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QUASI-ASYMPTOTIC BEHAVIOR OF BOEHMIANS

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Abstract

The notion of quasi-asymptotic behavior for the space of Boehmians is introduced and studied.

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1 Introduction

In this note we will consider a class of generalized functions known as Boehmians. The construction of this space is algebraic and uses convolution and delta sequences. The class of Boehmians is a generalization of regular operators [2], Schwartz distributions [1], and other spaces of generalized functions [4]. There are Boehmians which are not hyperfunctions and hyperfunctions which are not Boehmians.

There are several different types of asymptotics for generalized functions; equivalence at infinity (origin), S-asymptotics, and quasi-asymptotics, to name a few. Applications can be found in differential equations, Abelian and Tauberian theorems for integral transforms, and quantum physics.

The notion of equivalence at infinity (origin) for Boehmians was used to established some Abelian type theorems [6, 8, 11, 12, 13, 14]. S-asymptotic behavior of Boehmians has been investigated by Stanković [19] as well as the author [10].

Quasi-asymptotic behavior of Schwartz distributions was introduced in the early 1970s by Zavialov [21] and investigated by Vladimirov, Drozhzhinov and Zavialov (see [20] and references in [16]). More recently, Pilipović, Stanković, Vindas and others have continued the investigation. For an excellent account of quasi-asymptotic behavior of distributions, which includes a bibliography of more than two hundred references, the reader is referred to [16].

In this note, quasi-asymptotic behavior of Boehmians is investigated. This paper is organized as follows. Section 2 contains notation and the construction of the space of Boehmians. The notion of quasi-asymptotic behavior at infinity for a Boehmian is introduced and investigated in Section 3. In Section 4, after introducing the idea of quasi-asymptotic behavior at the origin, it is shown that this is a local property. In the last section, Section 5, two Abelian theorems of the final type for the Stieltjes transform are established.

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2 Boehmians

Let $C(\mathbb{R})$ denote the space of all continuous functions on the real line $\mathbb{R}$, and let $\mathcal{D}(\mathbb{R})$ denote the subspace of $C(\mathbb{R})$ of all infinitely differentiable functions with compact support.

The convolution of two continuous functions f and g with suitably restricted supports is given by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-t)g(t)\,dt. \tag{2.1}$$

Remark 2.1. Suitably restricted supports will mean either at least one of the supports is bounded or both are bounded on the left.

Let $\varphi_1, \varphi_2, \ldots \in \mathcal{D}(\mathbb{R})$. The sequence $\{\varphi_n\}$ is called a delta sequence provided:

(i) $\int \varphi_n = 1$, for all $n \in \mathbb{N}$,

(ii) $\varphi_n \geq 0$, for all $n \in \mathbb{N}$,

(iii) For every $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $\varphi_n(x) = 0$ for $|x| > \varepsilon$ and $n > n_\varepsilon$.

A pair of sequences (f_n, φ_n) is called a quotient of sequences if $f_n \in C(\mathbb{R})$ for $n \in \mathbb{N}$, $\{\varphi_n\}$ is a delta sequence, and $f_n * \varphi_k = f_k * \varphi_n$ for all $k, n \in \mathbb{N}$.

Two quotients of sequences (f_n, φ_n) and (g_n, ψ_n) are said to be equivalent if $f_n * \psi_k = g_k * \varphi_n$ for all $k, n \in \mathbb{N}$. A straightforward calculation shows that this is an equivalence relation. The equivalence classes are called Boehmians. The space of all Boehmians will be denoted by $\beta(\mathbb{R})$ and a typical element of $\beta(\mathbb{R})$ will be written as $W = \left[\frac{f_n}{\varphi_n}\right]$.

The operations of addition, scalar multiplication, and differentiation are defined as follows: $\left[\frac{f_n}{\varphi_n}\right] + \left[\frac{g_n}{\psi_n}\right] = \left[\frac{f_n*\psi_n + g_n*\varphi_n}{\varphi_n*\psi_n}\right]$, $\gamma\left[\frac{f_n}{\varphi_n}\right] = \left[\frac{\gamma f_n}{\varphi_n}\right]$, where $\gamma \in \mathbb{C}$, and $D^m\left[\frac{f_n}{\varphi_n}\right] = \left[\frac{f_n*\varphi_n^{(m)}}{\varphi_n*\varphi_n}\right]$, $m \in \mathbb{N}$.

Define the map $\iota : C(\mathbb{R}) \to \beta(\mathbb{R})$ by

$$\iota(f) = W^f, \text{ where } W^f = \left[\frac{f * \varphi_n}{\varphi_n}\right] \tag{2.2}$$

and $\{\varphi_n\}$ is any fixed delta sequence.

It is not difficult to show that the mapping ι is an injection which preserves the algebraic properties of $C(\mathbb{R})$. Thus, $C(\mathbb{R})$ can be identified with a proper subspace of $\beta(\mathbb{R})$. Likewise, the space of Schwartz distributions $\mathcal{D}'(\mathbb{R})$ [1] can be identified with a proper subspace of $\beta(\mathbb{R})$. Using this identification, the Dirac measure δ corresponds to the Boehmian $\left[\frac{\varphi_n}{\varphi_n}\right]$, where $\{\varphi_n\}$ is any delta sequence.

Defining a suitable product of a function and a Boehmian is not always possible [7]. However, multiplying by polynomials is possible [9].

The operator $\mathcal{M} : C(\mathbb{R}) \to C(\mathbb{R})$, given by $(\mathcal{M}f)(x) = xf(x)$, can be extended to $\beta(\mathbb{R})$. Let $W = \left[\frac{f_n}{\varphi_n}\right] \in \beta(\mathbb{R})$. Then, $\widetilde{\mathcal{M}}W$ is defined as follows.

$$\widetilde{\mathcal{M}}W = \left[\frac{\mathcal{M}f_n * \varphi_n - f_n * \mathcal{M}\varphi_n}{\varphi_n * \varphi_n}\right]. \tag{2.3}$$

Remarks 2.2.

1. $\widetilde{\mathcal{M}}W$ does not depend on the representative for W.
2. $\widetilde{\mathcal{M}}W \in \beta(\mathbb{R})$.
3. $\widetilde{\mathcal{M}}W^f = W^{\mathcal{M}f}$, $f \in C(\mathbb{R})$. That is the operators $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ agree on $C(\mathbb{R})$.

For $\psi \in \mathcal{D}(\mathbb{R})$ and $W = \left[\frac{f_n}{\varphi_n}\right] \in \beta(\mathbb{R})$, $W * \psi$ is defined as

$$\left[\frac{f_n}{\varphi_n}\right] * \psi = \left[\frac{f_n * \psi}{\varphi_n}\right]. \tag{2.4}$$

Definition 2.3. Let $W_n, W \in \beta(\mathbb{R})$ for $n = 1, 2, \dots$. We say that the sequence $\{W_n\}$ is δ-convergent to W if there exists a delta sequence $\{\varphi_n\}$ such that for each k and n, $W_n * \varphi_k$, $W * \varphi_k \in C(\mathbb{R})$, and for each k, $W_n * \varphi_k \to W * \varphi_k$ uniformly on compact sets as $n \to \infty$. This will be denoted by $\delta\text{-}\lim_{n\to\infty} W_n = W$.

Theorem 2.4. *The generalized derivative D and the operator $\widetilde{\mathcal{M}}$ satisfy the following.*

a) $DW^f = W^{f'}$, *f is differentiable with* $f' \in C(\mathbb{R})$.

b) $D(\widetilde{\mathcal{M}}W) = W + \widetilde{\mathcal{M}}(DW)$, $W \in \beta(\mathbb{R})$.

c) If $W_n, W \in \beta(\mathbb{R})$ $(n \in \mathbb{N})$ *and* $\delta\text{-}\lim_{n\to\infty} W_n = W$, *then*

   i) $\delta\text{-}\lim_{n\to\infty} \widetilde{\mathcal{M}}W_n = \widetilde{\mathcal{M}}W$.

   ii) $\delta\text{-}\lim_{n\to\infty} DW_n = DW$.

A Boehmian W is said to vanish on an open set $\Omega \subset \mathbb{R}$, denoted $W(x) = 0$ on Ω, provided that there exists a delta sequence $\{\varphi_n\}$ such that $W*\varphi_n \in C(\mathbb{R})$, $n \in \mathbb{N}$, and $W * \varphi_n \to 0$ uniformly on compact subsets of Ω as $n \to \infty$. The support of a Boehmian W, denoted supp W, is the complement of the largest open set on which W vanishes. The space of Boehmians with compact support is denoted by $\beta_c(\mathbb{R})$.

Theorem 2.5. *Let Ω be an open set in $\mathbb{R}$. Then, $W(x) = 0$ on Ω if and only if for each compact subset K of Ω there exists a delta sequence $\{\varphi_n\}$ such that for each $n \in \mathbb{N}$, $W * \varphi_n \in C(\mathbb{R})$, and $(W * \varphi_n)(x) = 0$, for all $x \in K$.*

3 Quasi-Asymptotic Behavior At Infinity

Let $W = \left[\frac{f_n}{\varphi_n}\right] \in \beta(\mathbb{R})$. For $\lambda > 0$, define $W_\lambda = \left[\frac{(f_n)_\lambda}{\lambda(\varphi_n)_\lambda}\right]$, where $(f_n)_\lambda(x) = f_n(\lambda x)$ and $(\varphi_n)_\lambda(x) = \varphi_n(\lambda x)$.

Remarks 3.1.

1. For each $\lambda > 0$, $\{\lambda(\varphi_n)_\lambda\}$ is a delta sequence and $W_\lambda \in \beta(\mathbb{R})$.
2. For each $\lambda > 0$, W_λ is independent of the representative used to represent W.

A real-valued function $L(x)$ is slowly varying at infinity [18] if it is positive, measurable on $[a, \infty)$ for some $a > 0$, and such that for each $\lambda > 0$,

$$\lim_{x \to \infty} \frac{L(\lambda x)}{L(x)} = 1. \tag{3.1}$$

The following properties are well-known.

1. For each $\alpha > 0$, $x^\alpha L(x) \to \infty$ as $x \to \infty$, and $x^{-\alpha} L(x) \to 0$ as $x \to \infty$.
2. Let $0 < a < b < \infty$. Then,
   (i) $\frac{L(\lambda x)}{L(\lambda)} \to 1$ uniformly on $[a, b]$ as $\lambda \to \infty$.
   (ii) L is bounded and integrable on $[a, b]$.

Throughout the sequel, $L(x)$ will denote a slowly varying function.

We will be concerned with the space $\beta_+(\mathbb{R})$ of Boehmians supported on the interval $[0, \infty)$. Suppose that $W = \left[\frac{f_n}{\varphi_n}\right] \in \beta_+(\mathbb{R})$. Then, the following are some useful properties for supports.

1. Let $a > 0$ and $\varphi \in \mathcal{D}(\mathbb{R})$ such that supp $\varphi \subset (-a, a)$. Then, supp $(W * \varphi) \subset (-a, \infty)$.
2. Let $\lambda > 0$. Then, supp $W_\lambda \subset [0, \infty)$.
3. There exists $b \geq 0$ such that supp $f_n \subset (-b, \infty)$, for all $n \in \mathbb{N}$. Moreover, for each $\varepsilon > 0$, there exists $n_\varepsilon > 0$, such that for all $n > n_\varepsilon$, supp $f_n \subset (-\varepsilon, \infty)$.
4. For each $f \in C(\mathbb{R})$, supp f = supp W^f, where $W^f = \left[\frac{f * \varphi_n}{\varphi_n}\right]$.

Convolution can be extended to a map from $\beta_+(\mathbb{R}) \times \beta_+(\mathbb{R})$ to $\beta_+(\mathbb{R})$ as follows.

Definition 3.2. Let $W, V \in \beta_+(\mathbb{R})$ with $W = \left[\frac{f_n}{\varphi_n}\right]$ and $V = \left[\frac{g_n}{\sigma_n}\right]$. Then,

$$W * V = \left[\frac{f_n}{\varphi_n}\right] * \left[\frac{g_n}{\sigma_n}\right] = \left[\frac{f_n * g_n}{\varphi_n * \sigma_n}\right].$$

This convolution is well defined. Moreover,

$$W * (V * U) = (W * V) * U, \text{ for all } W, V, U \in \beta_+(\mathbb{R}).$$

Definition 3.3. For $\alpha \in \mathbb{R}$ and $W, V \in \beta_+(\mathbb{R})$, $W \overset{q}{\sim} V$ at infinity related to $\lambda^\alpha L(\lambda)$ provided

$$\delta\text{ - }\lim_{\lambda\to\infty} \frac{W_\lambda}{\lambda^\alpha L(\lambda)} = V.$$

i.e. $\delta\text{-}\lim_{n\to\infty} \frac{W_{\lambda_n}}{\lambda_n^\alpha L(\lambda_n)} = V$, for every $\lambda_n \to \infty$.

The definition is independent of the representative of W.

Suppose that a sequence $\{f_n\}$ of distributions converges to f in $\mathcal{D}'(\mathbb{R})$. By using the Banach-Steinhaus Theorem, it can be shown that the sequence of infinitely differentiable functions $\{f_n * \varphi\}$ $(\varphi \in \mathcal{D}(\mathbb{R}))$ converges to $f * \varphi$ uniformly on compact sets of $\mathbb{R}$. And hence, using a similar argument, the sequence $\{W^{f_n}\}$ converges to W^f in $\beta(\mathbb{R})$. Thus, we obtain the following.

Theorem 3.4. *Let $f, g \in \mathcal{D}'_+(\mathbb{R})$ such that $f \overset{q}{\sim} g$ at infinity related to $\lambda^\alpha L(\lambda)$ (in $\mathcal{D}'(\mathbb{R})$). Then, $W^f \overset{q}{\sim} W^g$ at infinity related to $\lambda^\alpha L(\lambda)$ (in $\beta(\mathbb{R})$).*

By applying the results of Theorem 2.4 part (c) to the identities $DW_\lambda = \lambda(DW)_\lambda$ and $\lambda(\widetilde{\mathcal{M}}W_\lambda) = (\widetilde{\mathcal{M}}W)_\lambda$, we obtain the following.

Theorem 3.5. *Let $\alpha \in \mathbb{R}$ and $W, V \in \beta_+(\mathbb{R})$ such that $W \overset{q}{\sim} V$ at infinity related to $\lambda^\alpha L(\lambda)$. Then,*

a) $DW \overset{q}{\sim} DV$ at infinity related to $\lambda^{\alpha-1} L(\lambda)$.

b) $\widetilde{\mathcal{M}}W \overset{q}{\sim} \widetilde{\mathcal{M}}V$ at infinity related to $\lambda^{\alpha+1} L(\lambda)$.

Theorem 3.4 provides a justification for the following examples. However, for the verification of the first example, we will use Definition 3.3 . The second example then follows from the first by applying part a) of Theorem 3.5.

Examples 3.6.

1. Recall $\delta = \left[\frac{\varphi_n}{\varphi_n}\right]$. Then $\delta \overset{q}{\sim} \delta$ at infinity related to λ^{-1}. To see this, let $\lambda_n \to \infty$. Then, for each $k, n \in \mathbb{N}$,

$$\frac{\delta_{\lambda_n}}{\lambda_n^{-1}} * \varphi_k = \left[\frac{\lambda_n (\varphi_m)_{\lambda_n} * \varphi_k}{\lambda_n (\varphi_m)_{\lambda_n}}\right] = \varphi_k.$$

That is, for each $k \in \mathbb{N}$,

$$\frac{\delta_{\lambda_n}}{\lambda_n^{-1}} * \varphi_k = \varphi_k \to \varphi_k \text{ uniformly on compact sets as } n \to \infty.$$

Thus,

$$\delta\text{ - }\lim_{\lambda\to\infty} \frac{\delta_\lambda}{\lambda^{-1}} = \left[\frac{\varphi_k}{\varphi_k}\right] = \delta.$$

That is,

$$\delta \overset{q}{\sim} \delta \text{ at infinity related to } \lambda^{-1}.$$

2. $D\delta \overset{q}{\sim} D\delta$ at infinity related to λ^{-2}.

Theorem 3.7. *Let $U, W, V_1, V_2 \in \beta_+(\mathbb{R})$ such that $W \overset{q}{\sim} V_1$ at infinity related to $\lambda^{\alpha_1} L_1(\lambda)$, and $U \overset{q}{\sim} V_2$ at infinity related to $\lambda^{\alpha_2} L_2(\lambda)$. Then $W * U \overset{q}{\sim} V_1 * V_2$ at infinity related to $\lambda^{\alpha_1+\alpha_2+1}(L_1L_2)(\lambda)$.*

Proof. Let $\lambda_n \to \infty$ as $n \to \infty$. Then, there exist delta sequences $\{\varphi_n\}$ and $\{\psi_n\}$ such that for all $k, n \in \mathbb{N}$,

$$W_{\lambda_n} * \varphi_k,\ V_1 * \varphi_k,\ U_{\lambda_n} * \psi_k,\ V_2 * \psi_k \in C(\mathbb{R}),$$

and for each $k \in \mathbb{N}$,

$$\frac{W_{\lambda_n}}{\lambda_n^{\alpha_1} L_1(\lambda_n)} * \varphi_k \to V_1 * \varphi_k \text{ uniformly on compact sets as } n \to \infty, \text{and}$$

$$\frac{U_{\lambda_n}}{\lambda_n^{\alpha_2} L_2(\lambda_n)} * \psi_k \to V_2 * \psi_n \text{ uniformly on compact sets as } n \to \infty.$$

Let $\sigma_n = \varphi_n * \psi_n$, $n \in \mathbb{N}$. Then $\{\sigma_n\}$ is a delta sequence. The conclusion follows by observing that for all $k, n \in \mathbb{N}$,

$$\begin{aligned}
&\tfrac{(W*U)_{\lambda_n}}{\lambda_n^{\alpha_1+\alpha_2+1} L_1(\lambda_n) L_2(\lambda_n)} * \sigma_k - (V_1 * V_2) * \sigma_k \\
&\quad = \left(\tfrac{W_{\lambda_n}}{\lambda_n^{\alpha_1} L_1(\lambda_n)} - V_1\right) * \left(\tfrac{U_{\lambda_n}}{\lambda_n^{\alpha_2} L_2(\lambda_n)} * \sigma_k\right) + \left(\tfrac{U_{\lambda_n}}{\lambda_n^{\alpha_2} L_2(\lambda_n)} - V_2\right) * (V_1 * \sigma_k) \\
&\quad = \left(\tfrac{W_{\lambda_n}}{\lambda_n^{\alpha_1} L_1(\lambda_n)} * \varphi_k - V_1 * \varphi_k\right) * \left(\tfrac{U_{\lambda_n}}{\lambda_n^{\alpha_2} L_2(\lambda_n)} * \psi_k\right) \\
&\qquad + \left(\tfrac{U_{\lambda_n}}{\lambda_n^{\alpha_2} L_2(\lambda_n)} * \psi_k - V_2 * \psi_k\right) * (V_1 * \sigma_k).
\end{aligned}$$

□

Just as $\mathcal{D}'(\mathbb{R})$ can be identified with a subspace of $\beta(\mathbb{R})$, by using (2.2) with a delta sequence $\{\varphi_n\}$ consisting of ultradifferential functions, each space of Beurling ultradistributions [3] can be identified with a proper subspace of $\beta(\mathbb{R})$.

Before presenting the next theorem, we give a brief introduction to ultradifferential operators in $\beta(\mathbb{R})$ [12]. The delta sequences used in [12] are more general than the ones used in this paper. However, all the results needed for ultradifferential operators in this paper are still valid.

Let $\{M_n\}$ be a sequence of positive numbers which satisfies the following conditions.

(i) $M_0 = 1$,

(ii) $M_n^2 \le M_{n-1} M_{n+1}$, for all $n \in \mathbb{N}$,

(iii) $\sum_{n=0}^{\infty} \frac{M_n}{M_{n+1}} < \infty$.

An operator of the form

$$P(D) = \sum_{n=0}^{\infty} c_n D^n,$$

where $c_n \in \mathbb{C}$, is called an ultradifferential operator provided there exist constants $A > 0$, $B > 0$ such that $|c_n| \leq \frac{AB^n}{M_n}$, $n \in \mathbb{N}$. Unless otherwise stated, the sequence $\{c_n\}$ will be assumed to satisfy the previous inequality. Throughout this section, $P(D) = \sum_{n=0}^{\infty} c_n D^n$ (or just $P(D)$) will denote an ultradifferential operator.

The series $P(D)W := \sum_{n=0}^{\infty} c_n D^n W$ converges for all $W \in \beta(\mathbb{R})$. That is, there exists $V \in \beta(\mathbb{R})$ such that $\delta\text{-}\lim_{n\to\infty} \sum_{k=0}^{n} c_k D^k W = V$.

Properties

(i) $P(D)W \in \beta_+(\mathbb{R})$, for all $W \in \beta_+(\mathbb{R})$.

(ii) Let $W \in \beta(\mathbb{R})$ such that $W(x) = 0$ on (a, b). Then $P(D)W(x) = 0$ on (a, b).

(iii) $P(D)(W * V) = P(D)W * V = W * P(D)V$, for all $W, V \in \beta_+(\mathbb{R})$.

(iv) If $W_n, W \in \beta(\mathbb{R})$ $(n \in \mathbb{N})$ such that $\delta\text{-}\lim_{n\to\infty} W_n = W$, then $\delta\text{-}\lim_{n\to\infty} P(D)W_n = P(D)W$.

Theorem 3.8. *Let $\ell \in \mathbb{N} \cup \{0\}$ and $P(D)\delta = \sum_{m=\ell}^{\infty} c_m D^m \delta$ with $c_\ell \neq 0$. Then $P(D)\delta \overset{q}{\sim} c_\ell D^\ell \delta$ at infinity related to $\lambda^{-(\ell+1)}$.*

Proof. Since the sequence $\{M_n\}$ determines a non quasi-analytic class of functions [17] and the sequence $\{c_n\}$ satisfies the condition $|c_n| \leq \frac{AB^n}{M_n}$ $(n \in \mathbb{N})$, for some constants $A > 0$ and $B > 0$, there exists a delta sequence $\{\varphi_k\}$ such that for each $k \in \mathbb{N}$, $\sup\limits_{n\in\mathbb{N}, x\in\mathbb{R}} \left|2^n c_n \varphi_k^{(n)}(x)\right| = \gamma_k < \infty$.

Then, $P(D)\delta = \left[\frac{f_p}{\varphi_p}\right]$, where $f_p(x) = \sum_{m=\ell}^{\infty} c_m \varphi_p^{(m)}(x)$.
Now, for each p, k, and λ,

$$\begin{aligned}
(f_p)_\lambda * \varphi_k &= \sum_{m=\ell}^{\infty} c_m (\varphi_p^{(m)})_\lambda * \varphi_k \\
&= \sum_{m=\ell}^{\infty} \frac{c_m (\varphi_p(\lambda x))^{(m)} * \varphi_k}{\lambda^m} \\
&= \sum_{m=\ell}^{\infty} \frac{c_m (\varphi_p)_\lambda * \varphi_k^{(m)}}{\lambda^m} \\
&= \left(\sum_{m=\ell}^{\infty} \frac{c_m \varphi_k^{(m)}}{\lambda^m} \right) * (\varphi_p)_\lambda.
\end{aligned}$$

Therefore, for each k and λ

$$\begin{aligned}
\frac{(P(D)\delta)_\lambda}{\lambda^{-(\ell+1)}} * \varphi_k &= \lambda^{\ell+1}\left[\frac{f_p}{\varphi_p}\right]_\lambda * \varphi_k \\
&= \lambda^{\ell+1}\left[\frac{(f_p)_\lambda}{\lambda(\varphi_p)_\lambda}\right] * \varphi_k \\
&= \left[\frac{\left(\sum_{m=\ell}^{\infty}\frac{c_m\lambda^\ell\varphi_k^{(m)}}{\lambda^m}\right) * \lambda(\varphi_p)_\lambda}{\lambda(\varphi_p)_\lambda}\right] \\
&= c_\ell\varphi_k^{(\ell)} + \sum_{m=\ell+1}^{\infty}\frac{c_m\varphi_k^{(m)}}{\lambda^{m-\ell}}\,.
\end{aligned}$$

So, for each $k \in \mathbb{N}$,

$\frac{(P(D)\delta)_\lambda}{\lambda^{-(\ell+1)}} * \varphi_k \to c_\ell\varphi_k^{(\ell)}$ uniformly on compact sets as $\lambda \to \infty$, provided $\sum_{m=\ell+1}^{\infty}\frac{c_m\varphi_k^{(m)}(x)}{\lambda^{m-\ell}} \to 0$ uniformly on compact sets as $\lambda \to \infty$.

However, this follows from

$\left|\sum_{m=\ell+1}^{\infty}\frac{c_m\varphi_k^{(m)}(x)}{\lambda^{m-\ell}}\right| \le \sum_{m=\ell+1}^{\infty}\left|\frac{c_m\varphi_k^{(m)}(x)}{\lambda^{m-\ell}}\right| \le \frac{\gamma_k}{\lambda-1}$, for all $\lambda > 1$, $k \in \mathbb{N}$, and $x \in \mathbb{R}$.

Therefore,

$$P(D)\delta \overset{q}{\sim} c_\ell D^\ell\delta \text{ at infinity related to } \lambda^{-(\ell+1)}. \qquad \square$$

By applying the previous theorem to Theorem 3.7 and then using Property (iii) above, we obtain the following.

Theorem 3.9. *Let* $\ell \in \mathbb{N} \cup \{0\}$, $P(D) = \sum_{m=\ell}^{\infty} c_m D^m$ *with* $c_\ell \ne 0$, *and* $W \overset{q}{\sim} V$ *at infinity related to* $\lambda^\alpha L(\lambda)$. *Then,* $P(D)W \overset{q}{\sim} c_\ell D^\ell V$ *at infinity related to* $\lambda^{\alpha-\ell}L(\lambda)$.

Example 3.10. Let $P(D)\delta = \sum_{m=0}^{\infty}\frac{D^m\delta}{(2m)!}$. Then $P(D)\delta \overset{q}{\sim} \delta$ at infinity related to λ^{-1}. Note that $P(D)$ is an ultradifferential operator of class (M_p) if and only if the sequence satisfies the inequality $M_p \le AB^p(p!)^2$, for some constants $A, B > 0$.

Remark 3.11. Since both δ and $D\delta$ vanish on $(0,\infty)$, Example 3.6 illustrates that "quasi-asymptotic behavior at infinity" is not a local property of a Boehmian. The main goal of the next section is to show that this is not the case for "quasi-asymptotic behavior at zero".

4 Quasi-Asymptotic Behavior At The Origin

Quasi-asymptotic behavior at zero (from the right) can be defined in a similar way.

Let L be slowly varying at infinity and $L^*(\lambda) = L\left(\frac{1}{\lambda}\right)$, for $\lambda > 0$. For $\alpha \in \mathbb{R}$ and $W, V \in \beta_+(\mathbb{R})$, $W \overset{q}{\sim} V$ at 0^+ related to $\lambda^\alpha L^*(\lambda)$ provided $\delta\text{-}\lim_{\lambda\to 0^+} \frac{W_\lambda}{\lambda^\alpha L^*(\lambda)} = V$, i.e.,

$$\delta\text{-}\lim_{n\to\infty} \frac{W_{\lambda_n}}{\lambda_n^\alpha L^*(\lambda_n)} = V, \text{ for every } \lambda_n \to 0^+. \tag{4.1}$$

Quasi-asymptotic behavior at zero has similar properties as quasi-asymptotic behavior at infinity and with similar proofs. However, as the next theorem will show, unlike quasi-asymptotic behavior at infinity, quasi-asymptotic behavior at zero is a local property of a Boehmian.

Let $W, U \in \beta(\mathbb{R})$. Then, $W(x) = U(x)$ on (a, b) provided $W - U$ vanishes on the interval (a, b).

Theorem 4.1. *Let $a > 0$ and $W, V, U \in \beta_+(\mathbb{R})$ such that $W(x) = U(x)$ on $(-a, a)$. If $W \overset{q}{\sim} V$ at 0^+ related to $\lambda^\alpha L^*(\lambda)$, then $U \overset{q}{\sim} V$ at 0^+ related to $\lambda^\alpha L^*(\lambda)$.*

Proof. Let $\lambda_n \to 0^+$ as $n \to \infty$. Now,

$$\frac{U_{\lambda_n}}{\lambda_n^\alpha L^*(\lambda_n)} = \left(\frac{U_{\lambda_n}}{\lambda_n^\alpha L^*(\lambda_n)} - \frac{W_{\lambda_n}}{\lambda_n^\alpha L^*(\lambda_n)}\right) + \frac{W_{\lambda_n}}{\lambda_n^\alpha L^*(\lambda_n)}.$$

Since

$$\delta\text{-}\lim_{n\to\infty} \frac{W_{\lambda_n}}{\lambda_n^\alpha L^*(\lambda_n)} = V,$$

it suffices to show that $\delta\text{-}\lim_{n\to\infty} Q_n = 0$, where $Q_n = \frac{U_{\lambda_n}}{\lambda_n^\alpha L^*(\lambda_n)} - \frac{W_{\lambda_n}}{\lambda_n^\alpha L^*(\lambda_n)}$, $n \in \mathbb{N}$.

Now, since $W(x) = U(x)$ on $(-a, a)$, for each $m \in \mathbb{N}$ there exists $\ell_m \in \mathbb{N}$ such that for all $n \geq \ell_m$, $Q_n(x) = 0$ on $(-(m+1), m+1)$.

For each $n \in \mathbb{N}$, there exists a delta sequence $\{\varphi_{k,n}\}_{k=1}^\infty$ such that $Q_n * \varphi_{k,n} \in C(\mathbb{R})$, for all $k \in \mathbb{N}$. Moreover, by Theorem 2.5, the delta sequences $\{\varphi_{k,n}\}_{k=1}^\infty$ may be chosen such that for each $m \in \mathbb{N}$ and all $n \geq \ell_m$, $(Q_n * \varphi_{k,n})(x) = 0$ on $[-m, m]$, for all $k \in \mathbb{N}$.

Now, since for each $n \in \mathbb{N}$, supp $\varphi_{k,n} \to \{0\}$ as $k \to \infty$, it is possible, for each $j \in \mathbb{N}$, to choose a sequence of positive integers $\{p_{k,j}\}_{k=1}^\infty$ such that

$$\operatorname{supp} \varphi_{(p_{k,j}),j} \subseteq \left[\frac{-1}{k2^{j+1}}, \frac{1}{k2^{j+1}}\right].$$

For $k \in \mathbb{N}$, let $\psi_k = \prod_{j=1}^{\infty *} \varphi_{(p_{k,j}),j}$, where the product is convolution.

Notice that for each k, supp $\varphi_{(p_{k,j}),j} \subseteq \left[\frac{-1}{k2^{j+1}}, \frac{1}{k2^{j+1}}\right]$, $\sum_{j=1}^\infty \frac{1}{k2^j} < \infty$, and $\{\varphi_{(p_{k,j}),j}\}_{j=1}^\infty$ is a delta sequence. Thus, it follows that the infinite product converges uniformly for each $k \in \mathbb{N}$.

It also follows that $\{\psi_k\}$ is a delta sequence.

Now, for all $n, k \in \mathbb{N}$,

$$Q_n * \psi_k = Q_n * \varphi_{(p_{k,n}),n} * \prod_{j=1, j\neq n}^{\infty} {}^{*}\, \varphi_{(p_{k,j}),j}.$$

Thus, $Q_n * \psi_k \in C(\mathbb{R})$, for all $k, n \in \mathbb{N}$. Moreover, given $k \in \mathbb{N}$ and $m \in \mathbb{N}$, there exists $\ell \in \mathbb{N}$ (which depends on m) such that for all $n > \ell$, $(Q_n * \psi_k)(x) = 0$ on $[-m, m]$.

Therefore, $\delta\text{-}\lim_{n\to\infty} Q_n = 0$. This completes the proof of the theorem. $\square$

5 Final Value Theorems for the Stieltjes Transform

By utilizing the results from the previous sections, as well as obtaining some new results, we establish two Abelian theorems of the final type for the Stieltjes transform.

The Stieltjes transform of index r of a suitably restricted function f is given by

$$S_z^r f = \int_0^\infty \frac{f(t)dt}{(z+t)^{r+1}}\,, z \in \mathbb{C}\backslash(-\infty, 0]. \tag{5.1}$$

Since the restriction of the differential operator D to the space $\mathcal{D}'(\mathbb{R}) \subset \beta(\mathbb{R})$ agrees with the differential operator for $\mathcal{D}'(\mathbb{R})$, D will denote either operator. This should not cause any confusion.

Let $T \in J'(r)$, $r > -1$. That is, $T \in \mathcal{D}'(\mathbb{R})$ such that $T = D^k g$, where $k \in \mathbb{N}$, $g \in L^1_{loc}(\mathbb{R})$ with supp $g \subseteq [a, \infty)$ for some $a \geq 0$, and $g(t)t^{-r-k+\varepsilon}$ is bounded as $t \to \infty$ (for some $\varepsilon > 0$). Then, the Stieltjes transform of T is defined by $S_z^r T = \langle T_t, (z+t)^{-r-1}\rangle$, $z \in \mathbb{C}\backslash(-\infty, 0]$. Several authors ([5, 15]) have used the space $J'(r)$ to investigate the Stieltjes transform for distributions.

In this section we consider the space of Boehmians $B_r(\mathbb{R})$ which is a subspace of $\beta_+(\mathbb{R})$ (*Boehmians supported on* $[0, \infty)$) and contains $J'(r)$ as a proper subspace.

A Boehmian W is in $B_r(\mathbb{R})$ provided $W \in \beta_+(\mathbb{R})$ and $W(t) = D^k g(t)$ as $t \to \infty$ for some $k \in \mathbb{N}$, where $g \in L^1_{loc}(\mathbb{R})$ such that supp $g \subseteq [a, \infty)$ (some $a \geq 0$) and $g(t)t^{-r-k+\varepsilon}$ is bounded as $t \to \infty$ for some $\varepsilon > 0$. That is, $W = V + D^k g$, where supp $V \subseteq [0, a]$ and supp $g \subseteq [a, \infty)$ such that $g(t)t^{-r-k+\varepsilon}$ is bounded as $t \to \infty$.

Since the Stieltjes transform of a Boehmian is defined as the iterated Laplace transform [13], we give a brief introduction to the Laplace transform for a Boehmian, which also includes some new results.

The Laplace transform of a Boehmian $W = \left[\frac{f_n}{\varphi_n}\right] \in B_r(\mathbb{R})$ is given by

$$\widehat{W}(z) = \lim_{n\to\infty} \widehat{f_n}(z) = \lim_{n\to\infty} \int_{-\infty}^{\infty} e^{-zt} f_n(t)dt, \ \operatorname{Re} z > 0. \tag{5.2}$$

Remarks 5.1.

1. The Laplace transform operator on $B_r(\mathbb{R})$ has many of the same properties as the classical Laplace transform.

2. The Laplace transform for a Boehmian W is independent of the representative.

3. $\widehat{W}$ is an analytic function in the half-plane Re $z > 0$. Moreover, if $W \in \beta_c(\mathbb{R}) \cap \beta_+(\mathbb{R})$, then $\widehat{W}$ is an entire function.

Let $W \in J'(r)$. Since $J'(r) \subset B_r(\mathbb{R})$, the Laplace transform of W exists as an element of $J'(r)$ and as an element of $B_r(\mathbb{R})$. The Laplace transform is consistent on $J'(r)$. That is, these two notions agree on $J'(r)$.

Lemma 5.2. *Let $b > 0$. Suppose that δ-$\lim_{n\to\infty} W_n = W$ and supp $W_n \subset [0, b]$, $n \in \mathbb{N}$. Then, for each $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that for all $n > n_\varepsilon$, $\left|\widehat{W_n}(t) - \widehat{W}(t)\right| \leq e^{\varepsilon t}$, $t \geq 0$.*

Proof. Since δ-$\lim_{n\to\infty} W_n = W$, there exist a sequence $\{\varepsilon_n\}$ such that $\varepsilon_n \searrow 0$ and a delta sequence $\{\varphi_n\}$ such that supp $\varphi_n \subset (-\varepsilon_n, \varepsilon_n)$ $(n \in \mathbb{N})$, and for each $k \in \mathbb{N}$,

$$W_n * \varphi_k \to W * \varphi_k \tag{5.3}$$

uniformly on compact sets as $n \to \infty$. By using the properties of a delta sequence and the Mean Value Theorem, for each $k \in \mathbb{N}$, $\widehat{\varphi_k}(t) \geq e^{-\varepsilon_k t}$, for all $t \geq 0$. It follows that for $t \geq 0$,

$$\left|\widehat{W_n}(t) - \widehat{W}(t)\right| e^{-\varepsilon_k t} \leq e^{\varepsilon_k t} \int_{-\varepsilon_k}^{\gamma} |(W_n * \varphi_k)(x) - (W * \varphi_k)(x)|\, dx, \tag{5.4}$$

for some $\gamma > 0$ and all $k, n \in \mathbb{N}$.

By (5.3) and (5.4), the result follows. □

Theorem 5.3. *Let $b > 0$. Suppose that δ-$\lim_{n\to\infty} W_n = W$ and supp $W_n \subset [0, b]$, $n \in \mathbb{N}$. Then, $\widehat{W_n} \to \widehat{W}$ uniformly on compact subsets of $\mathbb{C}$ as $n \to \infty$.*

Proof. Since δ-$\lim_{n\to\infty} W_n = W$, there exist a sequence $\{\varepsilon_n\}$ such that $\varepsilon_n \searrow 0$ and a delta sequence $\{\varphi_n\}$ such that supp $\varphi_n \subset (-\varepsilon_n, \varepsilon_n)$ $(n \in \mathbb{N})$, and for each $k \in \mathbb{N}$,

$$W_n * \varphi_k \to W * \varphi_k$$

uniformly on compact subsets of $\mathbb{R}$ as $n \to \infty$.

Also, there exist $\gamma > 0$ and $B > 0$ such that

$$\left|\widehat{W_n}(z) - \widehat{W}(z)\right| |\widehat{\varphi_k}(z)| \leq e^{B|\mathrm{Re}\, z|} \int_{-\varepsilon_k}^{\gamma} |(W_n * \varphi_k)(t) - (W * \varphi_k)(t)|\, dt,$$

for all $k, n \in \mathbb{N}$ and $z \in \mathbb{C}$.

Now, let A be a compact subset of $\mathbb{C}$. Then, there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, $2|\widehat{\varphi_k}(z)| > 1$, for all $z \in A$.

Thus, for all $z \in A$,

$$|\widehat{W_n}(z) - \widehat{W}(z)| \leq M \int_{-\varepsilon_{k_0}}^{\gamma} |(W_n * \varphi_{k_0})(t) - (W * \varphi_{k_0})(t)|\, dt,$$

for some $M > 0$.

The result follows. □

For $r > -1$, the Stieltjes transform for $W = \left[\frac{f_n}{\varphi_n}\right] \in B_r(\mathbb{R})$ is given by

$$\Lambda_z^r W = \frac{1}{\Gamma(r+1)} \int_0^\infty e^{-zt}\, t^r\, \widehat{W}(t)\, dt, \ \text{Re } z > 0, \tag{5.5}$$

where Γ is the gamma function.

Remark 5.4. For $r > -1$, Λ_z^r is a linear injective mapping from $B_r(\mathbb{R})$ into the space of analytic functions in the half-plane Re $z > 0$. For more results concerning the Stieltjes transform on the space $B_r(\mathbb{R})$, see [13, 14].

Now, $J'(r) \subset B_r(\mathbb{R})$. Therefore, each element of $J'(r)$ has a Stieltjes transform as an element of $J'(r)$ and also as an element of $B_r(\mathbb{R})$. The two agree in the half-plane Re $z > 0$.

Theorem 5.5. *Let $\sigma > 0$. Suppose that δ-$\lim_{n\to\infty} W_n = W$ and, for some $b > 0$, supp $W_n \subset [0, b]$, $n \in \mathbb{N}$. Then, for $r > -1$, $\Lambda_z^r W_n \to \Lambda_z^r W$ uniformly in the half-plane $\mathrm{Re}\, z \geq \sigma$ as $n \to \infty$.*

Proof. Let $\sigma > 0$ and $r > -1$. By Lemma 5.2, there exists $n_0 > 0$ such that for all $n > n_0$,

$$|\widehat{W_n}(t) - \widehat{W}(t)| \leq e^{(\sigma/2)\, t}, \ \text{for all } t \geq 0.$$

Let $\varepsilon > 0$. Then, for some $\gamma > 0$,

$$4 \int_\gamma^\infty t^r e^{-(\sigma/2)\, t}\, dt < \varepsilon.$$

By Theorem 5.3, there exists $n_1 > 0$ such that for all $n > n_1$,

$$4\, |\widehat{W_n}(t) - \widehat{W}(t)| \int_0^\gamma x^r e^{-\sigma x}\, dx < \varepsilon, \ \text{for all } t \in [0, \gamma].$$

Now, by above, it follows that for $n > \max\{n_0, n_1\}$,

$$|\Lambda_z^r W_n - \Lambda_z^r W| < \frac{1}{\Gamma(r+1)} \int_0^\infty e^{-t\mathrm{Re}\, z}\, t^r\, |\widehat{W_n}(t) - \widehat{W}(t)|\, dt < \varepsilon,$$

for all z in the half-plaice $\mathrm{Re}\, z \geq \sigma$.

Thus, the result follows. $\square$

For $\nu > 0$, let q_ν be the continuous function supported on $[0, \infty)$ given by

$$q_\nu(t) = \frac{H(t) t^\nu}{\Gamma(\nu + 1)},$$

where H is the Heaviside function.

Then, for $\alpha \in \mathbb{R}$, the Boehmian $\Theta_{\alpha+1} \in \beta_+(\mathbb{R})$ defined below corresponds to the distribution $f_{\alpha+1} \in S'_+(\mathbb{R})$ [15, 16], where $S'_+(\mathbb{R})$ denotes the space of tempered distributions supported on $[0, \infty)$.

Let $\{\varphi_n\}$ be any fixed delta sequence.

$$\Theta_{\alpha+1} = \begin{cases} \left[\frac{q_\alpha * \varphi_n}{\varphi_n}\right] & \text{for } \alpha > 0 \\ D^k \Theta_{\alpha+k+1} & \text{for } \alpha \leq 0 \text{ and } \alpha + k + 1 > 0. \end{cases}$$

Also, for $r > \max\{\alpha, -1\}$,

$$\Lambda_z^r \Theta_{\alpha+1} = \frac{\Gamma(r-\alpha)}{\Gamma(r+1)} \frac{1}{z^{r-\alpha}}, \ \mathrm{Re}\, z > 0.$$

The distribution $f_{\alpha+1}$ is of fundamental importance in the study of quasi-asymptotic behavior of distributions [16].

Let $U \in \beta_+(\mathbb{R})$ and $V \in \beta_c(\mathbb{R}) \cap \beta_+(\mathbb{R})$ such that $V \overset{q}{\sim} U$ at infinity related to λ^α. By using the Laplace transform and the fact that $\mathrm{supp}\, V_\lambda \to \{0\}$ as $\lambda \to \infty$, it follows that $U = C\Theta_{\alpha+1}$, for some $C \in \mathbb{C}$. Moreover, if $\alpha \notin \{-1, -2, \ldots\}$, then $C = 0$.

For $\sigma > 0$, $\Omega_\sigma = \{z \in \mathbb{C} : |z| \leq \sigma, \mathrm{Re}\, z \leq 0\}$.

The following is an Abelian theorem of the final type for the Stieltjes transform for Boehmians having compact support.

Theorem 5.6. *Let $\sigma > 0$ and $W \in \beta_c(\mathbb{R})$ with $\mathrm{supp}\, W \subseteq [0, \sigma]$. Suppose that $W \overset{q}{\sim} C\Theta_{\alpha+1}$ at infinity related to λ^α. If $C \neq 0$, then for $r > -1$, the function $z \to z^{r-\alpha} \Lambda_z^r W$ can be analytically extended to the region $\mathbb{C} \backslash \Omega_\sigma$. Moreover,*

$$\lim_{\substack{z \to \infty \\ z \in \mathbb{C} \backslash \Omega_\sigma}} \frac{\Gamma(r+1)}{\Gamma(r-\alpha)} z^{r-\alpha} \Lambda_z^r W = C.$$

Proof. For $\lambda > 0$, $\mathrm{supp}\, W_\lambda \subseteq \left[0, \frac{\sigma}{\lambda}\right]$. Since $\delta\text{-}\lim_{\lambda \to \infty} \frac{W_\lambda}{\lambda^\alpha} = C\Theta_{\alpha+1}$ and $\mathrm{supp}\, W_\lambda \to \{0\}$ as $\lambda \to \infty$, thus, $\mathrm{supp}\, C\Theta_{\alpha+1} = \{0\}$. Since $C \neq 0$, it follows that $\alpha \in \{-1, -2, \ldots\}$.

Now, it is known [13] that for $W \in \beta_c(\mathbb{R})$, the function $z \to \Lambda_z^r W$ can be analytically extended to the region $\mathbb{C} \backslash (\Omega_\sigma \cup (-\infty, 0))$. Moreover, for $z \in \{z \in \mathbb{C} : |z| > \sigma\}$,

$$z^{r-\alpha} \Lambda_z^r W = \frac{1}{\Gamma(r+1)} \sum_{n=0}^{\infty} \frac{c_n \Gamma(n+r+1)}{z^{n+\alpha+1}}, \tag{5.6}$$

where $c_n = \frac{\widehat{W}(0)}{n!}$, $n \in \mathbb{N} \cup \{0\}$.

Now, since $\alpha \in \{-1, -2, \ldots\}$, the right-hand side of (5.6) is analytic in the region $z \in \{z \in \mathbb{C} : |z| > \sigma\}$. Therefore, the function $z \to z^{r-\alpha} \Lambda_z^r W$ can be analytically extended to $\mathbb{C} \backslash \Omega_\sigma$.

Since $\delta\text{-}\lim_{\lambda \to \infty} \frac{W_\lambda}{\lambda^\alpha} = C\Theta_{\alpha+1}$, by Theorem 5.5 it follows that for $\gamma > 0$

$$\lambda^{r-\alpha} \Lambda_{\lambda z}^r W \to \frac{\Gamma(r-\alpha) C}{\Gamma(r+1) z^{r-\alpha}}$$

uniformly in the half-plane $\operatorname{Re} z \geq \gamma$ as $\lambda \to \infty$.

Thus, for any z in the half-plane $\operatorname{Re} z > 0$,

$$\frac{\Gamma(r+1)}{\Gamma(r-\alpha)}\,(\lambda z)^{r-\alpha}\Lambda^r_{\lambda z}W \to C \quad \text{as } \lambda \to \infty. \tag{5.7}$$

Therefore, by (5.6) and (5.7), for $z \in \{z \in \mathbb{C} : |z| > \sigma, \operatorname{Re} z > 0\}$,

$$\frac{1}{\Gamma(r-\alpha)} \sum_{n=0}^{\infty} \frac{c_n \Gamma(n+r+1)}{(\lambda z)^{n+\alpha+1}} \to C \quad \text{as } \lambda \to \infty.$$

It follows that for $\alpha = -1$, $c_0 = C$. Otherwise, if $\alpha \in \{-2, -3, \ldots\}$, $c_0 = c_1 = \ldots = c_{-(\alpha+2)} = 0$ and $c_{-(\alpha+1)} = C$.

The conclusion follows. That is,

$$\frac{\Gamma(r+1)}{\Gamma(r-\alpha)}\, z^{r-\alpha}\Lambda^r_z W \to C \text{ as } z \to \infty,\ z \in \mathbb{C}\backslash\Omega_\sigma. \qquad \square$$

Let $\mathcal{A}$ denote the space of all Boehmians $W \in \beta_+(\mathbb{R})$ such that $W(t) = D^k g(t)$ on (b, ∞) for some $k \in \mathbb{N} \cup \{0\}$, $b > 0$, and $g \in L^1(\mathbb{R})$ with $\int_b^\infty g \neq 0$. Let $g_b(t) = H(t-b)g(t)$, for $t \in \mathbb{R}$.

For $\varepsilon > 0$, $\mathcal{O}_\varepsilon = \{z \in \mathbb{C} : |\arg z| \leq \pi - \varepsilon\}$.

We now establish the following Abelian theorem of the final type.

Theorem 5.7. *Let $W \in \mathcal{A}$ and $k \in \mathbb{N}$ $(k \neq 0)$ be as in the definition of $\mathcal{A}$. If $W \overset{q}{\sim} C\Theta_{\alpha+1}$ at infinity related to λ^α, where $C \neq 0$ and $\alpha \in \{-1, -2, \ldots, -k\}$, then for $r > -1$*

$$\lim_{\substack{z\to\infty \\ z \in \mathcal{O}_\varepsilon \cap (\mathbb{C}\backslash\Omega_\sigma)}} \frac{\Gamma(r+1)}{\Gamma(r-\alpha)}\, z^{r-\alpha}\Lambda^r_z W = C.$$

Proof. For $r > -1$, $W \in B_r(\mathbb{R})$. This follows by observing that

$$W = V + D^{k+1}(H * g_b),$$

where $V \in \beta_c(\mathbb{R})$ and $(H * g_b)(t)t^{-r-(k+1)+\varepsilon}$ is bounded for some $\varepsilon > 0$.

Now, since $g_b \in L^1(\mathbb{R})$, $g_b \overset{q}{\sim} A\Theta_{-1+1}$ (in $S'_+(\mathbb{R})$) at infinity related to λ^{-1}, where $A = \int g_b \neq 0$ [15]. Thus,

$$D^k g_b \overset{q}{\sim} A\Theta_{-(k+1)+1} \text{ (in } S'_+(\mathbb{R})) \text{ at infinity related to } \lambda^{-(k+1)}, \tag{5.8}$$

and hence,

$$D^k g_b \overset{q}{\sim} A\Theta_{-(k+1)+1} \text{ (in } \beta(\mathbb{R})) \text{ at infinity related to } \lambda^{-(k+1)}.$$

Notice that $\alpha + k + 1 > 0$ and $\delta\text{-}\lim_{\lambda\to\infty} \frac{W_\lambda}{\lambda^\alpha} = C\Theta_{\alpha+1}$. Since $\delta\text{-}\lim_{\lambda\to\infty} \frac{V_\lambda}{\lambda^\alpha} = \delta\text{-}\lim_{\lambda\to\infty} \left(\frac{W_\lambda}{\lambda^\alpha} - \frac{(D^k g_b)_\lambda}{\lambda^{-(k+1)}}\,\frac{1}{\lambda^{\alpha+k+1}}\right) = C\Theta_{\alpha+1}$, by Theorem 5.6, we have for $r > -1$

$$\lim_{\substack{z\to\infty \\ z \in \mathbb{C}\backslash\Omega_\sigma}} \frac{\Gamma(r+1)}{\Gamma(r-\alpha)}\, z^{r-\alpha}\Lambda^r_z V = C. \tag{5.9}$$

Now,

$$\begin{aligned}(5.10)\qquad \frac{\Gamma(r+1)}{\Gamma(r-\alpha)}\, z^{r-\alpha}\, \Lambda_z^r\, W &= \frac{\Gamma(r+1)}{\Gamma(r-\alpha)}\, z^{r-\alpha}\, \Lambda_z^r\, V \\ &\quad + \frac{\Gamma(r+k+1)}{\Gamma(r-\alpha)}\, \frac{\Gamma(r+1)}{\Gamma(r+k+1)}\, \frac{1}{z^{\alpha+k+1}}\, z^{r+k+1}\, \Lambda_z^r\, D^k g_b.\end{aligned}$$

By (5.8) we have

$$(5.11)\qquad \lim_{\substack{z\to\infty\\ z\in\mathcal{O}_\varepsilon}} \frac{\Gamma(r+1)}{\Gamma(r+k+1)}\, z^{r+k+1} \Lambda_z^r D^k g_b = A \quad [15].$$

By (5.9), (5.10), and (5.11),

$$\lim_{\substack{z\to\infty\\ z\in\mathcal{O}_\varepsilon\cap(\mathbb{C}\setminus\Omega_\sigma)}} \frac{\Gamma(r+1)}{\Gamma(r-\alpha)}\, z^{r-\alpha} \Lambda_z^r W = C. \qquad \square$$

Remarks 5.8.

1. Another type of asymptotics that has been utilized to establish Abelian type theorems of the final type for generalized functions is the notion of equivalence at infinity. Let $W \in B_r(\mathbb{R})$ and $\alpha \in \mathbb{R}\setminus\{-1,-2,\ldots\}$. Then $W(t) \overset{e}{\sim} Ct^\alpha$ ($C \in \mathbb{C}$) as $t \to \infty$ provided there exists $b > 0$, $k \in \mathbb{N}\cup\{0\}$, and $g \in L^1_{loc}(\mathbb{R})$ such that $W(t) = D^k g(t)$ on (b,∞) and $\frac{g(t)}{t^{\alpha+k}} \to \frac{C}{(\alpha+1)_k}$ as $t \to \infty$, where $(\alpha+1)_k = (\alpha+1)(\alpha+2)\ldots(\alpha+k)$.

   Using equivalence at infinity, Abelian type theorems for the Stieltjes transform for Boehmians have been established ([13, 14]). However, notice that in the definition, $\alpha \notin \{-1,-2,\ldots\}$. Moreover, the Abelian theorems require $\alpha > -1$. However, both the definition of quasi-asymptotic behavior and the final value theorem (Theorem 5.7) do not require $\alpha > -1$.
2. An example of a Boehmian with compact support and having quasi-asymptotic behavior at infinity, but which is not an ultradistribution, would be of interest. At this time, even an example of a Boehmian with compact support that is not an ultradistribution has not been found. However, there are known examples of Boehmians that are not ultradistributions.
3. It would be desirable to obtain a final value theorem similar to Theorem 5.7 without the condition $W(t) = D^k g(t)$ as $t \to \infty$, where $g \in L^1(\mathbb{R})$.

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