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A PARTIAL ANSWER TO A QUESTION OF Y. IKEDA, C. LIU AND Y. TANAKA

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Abstract. In this paper, we give a partial answer to the problem posed by Y. Ikeda, C. Liu and Y. Tanaka in [3].

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1. Introduction and Preliminaries

In 2002, Y. Ikeda, C. Liu and Y. Tanaka introduced the notion of σ-strong networks as a generalization of development in developable spaces, and considered certain quotient images of metric spaces in terms of σ-strong networks. By means of σ-strong networks, the authors proved that a sequential space X has a σ-point-finite strong *cs*-network if and only if X is a sequence-covering quotient compact image of a metric space, and posed the following question.

Question 1.1 ([3]). *Let X be a symmetric space with a σ-point-finite cs-network. Is X a quotient compact image of a metric space?*

In this paper, we give a partial answer to the Question 1.1.

Throughout this paper, all spaces are T_1 and regular, all maps are continuous and onto, $\mathbb{N}$ denotes the set of all natural numbers. Let $\mathcal{P}$ and $\mathcal{Q}$ be two families of subsets of X, we denote

$$(\mathcal{P})_x = \{P \in \mathcal{P} : x \in P\};$$

$$\mathcal{P} \bigwedge \mathcal{Q} = \{P \cap Q : P \in \mathcal{P}, Q \in \mathcal{Q}\}.$$

For a sequence $\{x_n\}$ converging to x, we say that $\{x_n\}$ is *eventually* in P, if $\{x\} \bigcup \{x_n : n \geq m\} \subset P$ for some $m \in \mathbb{N}$, and $\{x_n\}$ is *frequently* in P, if some subsequence of $\{x_n\}$ is eventually in P.

Definition 1.2. Let $\mathcal{P}$ be a family of subsets of a space X.

1. $\mathcal{P}$ is a *network at x* in X, if $x \in P$ for every $P \in \mathcal{P}$, and whenever $x \in U$ with U is open in X, then $x \in P \subset U$ for some $P \in \mathcal{P}$.

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2. $\mathcal{P}$ is a *cs-network* for X [7], if each sequence S converging to a point $x \in U$ with U open in X, S is eventually in $P \subset U$ for some $P \in \mathcal{P}$.

3. $\mathcal{P}$ is a *cs-cover* [9], if every convergent sequence is eventually in some $P \in \mathcal{P}$.

4. $\mathcal{P}$ is *point-finite* (resp., *point-countable*) [2], if each point $x \in X$ belongs to only finite (resp., countable) many members of $\mathcal{P}$.

5. X is *sequential* [2], if whenever A is a non-closed subset of X, then there is a sequence in A converging to a point not in A.

Definition 1.3. Let $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$ be a cover of a space X. Assume that $\mathcal{P}$ satisfies the following (1) and (2) for every $x \in X$.

1. $\mathcal{P}_x$ is a network at x.

2. If P_1, $P_2 \in \mathcal{P}_x$, then $P \subset P_1 \cap P_2$ for some $P \in \mathcal{P}_x$.

$\mathcal{P}$ is a *weak base* for X [1], if for $G \subset X$, G is open in X if and only if for every $x \in G$, there exists $P \in \mathcal{P}_x$ such that $P \subset G$; $\mathcal{P}_x$ is said to be a *weak neighborhood base* at x.

Definition 1.4 ([4, 5])**.** Let d be a d-function on a space X.

1. For each $x \in X$, $n \in \mathbb{N}$, let
$$S_n(x) = \Big\{y \in X : d(x, y) < \frac{1}{n}\Big\}.$$

2. For every $P \subset X$, put
$$d(P) = \sup\{d(x, y) : x, y \in P\}.$$

3. X is *symmetric*, if $\{S_n(x) : n \in \mathbb{N}\}$ is a weak neighborhood base at x for each $x \in X$.

4. X is *Cauchy symmetric*, if X is symmetric and every convergent sequence is d-Cauchy.

Remark 1.5 ([5]). X is Cauchy symmetric if and only if for each $x \in X$, $d\big(S_n(x)\big)$ converges to 0.

Definition 1.6. Let $\{\mathcal{P}_n : n \in \mathbb{N}\}$ be a sequence of covers of a space X such that $\mathcal{P}_{n+1}$ refines $\mathcal{P}_n$ for every $n \in \mathbb{N}$.

1. $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ is a *σ-strong network* for X [3], if $\{\mathtt{St}(x, \mathcal{P}_n) : n \in \mathbb{N}\}$ is a network at each point $x \in X$.

2. $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ is a *σ-point-finite strong network* for X [7], if it is a σ-strong network and each $\mathcal{P}_n$ is point-finite.

3. $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ is a *σ-point-finite strong network consisting of cs-covers* [7], if it is a σ-strong network and each $\mathcal{P}_n$ is a point-finite *cs*-cover.

Definition 1.7. Let $f : X \to Y$ be a map. Then

1. f is a *compact* map [2], if each $f^{-1}(y)$ is compact in X.

2. f is a *quotient* map [2], if whenever $f^{-1}(U)$ is open in X, then U is open in Y.

3. f is a *sequence-covering* map [8], if every convergent sequence of Y is the image of some convergent sequence of X.

For some undefined or related concepts, we refer the reader to [2, 6, 7].

2. Main Results

Theorem 2.1. *The following are equivalent for a space X.*

1. *X is a sequence-covering quotient compact image of a metric space;*

2. *X is a Cauchy symmetric with σ-point-finite cs-network.*

Proof. (1) $\Longrightarrow$ (2). By Theorem 9 and Theorem 12 in [3].

(2) $\Longrightarrow$ (1). Let X be a Cauchy symmetric and $\mathcal{U} = \bigcup\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a σ-point-finite *cs*-network for X. We can assume that each $\mathcal{U}_n$ is closed under finite intersections and $\mathcal{U}_n \subset \mathcal{U}_{n+1}$ for all $n \in \mathbb{N}$. So, $\mathcal{U}$ is closed under finite intersections. Put

$$\mathcal{P}_x = \big\{P \in \mathcal{U} : S_n(x) \subset P \ \text{ for some } \ n \in \mathbb{N}\big\}.$$

Claim. For each U open in X and $x \in U$, there exists $P \in \mathcal{P}_x$ such that $P \subset U$.

In fact, conversely assume that there exists U open in X and $x \in U$ such that $P \not\subset U$ for all $P \in \mathcal{P}_x$. Let

$$\{P \in \mathcal{P}_x : x \in P \subset U\} = \{P_m(x) : m \in \mathbb{N}\}.$$

Then $S_n(x) \not\subset P_m(x)$ for all $n, m \in \mathbb{N}$, so choose $x_{n,m} \in S_n(x) - P_m(x)$. For $n \geq m$, we denote $x_{n,m} = y_k$ with $k = m + n(n-1)/2$. Because $\{S_n(x)\}$ is a decreasing weak neighborhood base at x, the sequence $\{y_k : k \in \mathbb{N}\}$ converges to the point x in X. Thus, there exist $m, i \in \mathbb{N}$ such that

$$\{x\} \bigcup \{y_k : k \geq i\} \subset P_m(x) \subset U.$$

Take $j \geq i$ with $y_j = x_{n,m}$ for some $n \geq m$. Then $x_{n,m} \in P_m(x)$. This is a contradiction.

Then we have

(1) $\mathcal{P}_x$ is a network at x in X. Let U be an open subset of X and $x \in U$. Then there $P \in \mathcal{P}_x$ such that $P \subset U$ by the Claim.

(2) Let $P_1, P_2 \in \mathcal{P}_x$ and $P = P_1 \cap P_2$. Hence, there exist $n, m \in \mathbb{N}$ such that $S_m(x) \subset P_1$ and $S_n(x) \subset P_2$. If we put $k = \max\{m, n\}$, then $S_k(x) \subset P \in \mathcal{U}$. Thus, $P \in \mathcal{P}_x$ and $P \subset P_1 \cap P_2$.

(3) Let U be an open subset of X. By the Claim, there exists $P \in \mathcal{P}_x$ such that $P \subset U$. Conversely, if $U \subset X$ satisfies that for each $x \in U$, there exists $P \in \mathcal{P}_x$ with $P \subset U$, then for each $x \in U$, there exists $n \in \mathbb{N}$ such that $S_n(x) \subset U$. Because $\{S_n(x)\}$ is a weak neighborhood at x for all $x \in X$, U is open in X.

Therefore, $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$ is a weak base for X and $\mathcal{P} \subset \mathcal{U}$.

Now, for each $n \in \mathbb{N}$, put $\mathcal{P}_n = \mathcal{U}_n \cap \mathcal{P}$. Then, $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ and $\mathcal{P}_n \subset \mathcal{P}_{n+1}$ for all $n \in \mathbb{N}$. Since $\mathcal{U}$ is a σ-point-finite *cs*-network, $\mathcal{P}$ is a σ-point-finite weak base.

Next, for each $m, n \in \mathbb{N}$, put

$$\mathcal{Q}_{m,n}(x) = \left\{P \in \mathcal{P}_m \cap \mathcal{P}_x : S_m(x) \subset P \text{ and } d(P) < \frac{1}{n}\right\};$$

$$A_{m,n} = \{x \in X : \mathcal{Q}_{m,n}(x) = \emptyset\};$$

$$B_{m,n} = X - A_{m,n};$$

$$\mathcal{Q}_{m,n} = \bigcup\{\mathcal{Q}_{m,n}(x) : x \in B_{m,n}\};$$

$$\mathcal{F}_{m,n} = \mathcal{Q}_{m,n} \bigcup \{A_{m,n}\}.$$

Then, each $\mathcal{F}_{m,n}$ is point-finite. Furthermore, we have

(i) *Each $\mathcal{F}_{m,n}$ is a cs-cover for X.*

Let $x \in X$ and $S = \{x_i : i \in \mathbb{N}\}$ be a sequence converging to x in X, then

Case 1. If $x \in B_{m,n}$, then there is $P \in \mathcal{Q}_{m,n}(x)$ such that $S_m(x) \subset P$. Hence, S is eventually in $P \in \mathcal{F}_{m,n}$.

Case 2. If $x \notin B_{m,n}$ and $S \cap B_{m,n}$ is finite, then S is eventually in $A_{m,n} \in \mathcal{F}_{m,n}$.

Case 3. If $x \notin B_{m,n}$ and $S \cap B_{m,n}$ is infinite, then we can assume that

$$S \cap B_{m,n} = \{x_{i_k} : k \in \mathbb{N}\}.$$

Since X is Cauchy symmetric and S converges to x, there exists $n_0 \in \mathbb{N}$ such that

$$d(x_i, x_j) < \frac{1}{m} \text{ and } d(x, x_i) < \frac{1}{m} \text{ for every } i, j \geq n_0.$$

Now, we pick $k_0 \in \mathbb{N}$ such that $i_{k_0} \geq n_0$. Because

$$d(x_{i_{k_0}}, x) < \frac{1}{m} \text{ and } d(x_{i_{k_0}}, x_i) < \frac{1}{m} \text{ for every } i \geq n_0,$$

it implies that S is eventually in $S_m(x_{i_{k_0}})$. Furthermore, since $x_{i_{k_0}} \in B_{m,n}$, we get $S_m(x_{i_{k_0}}) \subset P$ for some $P \in Q_{m,n}(x_{i_{k_0}})$. Hence, $P \in \mathcal{F}_{m,n}$ and S is eventually in P.

Therefore, each $\mathcal{F}_{m,n}$ is a *cs*-cover for X.

(ii) $\{\mathtt{St}(x, \mathcal{F}_{m,n}) : m, n \in \mathbb{N}\}$ *is a network at* x.

Let $x \in U$ with U is open in X. Then, $S_n(x) \subset U$ for some $n \in \mathbb{N}$. Since X is Cauchy symmetric, it follows from Remark 1.5 that there exists $j \in \mathbb{N}$ such that $d\big(S_j(x)\big) < 1/n$. Furthermore, we have $P \subset S_j(x)$ for some $P \in \mathcal{P}_x$. Indeed, since $\mathcal{P}$ is point-countable, we can put

$$\mathcal{P}_x = \{P_n(x) : n \in \mathbb{N}\}.$$

On the other hand, because $\mathcal{P}$ is a weak base, we can choose a sequence $\{n_i : i \in \mathbb{N}\}$ such that $\{P_{n_i}(x) : i \in \mathbb{N}\}$ is a decreasing network at x. Then, there exists $i \in \mathbb{N}$ such that $P_{n_i}(x) \subset S_n(x)$. Thus, $P \in \mathcal{P}_k$ for some $k \in \mathbb{N}$.

Because P is a sequential neighborhood at x, there exists $i \in \mathbb{N}$ such that $S_i(x) \subset P$. If not, for each $n \in \mathbb{N}$, there exists $x_n \in S_n(x) - P$. Hence, $\{x_n\}$ converges to x. Then, there exists $m \in \mathbb{N}$ such that $x_n \in P$ for every $n \geq m$. This is a contradiction.

Denote $m = \max\{i, k\}$, then

$$S_m(x) \subset S_i(x) \subset P \in \mathcal{P}_k \subset \mathcal{P}_m.$$

Since $d(P) < 1/n$, it implies that $P \in \mathcal{F}_{m,n}$. Then, we have $\mathtt{St}(x, \mathcal{F}_{m,n}) \subset S_n(x)$. It follows that $\{\mathtt{St}(x, \mathcal{F}_{m,n}) : m, n \in \mathbb{N}\}$ is a network at x.

Finally, we write

$$\{\mathcal{F}_{m,n} : m, n \in \mathbb{N}\} = \{\mathcal{H}_n : n \in \mathbb{N}\},$$

and for each $n \in \mathbb{N}$, put

$$\mathcal{G}_n = \bigwedge\{\mathcal{H}_i : i \leq n\}.$$

Then, $\bigcup\{\mathcal{G}_n : n \in \mathbb{N}\}$ is a σ-point-finite strong network consisting of *cs*-covers of X.

By Theorem 12 in [3], X is a sequential space. It follows from Theorem 9 in [3] that X is a sequence-covering quotient compact image of a metric space. $\square$

Remark 2.2. By Theorem 2.1, we get a partial answer to the Question 1.1.

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