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UNIQUENESS OF MEROMORPHIC SOLUTION OF A NON-LINEAR DIFFERENTIAL EQUATION[1]

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Abstract. In the paper we shall concentrate on the the uniqueness property of the solution of a specific type of differential equation as obtained from the conclusion of Brück Conjecture by radically improving, extending and generalizing a result of Bhoosnurmath-Kulkarni-Prabhu. Some examples have been given in the paper to show that one condition in our main result is sharp and a number of examples have been exhibited to show that one condition used in the paper is essential.

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1. Introduction, Definitions and Results

Let f and g be two non-constant meromorphic functions defined in the open complex plane $\mathbb{C}$. If for some $a \in \mathbb{C} \cup \{\infty\}$, $f-a$ and $g-a$ have the same set of zeros with the same multiplicities, we say that f and g share the value a CM (counting multiplicities), and if we do not consider the multiplicities then f and g are said to share the value a IM (ignoring multiplicities).

A meromorphic function a is said to be a small function of f provided that $T(r,a) = S(r,f)$, that is $T(r,a) = o(T(r,f))$ as $r \longrightarrow \infty$, outside of a possible exceptional set of finite linear measure.

In 1979 Mues and Steinmetz [15] proved the following theorem.

Theorem A. *[15] Let f be a non-constant entire function. If f and $f^{'}$ share two distinct values a, b IM then $f^{'} \equiv f$.*

In 1996, for one CM shared values of entire function with its first derivative Brück proposed the following famous conjecture [4]:

Conjecture: *Let f be a non-constant entire function such that the hyper order $\rho_2(f)$ of f is not a positive integer or infinite. If f and $f^{'}$ share a finite value a CM, then $\frac{f^{'}-a}{f-a} = c$, where c is a non zero constant.*

Brück himself proved the conjecture for $a = 0$. For $a \neq 0$ following result was obtained in [4].

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Theorem B. *[4] Let f be a non-constant entire function. If f and $f^{'}$ share the value 1 CM and if $N(r,0;f^{'})=S(r,f)$ then $\frac{f^{'}-1}{f-1}$ is a nonzero constant.*

From the following example we see that it is impossible to replace the value 1 of *Theorem B* by simply a small function $a(\not\equiv 0,\infty)$.

Example 1.1. Let $f=1+e^{e^{z}}$ and $a(z)=\frac{1}{1-e^{-z}}$.

By *Lemma 2.6* of [7], [p. 50] we know that a is a small function of f. Also it can be easily seen that f and $f^{'}$ share a CM and $N(r,0;f^{'})=0$ but $f-a\neq c\,(f^{'}-a)$ for every nonzero constant c. We note that $f-a=e^{-z}\,(f^{'}-a)$. So in order to replace the value 1 by a small function some extra conditions are required.

For entire function of finite order removing the condition $N(r,0;f^{'})=0$ in *Theorem B*, Yang [16] improved the same in the following manner.

Theorem C. *[16] Let f be a non-constant entire function of finite order and let $a(\neq 0)$ be a finite constant. If f, $f^{(k)}$ share the value a CM then $\frac{f^{(k)}-a}{f-a}$ is a nonzero constant, where $k(\geq 1)$ is an integer.*

The following examples show that in *Theorem B* one can not simultaneously replace "CM" by "IM" and "entire function" by "meromorphic function".

Example 1.2. $f(z)=1+\tan z$.

Clearly $f(z)-1=\tan z$ and $f^{'}(z)-1=\tan^{2}z$ share 1 IM and $N(r,0;f^{'})=0$. But the conclusion of *Theorem B* ceases to hold.

Example 1.3. $f(z)=\frac{2}{1-e^{-2z}}$.

Clearly $f^{'}(z)=-\frac{4e^{-2z}}{(1-e^{-2z})^{2}}$. Here $f-1=\frac{1+e^{-2z}}{1-e^{-2z}}$ and $f'-1=-\frac{(1+e^{-2z})^{2}}{(1-e^{-2z})^{2}}$. Here $N(r,0;f^{'})=0$ but the conclusion of *Theorem B* does not hold.

Zhang [18] extended *Theorem B* to meromorphic functions and also studied the value sharing of a meromorphic function with its k-th derivative counterpart.

Meanwhile, a new notion of scalings between CM and IM, known as weighted sharing, appeared in the uniqueness literature. Below we are giving the definition.

Definition 1.4. [8, 9] Let k be a nonnegative integer or infinity. For $a\in\mathbb{C}\cup\{\infty\}$ we denote by $E_{k}(a;f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m\leq k$ and $k+1$ times if $m>k$. If $E_{k}(a;f)=E_{k}(a;g)$, we say that f,g share the value a with weight k.

The definition implies that if f, g share a value a with weight k then z_0 is an a-point of f with multiplicity m $(\leq k)$ if and only if it is an a-point of g with multiplicity m $(\leq k)$ and z_0 is an a-point of f with multiplicity m $(>k)$ if and only if it is an a-point of g with multiplicity n $(>k)$, where m is not necessarily equal to n.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k), then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

If a is a small function we define that f and g share a IM or a CM or with weight l iff $f - a$ and $g - a$ share $(0, 0)$ or $(0, \infty)$ or $(0, l)$ respectively.

Though we use the standard notations and definitions of the value distribution theory available in [7], we explain some definitions and notations which are used in the paper.

Definition 1.5. [10]Let p be a positive integer and $a \in \mathbb{C} \cup \{\infty\}$.

(i) $N(r, a; f \mid\geq p)$ $(\overline{N}(r, a; f \mid\geq p))$ denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not less than p.

(ii) $N(r, a; f \mid\leq p)$ $(\overline{N}(r, a; f \mid\leq p))$ denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not greater than p.

With the notion of weighted sharing of values, the results of Zhang [18] were improved by Lahiri-Sarkar [10]. In 2005, Zhang [19] further extended the result of Lahiri-Sarkar [10] to a small function. Further investigations analogous to Brück conjecture can be found in the work of Zhang and Lü [20], Liu [11], Li and Yang [12] et. al. So we see that the Brüück result and the research which followed has a long history. Several special forms on the Brück conjecture such as Nevanlinna deficiency, small functions, power functions etc. were investigated by many authors.

If we carefully observe the conclusion of Brück's result and the subsequent ones, we see that for an appropriate constant or small function a, the relation between a function f and its k-th derivative counterpart are determined by $\frac{f^{(k)}-a}{f-a} = c$ for some constant $c \in \mathbb{C}/\{0\}$. In particular, if $c = 1$ then $f = f^{(k)}$, which gives more specific form of the function.

To the knowledge of the authors K.T.Yu [17] was the first to show that the above specific type of relation between an entire or non entire meromorphic function with its k-th derivative holds. Yu [17] did not assume any restriction on the growth of the function to serve his purpose, rather to achieve his goal he resorted to the deficiencies of the value 0 of the function. We first recall the results of Yu [17]

Theorem A. *Let f be a non-constant entire function and $a(\not\equiv 0, \infty)$ be a small function of f. If $f - a$ and $f^{(k)} - a$ share the value 0 CM and $\delta(0; f) > \frac{3}{4}$, then $f \equiv f^{(k)}$, where k is a positive integer.*

Theorem B. *Let f be a non-constant non-entire meromorphic function and $a(\not\equiv 0, \infty)$ be a small function of f. If*

i) f and a have no common poles.

ii) $f - a$ *and* $f^{(k)} - a$ *share the value* 0 *CM.*

iii) $4\delta(0; f) + 2\Theta(\infty; f) > 19 + 2k$

then $f \equiv f^{(k)}$ *where* k *is a positive integer.*

Later Yu's results have been improved, extended and generalized by many authors such as Liu-Gu [14], Lahiri-Sarkar [10], Lin-Lin [13], Banerjee [1]-[2], Zhang [19] etc.

In this paper we consider the uniqueness of a meromorphic function with its derivative from a different angle than those stated so far. One can easily observe that the conclusion of Brück's conjecture is nothing but a differential equation. So it will be interesting to know about the uniqueness property of the solution of this type or more generalized differential equation of the same form without any sharing conditions. In this direction, in 2007, Bhoosnurmath-Kulkarni-Prabhu [3] made some progress. Before demonstrating their result we first recall the following definition.

Definition 1.6. Let $n_{0j}, n_{1j}, \ldots, n_{kj}$ be non negative integers.
The expression $M_j[f] = (f)^{n_{0j}}(f^{(1)})^{n_{1j}} \ldots (f^{(k)})^{n_{kj}}$ is called a differential monomial generated by f of degree $d(M_j) = \sum\limits_{i=0}^{k} n_{ij}$ and weight $\Gamma_{M_j} = \sum\limits_{i=0}^{k} (i+1)n_{ij}$.

The sum $P[f] = \sum\limits_{j=1}^{t} b_j M_j[f]$ is called a differential polynomial generated by f of degree $\overline{d}(P) = max\{d(M_j) : 1 \leq j \leq t\}$ and weight $\Gamma_P = max\{\Gamma_{M_j} : 1 \leq j \leq t\}$, where $T(r, b_j) = S(r, f)$ for $j = 1, 2, \ldots, t$.

The numbers $\underline{d}(P) = min\{d(M_j) : 1 \leq j \leq t\}$ and k(the highest order of the derivative of f in $P[f]$) are called respectively the lower degree and order of $P[f]$.

$P[f]$ is said to be homogeneous if $\overline{d}(P)$=$\underline{d}(P)$.

$P[f]$ is called a Linear Differential Polynomial generated by f if $\overline{d}(P) = 1$. Otherwise $P[f]$ is called Non-linear Differential Polynomial. We denote by $Q = max\ \{\Gamma_{M_j} - d(M_j) : 1 \leq j \leq t\} = max\ \{n_{1j} + 2n_{2j} + \ldots + kn_{kj} : 1 \leq j \leq t\}$.

Extending some previous theorems, Bhoosnurmath-Kulkarni-Prabhu [3] obtained the following theorem.

Theorem C. *[3] Let* f *be a non-constant transcendental meromorphic function such that* $N(r, \infty; f) + N(r, 0; f) = S(r, f)$. *Let* $a \equiv a(z)$ $(\not\equiv 0, \infty)$ *be a small meromorphic function and* $P[f]$ *be a homogeneous differential polynomial in* f. *Suppose that* f *satisfies the equation*

$$cP[f] - f - (c - 1)a = 0,$$

where c *is a non-zero constant then* $f \equiv P[f]$.

In this paper we shall improve, extend, generalize above result to a large extent. The following theorem is the main result of the paper.

Theorem 1.7. *Let f be a non-constant meromorphic function such that $\overline{N}(r,0;f \mid \leq k) = S(r,f)$ and $P[f]$ be a differential polynomial in f. Let $a \equiv a(z)$ $(\not\equiv 0,\infty)$ be a small meromorphic function. Suppose that f satisfies the equation*

$$cP[f] - f - (c-1)a = 0, \tag{1.1}$$

where c is a non-zero constant and $P[f]$ contains at least one derivative. If $2\underline{d}(P) > \overline{d}(P)$, then $f \equiv P[f]$.

The following example shows that under the condition $2\underline{d}(P) < \overline{d}(P)$, the conclusion of *Theorem 1.7* ceases to hold.

Example 1.8. Let $f = e^z$ and $P[f] = (f^{''})^2 - ff^{'} + 2f - 1$. Then $\underline{d}(P) = 0$, $\overline{d}(P) = 2$. Here $f \not\equiv P[f]$. We note that $\frac{f-1}{P[f]-1} = \frac{1}{2}$.

However in the following example, we see that when $2\underline{d}(P) = \overline{d}(P)$, the conclusion of *Theorem 1.7* holds.

Example 1.9. Let $f = e^z$ and $P[f] = (f^{''})^2 - ff^{'} + f$. Then $\underline{d}(P) = 1$, $\overline{d}(P) = 2$. Here $f \equiv P[f]$. We note that for any small function a, $\frac{f-a}{P[f]-a} = 1$.

So the following question is inevitable.

Question 1.10. Is the condition $2\underline{d}(P) > \overline{d}(P)$ sharp in *Theorem 1.7* ?

Following examples show that $\overline{N}(r,0;f \mid \leq k) = S(r,f)$ can not be removed in *Theorem 1.7*.

Example 1.11. Let $f = e^{2z} + \frac{b}{2}$, where b is a non-zero constant and $P[f] = f^{'}$. Then $2\underline{d}(P) > \overline{d}(P)$. Here $f \not\equiv P[f]$. We note that $\frac{f-b}{P[f]-b} = \frac{1}{2}$.

Example 1.12. Let $k \geq 3$ and let $b \neq 1$ be a $(k-1)$th root of unity. Let $f = e^{bz} + b - 1$ and $P[f] = f^{(k)}$. Then $2\underline{d}(P) > \overline{d}(P)$. Here $f \not\equiv P[f]$. We note that $\frac{f-b}{P[f]-b} = \frac{1}{b} \neq 1$.

Example 1.13. Let $f = e^{bz} + \frac{b-1}{b}z + \frac{b-1}{b^2}$, where $b \neq 0,\ 1$ is a constant, $a(z) = z$ and $P[f] = f^{'}$. Then $2\underline{d}(P) > \overline{d}(P)$. Here $f \not\equiv P[f]$. We note that $\frac{f-a}{P[f]-a} = \frac{1}{b} \neq 1$.

Example 1.14. Let $f = 2e^{z/2} + \frac{z^2}{2}$, $a(z) = 2z - \frac{z^2}{2}$ and $P[f] = f^{'}$. Then $2\underline{d}(P) > \overline{d}(P)$. Here $f \not\equiv P[f]$. We note that $\frac{f-a}{P[f]-a} = 2$.

Example 1.15. Let $f = e^z - 1$ and $P[f] = f^{''} - if = (1-i)e^z + i$. Then $\underline{d}(P) = 1 = \overline{d}(P)$. Here $f \not\equiv P[f]$. We note that $\frac{f+i}{P[f]+i} = \frac{1+i}{2}$.

Example 1.16. Let $f = e^{-z} + z$ and $P[f] = f^{''} + f = 2e^{-z} + z$. Then $\underline{d}(P) = 1 = \overline{d}(P)$. Here $f \not\equiv P[f]$. We note that $\frac{f-z}{P[f]-z} = \frac{1}{2}$.

We see that under the hypothesis of *Theorem 1.7*, the condition $\overline{N}(r,0;f \mid \leq k) = S(r,f)$ can not be removed. So we pose the following open question:

Question 1.17. Can the condition $\overline{N}(r,0;f \mid \leq k) = S(r,f)$ be removed in *Theorem 1.7* ?

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1. *[5] Let f be a meromorphic function and $P[f]$ be a differential polynomial. Then*

$$m\left(r, \frac{P[f]}{f^{\overline{d}(P)}}\right) \leq (\overline{d}(P) - \underline{d}(P))m\left(r, \frac{1}{f}\right) + S(r,f).$$

Lemma 2.2. *Let f be a meromorphic function and $P[f]$ be a differential polynomial. Then we have*

$$\begin{aligned} & N\left(r, \infty; \frac{P[f]}{f^{\overline{d}(P)}}\right) \\ \leq\ & (\Gamma_P - \overline{d}(P))\,\overline{N}(r,\infty; f) + (\overline{d}(P) - \underline{d}(P))\,N(r,0;f\mid\geq k+1) \\ & +Q\overline{N}(r,0;f\mid\geq k+1) + \overline{d}(P)N(r,0;f\mid\leq k) + S(r,f). \end{aligned}$$

Proof. Let z_0 be a pole of f of order r, such that $b_j(z_0) \neq 0, \infty; 1 \leq j \leq t$. Then it would be a pole of $P[f]$ of order at most $r\overline{d}(P) + \Gamma_P - \overline{d}(P)$. Since z_0 is a pole of $f^{\overline{d}(P)}$ of order $r\overline{d}(P)$, it follows that z_0 would be a pole of $\frac{P[f]}{f^{\overline{d}(P)}}$ of order at most $\Gamma_P - \overline{d}(P)$. Next suppose z_1 is a zero of f of order $s(> k)$, such that $b_j(z_1) \neq 0, \infty; 1 \leq j \leq t$. Clearly it would be a zero of $M_j(f)$ of order $s \cdot n_{0j} + (s-1)n_{1j} + \ldots + (s-k)n_{kj} = s \cdot d(M_j) - (\Gamma_{M_j} - d(M_j))$. Hence z_1 is a pole of $\frac{M_j[f]}{f^{\overline{d}(P)}}$ of order

$$s \cdot \overline{d}(P) - s \cdot d(M_j) + (\Gamma_{M_j} - d(M_j)) = s(\overline{d}(P) - d(M_j)) + (\Gamma_{M_j} - d(M_j)).$$

So z_1 would be a pole of $\frac{P[f]}{f^{\overline{d}(P)}}$ of order at most

$$\max\{s(\overline{d}(P) - d(M_j)) + (\Gamma_{M_j} - d(M_j)) : 1 \leq j \leq t)\} = s(\overline{d}(P) - \underline{d}(P)) + Q.$$

If z_1 is a zero of f of order $s \leq k$, such that $b_j(z_1) \neq 0, \infty : 1 \leq j \leq t$ then it would be a pole of $\frac{P[f]}{f^{\overline{d}(P)}}$ of order $s\overline{d}(P)$. Since the poles of $\frac{P[f]}{f^{\overline{d}(P)}}$ come from the poles or zeros of f and poles or zeros of $b_j(z)$'s only, it follows that

$$\begin{aligned} & N\left(r, \infty; \frac{P[f]}{f^{\overline{d}(P)}}\right) \\ \leq\ & (\Gamma_P - \overline{d}(P))\,\overline{N}(r,\infty; f) + (\overline{d}(P) - \underline{d}(P))\,N(r,0;f\mid\geq k+1) \\ & +Q\,\overline{N}(r,0;f\mid\geq k+1) + \overline{d}(P)N(r,0;f\mid\leq k) + S(r,f). \end{aligned}$$

□

Lemma 2.3. *[6] Let $P[f]$ be a differential polynomial. Then*

$$T(r, P[f]) \leq \Gamma_P T(r,f) + S(r,f).$$

Lemma 2.4. *Let f be a non-constant meromorphic function and $P[f]$ be a differential polynomial. Then $S(r, P[f])$ can be replaced by $S(r, f)$.*

Proof. From *Lemma 2.3* it is clear that $T(r, P[f]) = O(T(r, f))$ and so the lemma follows. □

Lemma 2.5. *Let f be a non-constant meromorphic function and $P[f]$ be a differential polynomial. Then*

$$\begin{aligned} & N(r,0;P[f]) \\ \leq\ & \frac{\overline{d}(P)-\underline{d}(P)}{\underline{d}(P)} m\left(r,\frac{1}{P[f]}\right) + \left(\Gamma_P - \overline{d}(P)\right)\overline{N}(r,\infty;f) \\ & +\left(\overline{d}(P)-\underline{d}(P)\right) N(r,0;f\mid\geq k+1) + Q\overline{N}(r,0;f\mid\geq k+1) \\ & +\overline{d}(P)N(r,0;f\mid\leq k) + \overline{d}(P)\,N(r,0;f) + S(r,f). \end{aligned}$$

Proof. For a fixed value of r, let $E_1 = \{\theta \in [0, 2\pi] : \left|f(re^{i\theta})\right| \leq 1\}$ and E_2 be its complement. Since by definition

$$\sum_{i=0}^{k} n_{ij} \geq \underline{d}(P),$$

for every $j = 1, 2, \ldots, t$, it follows that on E_1

$$\left|\frac{P[f]}{f^{\underline{d}(P)}}\right| \leq \sum_{j=1}^{t} |b_j(z)| \prod_{i=1}^{k} \left|\frac{f^{(i)}}{f}\right|^{n_{ij}} |f|^{\sum\limits_{i=0}^{k} n_{ij}-\underline{d}(P)} \leq \sum_{j=1}^{t} |b_j(z)| \prod_{i=1}^{k} \left|\frac{f^{(i)}}{f}\right|^{n_{ij}}.$$

Also we note that

$$\frac{1}{f^{\underline{d}(P)}} = \frac{P[f]}{f^{\underline{d}(P)}}\,\frac{1}{P[f]}.$$

Since on E_2, $\frac{1}{|f(z)|} < 1$, we have

$$\begin{aligned} & \underline{d}(P)m\left(r,\frac{1}{f}\right) \\ =\ & \frac{1}{2\pi}\int\limits_{E_1} \log^+ \frac{1}{|f(re^{i\theta})|^{\underline{d}(P)}}\,d\theta + \frac{1}{2\pi}\int\limits_{E_2} \log^+ \frac{1}{|f(re^{i\theta})|^{\underline{d}(P)}}\,d\theta \\ \leq\ & \frac{1}{2\pi}\sum_{j=1}^{t}\left[\int\limits_{E_1} \log^+ |b_j(z)|\,d\theta + \sum_{i=1}^{k}\int\limits_{E_1} \log^+ \left|\frac{f^{(i)}}{f}\right|^{n_{ij}} d\theta\right] \\ & + \frac{1}{2\pi}\int\limits_{E_1} \log^+ \left|\frac{1}{P[f(re^{i\theta})]}\right| d\theta \\ \leq\ & \frac{1}{2\pi}\int\limits_{0}^{2\pi} \log^+ \left|\frac{1}{P[f(re^{i\theta})]}\right| d\theta + S(r,f) = m\left(r,\frac{1}{P[f]}\right) + S(r,f). \end{aligned}$$

So using *Lemmas 2.1*, *2.2* and the first fundamental theorem we get

$$\begin{aligned}
& N(r,0;P[f]) \\
\leq\; & N\left(r,\infty;\frac{f^{\overline{d}(P)}}{P[f]}\right)+\overline{d}(P)N(r,0;f) \\
\leq\; & m\left(r,\frac{P[f]}{f^{\overline{d}(P)}}\right)+N\left(r,\infty;\frac{P[f]}{f^{\overline{d}(P)}}\right)+\overline{d}(P)N(r,0;f)+S(r,f) \\
\leq\; & (\overline{d}(P)-\underline{d}(P))m\left(r,\frac{1}{f}\right)+(\Gamma_P-\overline{d}(P))\,\overline{N}(r,\infty;f) \\
& +(\overline{d}(P)-\underline{d}(P))\,N(r,0;f\mid\geq k+1)+Q\overline{N}(r,0;f\mid\geq k+1) \\
& +\overline{d}(P)N(r,0;f\mid\leq k)+\overline{d}(P)N(r,0;f)+S(r,f) \\
\leq\; & \frac{(\overline{d}(P)-\underline{d}(P))}{\underline{d}(P)}m\left(r,\frac{1}{P[f]}\right)+(\Gamma_P-\overline{d}(P))\,\overline{N}(r,\infty;f) \\
& +(\overline{d}(P)-\underline{d}(P))\,N(r,0;f\mid\geq k+1)+Q\overline{N}(r,0;f\mid\geq k+1) \\
& +\overline{d}(P)N(r,0;f\mid\leq k)+\overline{d}(P)N(r,0;f)+S(r,f).
\end{aligned}$$

□

3. Proof of the theorem

Proof of Theorem 1.7. We know from (1.1) that $\frac{f-a}{P[f]-a}=c$. We assume that $c\neq 1$, since otherwise we have nothing to prove. This implies $f-a$ and $P[f]-a$ share $(0,\infty)$. Let $F=\frac{f}{a}$ and $G=\frac{P[f]}{a}$. Then $F-1=\frac{f-a}{a}$ $G-1=\frac{P[f]-a}{a}$. Since $f-a$ and $P[f]-a$ share $(0,\infty)$ it follows that F, G share $(1,\infty)$ except the zeros and poles of $a(z)$. If z_0 $(a(z_0),b_j(z_0))\not\equiv 0,\infty:1\leq j\leq t)$ is a pole of f of order r then we know it is a pole of $P[f]$ of order $\max\{rd(M_j)+(\Gamma_{M_j}-d(M_j)):1\leq j\leq t\}$, which is a contradiction. So

$$\overline{N}(r,\infty;f)\leq N(r,\infty;a)+\overline{N}(r,0;a)+\sum_{j=1}^{t}N(r,\infty;b_j)+\sum_{j=1}^{t}N(r,0;b_j)=S(r,f).$$

and so

$$\overline{N}(r,\infty;F)=S(r,f).$$

Suppose z_1 $(a(z_1),b_j(z_1))\not\equiv 0,\infty:1\leq j\leq t)$ is a zero of f of multiplicity $s\geq k+1$. Clearly it would be a zero of $P[f]$ of order

$$\begin{aligned}
\min\quad & \{(s+1)d(M_j)-\Gamma_{M_j}:1\leq j\leq t\}=\min\{\sum_{i=0}^{k}(s-i)n_{ij}:1\leq j\leq t\} \\
\geq\quad & \underline{d}(P)\geq 1,
\end{aligned}$$

which contradicts the fact that $c\neq 1$. Hence $N(r,0;f\mid\geq k+1)=S(r,f)$. So from the given condition we know that

$$N(r,0;f)\leq k\overline{N}(r,0;f\mid\leq k)+N(r,0;f\mid\geq k+1)=S(r,f).$$

Since $\frac{f-a}{P[f]-a} = c \Rightarrow \frac{F-1}{G-1} = c$, we get

$$G - 1 = \frac{1}{c}(F - 1).$$

As $c \neq 1$, we have

$$F = c\left(G - 1 + \frac{1}{c}\right)$$

and so

$$\overline{N}(r, 0; F) = \overline{N}\left(r, 1 - \frac{1}{c}; G\right) + S(r, f). \tag{3.1}$$

So by the second fundamental theorem, *Lemma 2.4*, (3.1) and noting that $\overline{N}(r, \infty; G) = \overline{N}(r, \infty; F) + S(r, f)$, we get

$$\begin{aligned} & T(r, G) \\ \leq\ & \overline{N}(r, \infty; G) + \overline{N}(r, 0; G) + \overline{N}(r, 1 - \frac{1}{c}; G) + S(r, G) \\ \leq\ & N(r, 0; P[f]) + \overline{N}(r, 0; f) + S(r, f) \\ \leq\ & N(r, 0; P[f]) + S(r, f). \end{aligned} \tag{3.2}$$

Hence by *Lemma 2.5* we have

$$\frac{2\underline{d}(P) - \overline{d}(P)}{\underline{d}(P)} T(r, P[f]) \leq S(r, f). \tag{3.3}$$

Since $2\underline{d}(P) > \overline{d}(P)$ (3.3) leads to a contradiction.
This proves the theorem. □

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