)\leq\xi(1/n)/(1/n)$.
Now let us estimate J_4. By using Lemma 2.2, we write

$$J_4 = O\left(\int_{\pi/n}^{\pi}|\Psi(t)|\left(\frac{t^{-2}}{n+1}\right)dt\right)+O\left(\int_{\pi/n}^{\pi}|\Psi(t)|\left(t^{-1}\right)dt\right) =: J_4^1+J_4^2. \tag{2.11}$$

We shall evaluate J_4^1 and J_4^2 in a manner similar to the evaluation of J_3, respectively. Using Hölder's inequality, the (2.8) and Jordan's inequality, we have

$$\begin{aligned}
J_4^1 &= O(n^{-1})\left(\int_{\pi/n}^{\pi}\left(\frac{|\Psi(t)|t^{-\delta}sin^\beta(t/2)}{\xi(t)}\right)^p dt\right)^{1/p}\left(\int_{\pi/n}^{\pi}\left(\frac{\xi(t)t^{\delta-2}}{sin^\beta(t/2)}\right)^q dt\right)^{1/q} \\
&= O(n^{\delta-1})\left(\int_{\pi/n}^{\pi}\left(\frac{\xi(t)t^{\delta-2}}{sin^\beta(t/2)}\right)^q dt\right)^{1/q} = O(n^{\delta-1})\left(\int_{\pi/n}^{\pi}\left(\xi(t)t^{\delta-\beta-2}\right)^q dt\right)^{1/q} \\
&= O(n^{\delta-1})\left(\int_{1/\pi}^{n/\pi}\left(\xi(1/x)x^{\beta-\delta+2}\right)^q x^{-2}dx\right)^{1/q} \qquad ; x\xi(1/x) < \frac{n}{\pi}\xi(\pi/n) \\
&= O\left(\xi(\frac{\pi}{n})n^\delta\right)\left(\int_{1/\pi}^{n/\pi} x^{\beta q-\delta q+q-2}dx\right)^{1/q} = O\left(\xi(\frac{\pi}{n})n^\delta n^{1-\delta+\beta-(1/q)}\right)
\end{aligned}$$

$$= O\left(n^{\beta+1/p}\xi(\frac{1}{n})\right), \tag{2.12}$$

since $p^{-1}+q^{-1}=1$ and $\xi(\pi/n)/(\pi/n)\le \xi(1/n)/(1/n)$.

$$\begin{aligned}
J_4^2 &= O(1)\left(\int_{\pi/n}^{\pi}\left(\frac{|\Psi(t)|t^{-\delta}sin^{\beta}(t/2)}{\xi(t)}\right)^p dt\right)^{1/p}\left(\int_{\pi/n}^{\pi}\left(\frac{\xi(t)t^{\delta-1}}{sin^{\beta}(t/2)}\right)^q dt\right)^{1/q}\\
&= O(n^{\delta})\left(\int_{\pi/n}^{\pi}\left(\xi(t)t^{\delta-1-\beta}\right)^q dt\right)^{1/q}\\
&= O(n^{\delta})\left(\int_{1/\pi}^{n/\pi}\left(\xi(1/x)x^{\beta-\delta+1}\right)^q x^{-2}dx\right)^{1/q} \qquad ; x\xi(1/x)<\frac{n}{\pi}\xi(\pi/n)\\
&= O\left(\xi(\frac{\pi}{n})n^{\delta+1}\right)\left(\int_{1/\pi}^{n/\pi}x^{\beta q-\delta q-2}dx\right)^{1/q} = O\left(\xi(\frac{\pi}{n})n^{\delta+1}n^{\beta-\delta-(1/q)}\right)
\end{aligned}$$

$$= O\left(n^{\beta+1/p}\xi(\frac{1}{n})\right), \tag{2.13}$$

since $p^{-1}+q^{-1}=1$ and $\xi(\pi/n)/(\pi/n)\le\xi(1/n)/(1/n)$. Combining (2.9)-(2.13), we get

$$\|t_n^{CT}(f)-f(x)\|_p = O(n^{\beta+1/p}\xi(\frac{1}{n})).$$

□

3. In case of conjugate Fourier series

As mentioned above, the problems on determining the degree of approximation by summability methods have been studied by many mathematicians. Qureshi has determined the degree of approximation to functions which belong to the classes $Lip\alpha$ and $Lip(\alpha,p)$ by means of conjugate series in [19] and [22], respectively. In subsequent years, similar investigations have been made in researches such as in [7], [9], [16] and [17].

The following two theorems are related with the degree of approximation to conjugate of functions belonging to the classes $W(L^p,\xi(t))$ and $Lip\alpha$ by $(C^1\cdot T)$ matrix means of conjugate of their Fourier series and are more general than $(C,1)(E,1)$ and $(C,1)(E,q)$. Not only for $(C,1)(E,1)$ and $(C,1)(E,q)$ means but also different results are obtained for other means.

The following notations will be used throughout this section and auxiliary results:

$$\rho(x,t) := \rho(t) = f(x+t)+f(x-t)$$

and

$$\tilde{K}_T(n,t) := \frac{1}{2\pi(n+1)}\sum_{m=0}^{n}\sum_{k=0}^{m}a_{m,k}\frac{cos(k+\frac{1}{2})t}{sin(\frac{t}{2})}.$$

Lemma 3.1. *For $0 < t \leq \pi/n$, we have $\tilde{K}_T(n,t) = O(1/t)$.*

Proof. For $0 < t \leq \pi/n$, by $(sin(t/2))^{-1} \leq \pi/t$ and $|cos(2k+1)t| \leq 1$, we have

$$|\tilde{K}_T(n,t)| \leq \frac{1}{2\pi(n+1)} \sum_{m=0}^{n} \sum_{k=0}^{m} a_{m,k} \left| \frac{cos(k+\frac{1}{2})t}{sin(\frac{t}{2})} \right|.$$

$$\leq \frac{1}{2\pi t(n+1)} \sum_{m=0}^{n} 1 = O(1/t)$$

by (1.3). □

Lemma 3.2. *For $\pi/n < t \leq \pi$ and any n, we have*

$$\tilde{K}_T(n,t) = O\left(\frac{t^{-2}}{n+1}\right) + O(t^{-1}).$$

Proof. This lemma can be proved by using an argument similar to that of Lemma 2.2. □

Theorem 3.3. *Let $f \in L$ and let $T \equiv (a_{n,k})$ be a lower triangular regular matrix with nonnegative entries and row sums 1. If $f \in Lip\alpha$ $(0 < \alpha \leq 1)$, then the degree of approximation of the conjugate function $\tilde{f}$ by the $(C^1 \cdot T)$ means of its conjugate Fourier series is given by*

$$\|\widetilde{t_n^{CT}}(f) - \tilde{f}(x)\|_\infty = \begin{cases} O(n^{-\alpha}), & 0 < \alpha < 1; \\ O(\frac{logn}{n}), & \alpha = 1. \end{cases} \tag{3.1}$$

Proof. We have, by (1.2),

$$\tilde{s}_n(f,x) - \tilde{f}(x) = \frac{1}{2\pi} \int_0^\pi \rho(t) \left(\frac{cos(n+\frac{1}{2})t}{sin(\frac{t}{2})} \right) dt. \tag{3.2}$$

Taking into consideration (3.2) and $\widetilde{t_n^{CT}}(f)$ that $(C^1 \cdot T)$ means of $\tilde{s}_n(f)$, we write

$$|\widetilde{t_n^{CT}}(f) - f(x)| = \frac{1}{2\pi(n+1)} \left| \int_0^\pi \rho(t) \sum_{m=0}^{n} \sum_{k=0}^{m} a_{m,k} \left(\frac{cos(k+\frac{1}{2})t}{sin(\frac{t}{2})} \right) dt \right|. \tag{3.3}$$

Using Lemma 3.1 and Lemma 3.2, and proceeding as in the proof of Theorem 2.3 in (3.3) , we get (3.1). □

Theorem 3.4. *Let $f \in L$ and $\xi(t)$ be a positive increasing function. If $f \in W(L^p, \xi(t))$ with $0 \leq \beta \leq 1 - 1/p$, then the degree of approximation of the conjugate function $\tilde{f}$ by the $(C^1 \cdot T)$ means of its conjugate Fourier series is given by*

$$\|\widetilde{t_n^{CT}}(f) - \tilde{f}(x)\|_p = O(n^{\beta+1/p}\xi(\frac{1}{n})), \tag{3.4}$$

provided that the function $\xi(t)$ satisfies the following conditions:

$$\left\{\frac{\xi(t)}{t}\right\}$$

is a decreasing function and

$$\left\{\int_0^{\pi/n} \left(\frac{|\rho(t)| sin^\beta(t/2)}{\xi(t)}\right)^p dt\right\}^{1/p} = O(1) \tag{3.5}$$

$$\left\{\int_{\pi/n}^{\pi} \left(\frac{|\rho(t)| t^{-\delta}}{\xi(t)}\right)^p dt\right\}^{1/p} = O(n^\delta), \tag{3.6}$$

where δ is an arbitrary number such that $q(\beta-\delta)-1>0$, $p^{-1}+q^{-1}=1$, $p \geq 1$, and (3.5) and (3.6) hold uniformly in x.

Proof. Taking into account Lemma 3.1 and Lemma 3.2, and proceeding as in the proof of Theorem 2.4 in (3.3), we obtain (3.4). □

4. Corollaries and remarks

Using results given in Section 2 and Section 3, we observe the following corollaries and remarks.

Corollary 4.1. *If $\beta = 0$, then the weighted class $W(L^p, \xi(t))$ reduces to the class $Lip(\xi(t), p)$. Therefore, for $f \in Lip(\xi(t), p)$, we have*

$$\|t_n^{CT}(f) - f(x)\|_p = O(n^{1/p}\xi\left(\frac{1}{n}\right))$$

and

$$\|\widetilde{t_n^{CT}}(f) - \tilde{f}(x)\|_p = O(n^{1/p}\xi\left(\frac{1}{n}\right))$$

with respect to Theorem 2.4 and Theorem 3.4, respectively.

Corollary 4.2. *If $\beta = 0$ and $\xi(t) = t^\alpha$, $(0 < \alpha \leq 1)$, then the weighted class $W(L^p, \xi(t))$ reduces to the $Lip(\alpha, p)$ class. Therefore, for $f \in Lip(\alpha, p)$, $(1/p < \alpha)$, we have*

$$\|t_n^{CT}(f) - f(x)\|_p = O(n^{1/p-\alpha})$$

and

$$\|\widetilde{t_n^{CT}}(f) - \tilde{f}(x)\|_p = O(n^{1/p-\alpha})$$

with respect to Theorem 2.4 and Theorem 3.4, respectively.

Corollary 4.3. *If $p \to \infty$ in Corollary 4.2, then for $f \in Lip\alpha$, $(0 < \alpha < 1)$ we have*

$$\|t_n^{CT}(f) - f(x)\|_\infty = O(n^{-\alpha})$$

and

$$\|\widetilde{t_n^{CT}}(f) - \tilde{f}(x)\|_\infty = O(n^{-\alpha})$$

with respect to Theorem 2.4 and Theorem 3.4, respectively.

Remark 4.4. If $T \equiv (a_{m,k})$ is a Nörlund matrix, then the $(C^1 \cdot T)$ means give us the Cesàro- Nörlund $(C^1 \cdot N_p)$ means. Accordingly, our main theorems coincide with Theorem 2.3 and Theorem 2.4 in [23]. Moreover, our main results generalize the main results in [8] and [23].

Remark 4.5. If $a_{m,k} = \frac{1}{2^m}\binom{m}{k}$, then the $(C^1 \cdot T)$ means give us the $(C, 1)(E, 1)$ product means. In this case our main results are reduced to the $(C, 1)(E, 1)$ product means and the results given in Section 2 and Section 3 coincide with the results in [6] and [16], respectively.

Remark 4.6. If $a_{m,k} = \frac{1}{(1+q)^m}\binom{m}{k}q^{m-k}$, then the $(C^1 \cdot T)$ means give us the $(C, 1)(E, q)$ product means. Therefore, the results mentioned in Section 2 and Section 3 are reduced the main results in [10], [16] and [17].

Acknowledgement

This research was partially supported by the The Council of Higher Education of Turkey under a grant.

References

[1] Boos, J. and Cass, P., Classical and Modern Methods in Summability. Oxford University Press 2000, 586 pp.

[2] Chandra, P., Trigonometric approximation of functions in L_p-norm. Journal of Mathematical Analysis and Applications 275 (2002), 13–26.

[3] Değer, U., Dağadur, İ. and Küçükaslan, M., Approximation by trigonometric polynomials to functions in L_p-norm. Proc. Jangjeon Math. Soc. (15)2 (2012), 203-213.

[4] Fadden, L. Mc., Absolute Nörlund summability. Duke Math. J, Narosa 9 (1942), 168–207.

[5] Khan, H. H., On degree of approximation of functions belonging to the class $Lip(\alpha, p)$. Indian J. Pure Appl. Math. (5)2 (1974), 132–136.

[6] Lal, S., On degree of approximation of functions belonging to the weighted $(L_p, \xi(t))$ class by $(C, 1)(E, 1)$ means. Tamkang Journal of Mathematics 30 (1999), 47–52.

[7] Lal, S., On the degree of approximation of conjugate of a function belonging to the weighted $W(L_p, \xi(t))$ class by matrix summability means of conjugate series of a Fourier series. Tamkang Journal of Mathematics (31)4 (2000), 47–52.

[8] Lal, S., Approximation of functions belonging to the generalized Lipschitz class by $(C^1 \cdot N_p)$ summability method of Fourier series. Applied Mathematics and Computation (209)2 (2009), 346–350.

[9] Lal, S. and Kushwaha, J. K., Approximation of conjugate of functions belonging to the generalized Lipschitz class by lower triangular matrix means. Int. Journal of Math. Analysis (3)21 (2009), 1031–1041.

[10] Lal, S. and Kushwaha, J. K., Degree of Approximation of Lipschitz function by product summability method. International Mathematical Forum (4)43 (2009), 2101–2107.

[11] Leindler, L., Trigonometric approximation in L_p-norm. Journal of Mathematical Analysis and Applications 302 (2005), 129–136.

[12] Mazhar, S. M. and Totik, V., Approximation of continuous functions by T-means of Fourier series. J. Approx. Theory (60)2 (1990), 174–182.

[13] Mohapatra, R. N. and Szal, B., On trigonometric approximation of functions in the L_p-norm. arXiv:1205.5869v1 [math.CA], 2012.

[14] Mittal, M. L., Rhoades, B. E., Mishra, V. N. and Singh, U., Using infinite matrices to approximate functions of class $Lip(\alpha, p)$ using trigonometric polynomials. Journal of Mathematical Analysis and Applications (326)1 (2007), 667-676.

[15] Mittal, M. L., Rhoades, B. E., Sonker, S. and Singh, U., Approximation of signals of class $Lip(\alpha, p)$ by linear operators. Applied Mathematics and Computation (217)9 (2011), 4483-4489.

[16] Nigam, H. K., A study on approximation of conjugate of functions belonging to Lipschitz class and generalized Lipschitz class by product summability means of conjugate series of Fourier series. Thai Journal of Mathematics 10(2) (2012), 275–287.

[17] Nigam, H. K. and Sharma, A., On approximation of conjugate of functions belonging to different classes by product means. International Journal of Pure and Applied Mathematics, 76(2) (2012), 303–316.

[18] Quade, E. S., Trigonometric approximation in the mean. Duke Math. J. 3 (1937), 529–542.

[19] Qureshi, K., On the degree of approximation of a function belonging to the Lipschitz class by means of conjugate series. Indian J. Pure Appl. Math. (12)9 (1981), 1120-1123.

[20] Qureshi, K., On the degree of approximation of a function belonging to the class $Lip\alpha$. Indian J. Pure Appl. Math. (13)8 (1982), 560–563.

[21] Qureshi, K., On the degree of approximation of a function belonging to weighted $W(L_r, \xi(t))$ class. Indian J. Pure Appl. Math. 13 (1982), 471-475.

[22] Qureshi, K., On the degree of approximation of a function belonging to the class $Lip(\alpha, p)$ by means of conjugate series. Indian J. Pure Appl. Math. (13)5 (1982), 560-563.

[23] Sing, U., Mittal, M. L. and Sonker, S., Trigonometric Approximation of Signals(Functions) Belonging to $W(L^r, \xi(t))$ Class by Matrix $(C^1 \cdot N_p)$ Operator. International Journal of Mathematics and Mathematical Sciences, Vol. (2012) Article ID 964101 (2012), 11 p.

[24] Toeplitz, O., Uberallagemeine lineara Mittel bil dunger. P.M.F. 22 (1913), 113–119.

[25] Zygmund, A., Trigonometric Series. Vol. I, Cambridge: Cambridge University Press, 1959.

Received by the editors March 4, 2013