Novi Sad J. Math.
Vol. 47, No. 1, 2017, 89-113

NONLINEAR NEUTRAL INTEGRO-DIFFERENTIAL EQUATIONS, STABILITY BY FIXED POINT AND INVERSES OF DELAYS

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Abstract. A class of second order nonlinear neutral integro-differential equations with variable delays is investigated. We give new conditions ensuring that the zero solution is asymptotically stable by means of the fixed point theory. Our work extends and improves previous results in the literature. An example is given to illustrate our claim.

AMS Mathematics Subject Classification (2010): 34K20; 34K30; 34K40

Key words and phrases: fixed points; stability; neutral integro-differential equations; variable delays

1. Introduction

Time-delay systems constitute basic mathematical models of real phenomena such as nuclear reactors, chemical engineering systems, biological systems, and population dynamics models. Such systems are often sources of instability and degradation in control performance in many control problems. For more than 100 years, the Lyapunov direct method has been the ultimate key tool to study stability problems. The direct method was widely used to study the stability of solutions of ordinary differential equations and functional differential equations. Nevertheless, the pointwise nature of this method and the construction of the Lyapunov functionals have been, and still are, an arduous task (see [7]). Recently, many authors have realized that the fixed point theory can be used to study the stability of the solution (see [1]-[11], [14], [16]-[18]). Levin and Nohel [15] studied the following nonlinear systems of differential equations of Liénard form

$$\ddot{x} + h\left(t, x, \dot{x}\right)\dot{x} + f\left(x\right) = a(t). \tag{1.1}$$

They obtained, by constructing a proper Lyapunov function, conditions under which all solutions of 1.1 tend to zero as $t \to \infty$. In [10], Burton considered the following delay equation

$$\ddot{x} + f\left(t, x, \dot{x}\right)\dot{x} + b(t)g\left(x\left(t - L\right)\right) = 0, \tag{1.2}$$

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where L is a positive constant. By using the fixed point theory, he gave sufficient conditions for each solution $x(t)$ to satisfy $(x(t), \dot{x}(t)) \longrightarrow 0$ as $t \to \infty$. D. Pi (see [16, 18]) studied the asymptotic stability of the following two equations with delays

$$\ddot{x} + f(t, x, \dot{x})\dot{x} + b(t)g(x(t - r(t))) = 0, \tag{1.3}$$

$$\ddot{x} + f(t, x, \dot{x})\dot{x} + \sum_{j=1}^{N} \int_{t-\tau_j(t)}^{t} a_j(t, s)g_j(s, x(s))\, ds = 0. \tag{1.4}$$

Nevertheless, Pi results (see [17, 18]) rely on the introduction of an arbitrary and unknown continuous function h which is contested by the public of this domain because no one has had any real success at finding such a function. Many other interesting results on fixed points and stability properties can be found in the references ([1]-[8]). In this paper, we consider the equation

$$\begin{aligned} &\ddot{x} + f(t, x, \dot{x})\dot{x} + \sum_{j=1}^{N} \int_{t-\tau_j(t)}^{t} a_j(t, s)g_j(s, x(s))\, ds \\ &\quad + \sum_{j=1}^{N} b_j(t)x'(t - \tau_j(t)) = 0, \end{aligned} \tag{1.5}$$

for $t \geq 0$. Where, for $j = \overline{1, N}$, functions $\tau_j : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, $a_j(\cdot, \cdot) : \mathbb{R}^+ \times [-\tau_j(0), \infty) \longrightarrow \mathbb{R}$, $f : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}^+$, $b_j : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ and $g_j(\cdot, \cdot) : [-\tau_j(0), \infty) \times \mathbb{R} \longrightarrow \mathbb{R}$ are all continuous functions. We assume that, for $j = \overline{1, N}$, τ_j is twice differentiable and

$$\tau_j'(t) \neq 1 \text{ for all } t \geq 0, \tag{1.6}$$

$$t - \tau_j(t) \longrightarrow \infty \text{ as } t \longrightarrow \infty, j = \overline{1, N}, \tag{1.7}$$

and that

$$t - \tau_j(t) : \mathbb{R}^+ \longrightarrow [-\tau_j(0), \infty), \tag{1.8}$$

is strictly increasing so that $t - \tau_j(t)$ is one-to-one; so it has an inverse which we denote by $p_j(t)$ and which we call, for ease of the terminology, 'inverse of delay'. For each $t_0 \geq 0$, define $m_j(t_0) =: \inf\{s - \tau_j(s) : s \geq t_0\}$, $j = \overline{1, N}$ and let $m(t_0) = \min\{m_j(t_0), j = \overline{1, N}\}$. Let $\mathcal{C}(t_0) := \mathcal{C}([m(t_0), t_0], \mathbb{R})$ be the space of continuous functions endowed with function supremum norm $\|\cdot\|$, that is, for $\psi \in \mathcal{C}(t_0)$, $\|\psi\| := \sup\{|\psi(s)| : m(t_0) \leq s \leq t_0\}$. We will also use $\|\varphi\| := \sup\{|\varphi(s)| : s \in [m(t_0), \infty)\}$ to express the supremum of continuous bounded functions on $[m(t_0), \infty)$ later. It is well known (see [13]) that, for a given continuous function ψ and a number y_0, there exists a solution for equation (1.5) on an interval $[-m(t_0), T)$, and if the solution remains bounded, then $T = \infty$. We denote by $(x(t), y(t))$ the solution $(x(t, t_0, \psi), y(t, t_0, \psi))$.

Denote by $A(t) := f((t, x(t), y(t))$. We can rewrite equation (1.5) as

$$\left\{\begin{array}{l} \dot{x}(t) = y(t), \\ \dot{y}(t) = -A(t)\, y(t) - \sum\limits_{j=1}^{N} \int_{t-\tau_j(t)}^{t} a_j(t,s) g_j(s, x(s))\, ds \\ \qquad - \sum\limits_{j=1}^{N} \omega_j(t) \frac{d}{dt} x(t - \tau_j(t)), \end{array}\right. \tag{1.9}$$

with

$$\omega_j(t) = \frac{b_j(t)}{1 - \tau_j'(t)}. \tag{1.10}$$

Our purpose is to give a necessary and sufficient condition ensuring that the zero solution of the above equation is asymptotically stable. To our knowledge the considered equation has not yet been studied by any method. Further, being free of the famous unknown function $h(t)$ that has weakened previous particular results, we hope that this work is clean and interesting.

2. Preliminaries

Some asymptotic properties on integral equations are needed in this work. So, let f be a real or complex-valued function of the variable $t > 0$ and p be a real or a complex parameter such that $\mathrm{Re}(p) > 0$. We define the Laplace transform (see [19], [12]) of f as

$$F(p) = \mathfrak{L}(f(t))_{(p)} = \int_0^{\infty} e^{-pt} f(t)\, dt. \tag{2.1}$$

We also indicate the Laplace transform (2.1) of the power function t^{γ} is given by

$$\mathfrak{L}(t^{\gamma})_{(p)} = \int_0^{\infty} e^{-pt} t^{\gamma} dt = \frac{\Gamma(\gamma+1)}{p^{\gamma+1}},\ \gamma > -1,\ p > 0, \tag{2.2}$$

with Gamma function $\Gamma(z)$ is defined by the integral

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt = \mathfrak{L}\left(t^{z-1}\right)_{(1)}, \tag{2.3}$$

which converges in the right half of the complex plane $\mathrm{Re}(z) > 0$. Now, let $-\infty \le \alpha < \beta \le +\infty$, $\varphi : [\alpha, \beta] \to \mathbb{R}$ and define for $\lambda \in \mathbb{R}$ the integral

$$F(\lambda) = \int_{\alpha}^{\beta} e^{-\lambda \varphi(t)} f(t)\, dt. \tag{2.4}$$

We assume that there exists a constant $\lambda_0 > 0$ such that for every $\lambda \ge \lambda_0$ we have

$$\int_{\alpha}^{\beta} e^{-\lambda \varphi(t)} |f(t)|\, dt < \infty. \tag{2.5}$$

The following theorem is crucial to reach our goal.

Theorem 2.1. *Let $\varphi : [\alpha, \beta[\longrightarrow \mathbb{R}$ be a function such that φ is of class $\mathcal{C}^1$, $\varphi' > 0$ on $[\alpha, \beta[$. Assume that f is function continuous at α and $f(\alpha) \neq 0$. Then,*

$$F(\lambda) \sim \frac{f(\alpha)}{\varphi'(\alpha)} \frac{1}{\lambda} e^{-\lambda\varphi(\alpha)}, \quad (\lambda \longrightarrow +\infty). \tag{2.6}$$

Proof. **(a)** To begin with, $\varphi(t) = t$, $\alpha = 0$;

$$F(\lambda) = \int_0^\beta e^{-\lambda t} f(t)\, dt. \tag{2.7}$$

We check that $F(\lambda)$ satisfies the property (2.6). Indeed, since f is continuous at $\alpha = 0$, then for any given $\varepsilon > 0$, one can choose η sufficiently small, such that

$$|f(t) - f(0)| \leq \varepsilon, \text{ for } 0 \leq t \leq \eta. \tag{2.8}$$

Next, we decompose $F(\lambda)$ in the following manner

$$F(\lambda) = f(0) \int_0^\eta e^{-\lambda t} dt + \int_0^\eta e^{-\lambda t} (f(t) - f(0))\, dt + \int_\eta^\beta e^{-\lambda t} f(t)\, dt. \tag{2.9}$$

From (2.9) we can establish the following estimates

$$\int_0^\eta e^{-\lambda t} dt = \frac{1}{\lambda} \left(1 - e^{-\lambda\eta}\right), \tag{2.10}$$

$$\int_0^\eta e^{-\lambda t} (f(t) - f(0))\, dt \leq \varepsilon \int_0^\infty e^{-\lambda t} dt = \frac{\varepsilon}{\lambda}. \tag{2.11}$$

For $t \geq \eta$ we have $(\lambda - \lambda_0)(t - \eta) \geq 0$. Consequently,

$$\int_\eta^\beta e^{-\lambda t} f(t)\, dt \leq e^{-\eta(\lambda - \lambda_0)} \int_\eta^\beta e^{-\lambda_0 t} f(t)\, dt. \tag{2.12}$$

(b) Let us return to the general case. For this purpose, consider the function

$$g : [\alpha, \beta[\longrightarrow [0, \beta_0[, \quad t \longmapsto g(t) := \varphi(t) - \varphi(\alpha), \tag{2.13}$$

where $\beta_0 = \varphi(\beta) - \varphi(\alpha)$. We observe that g is bijective on $[\alpha, \beta[$. Denote the reciprocal function of g by

$$\psi : [0, \beta_0[\longrightarrow [\alpha, \beta[, \quad u \longmapsto \psi(u). \tag{2.14}$$

The change of variables $t = \psi(u)$ yields the integral formula

$$F(\lambda) = e^{-\lambda\varphi(\alpha)} \int_0^{\beta_0} e^{-\lambda u} f(\psi(u))\, \psi'(u)\, du. \tag{2.15}$$

We see that

$$\frac{d\psi(u)}{dt} = \psi'(\varphi(t) - \varphi(\alpha))\varphi'(t) = 1 \text{ and } \psi'(0) = \frac{1}{\varphi'(\alpha)}. \tag{2.16}$$

Define

$$\tilde{f}(u) := f(\psi(u))\psi'(u). \tag{2.17}$$

Clearly, the function $\tilde{f}$ is continuous at 0. Moreover,

$$\tilde{f}(0) = f(\psi(0))\psi'(0) = \frac{f(\alpha)}{\varphi'(\alpha)}. \tag{2.18}$$

Repeated application of **(a)** yields

$$\begin{aligned} F(\lambda) &= e^{-\lambda\varphi(\alpha)}\tilde{f}(0)\int_0^\eta e^{-\lambda u}du + e^{-\lambda\varphi(\alpha)}\int_0^\eta e^{-\lambda u}\left(\tilde{f}(u) - \tilde{f}(0)\right)du \\ &\quad + e^{-\lambda\varphi(\alpha)}\int_\eta^\beta e^{-\lambda u}\tilde{f}(u)\,du \\ &\le e^{-\lambda\varphi(\alpha)}\frac{f(\alpha)}{\varphi'(\alpha)}\frac{1}{\lambda}\left(1 - e^{-\lambda\eta}\right) + \frac{\varepsilon}{\lambda}e^{-\lambda\varphi(\alpha)} \\ &\quad + e^{-\lambda\varphi(\alpha)}e^{-\eta(\lambda-\lambda_0)}\int_\eta^{\beta_0} e^{-\lambda_0 u}\tilde{f}(u)\,du. \end{aligned} \tag{2.19}$$

□

Stability definitions, fixed point technique and more details on delay differential equations can be found in ([13, 7]).

Definition 2.2. The zero solution of (1.9) is stable if for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that $[\psi \in \mathcal{C}(t_0), y_0 \in \mathbb{R}, \|\psi\| + |y_0| < \delta]$ implies that $|x(t, t_0, \psi)| + |y(t, t_0, \psi)| < \varepsilon$ for $t \geq t_0$.

Definition 2.3. The zero solution of (1.9) is asymptotically stable if it is stable and there is a $\delta_1 = \delta_1(t_0) > 0$ such that $[\psi \in \mathcal{C}(t_0), y_0 \in \mathbb{R}, \|\psi\| + |y_0| < \delta_1]$ implies that $|x(t, \psi, y_0)| + |y(t, \psi, y_0)| \longrightarrow 0$ as $t \longrightarrow \infty$.

3. Main Results

In this section, we will prove Theorem 3.5 and Theorem 3.6 on stability and asymptotic stability, respectively, for equation (1.5) by using the fixed point theory. But our equation is nonlinear and has no non trivial edo term so the inversion of that equation needs some preparations to be domesticated. Lemma 3.1 and Lemma 3.2 are the subject of these aesthetic works. We use the variation of parameters and then transform the given equation and in Lemma 3.3 we invert it and give the expression of the solutions of (1.5). The proof of these theorems depends on Theorem 2.1.

Lemma 3.1. *Let $p_j : [-\tau_j(0), \infty) \longrightarrow \mathbb{R}^+$ denote the inverse of $t - \tau_j(t)$. Then the second equation of (1.9) is equivalent to*

$$\begin{aligned}\dot{x}(t) &= -\hat{D}(t)\,x(t) + B(t)\\&\quad - \sum_{j=1}^{N}\int_{t_0}^{t} e^{-\int_s^t A(u)du}\int_{s-\tau_j(s)}^{s} a_j(s,v)g_j(v, x(v))\,dvds\\&\quad + \sum_{j=1}^{N}\frac{d}{dt}\int_{t-\tau(t)}^{t}\hat{D}_j(s)\,x(s)\,ds\\&\quad + \sum_{j=1}^{N}\int_{t_0}^{t} e^{-\int_s^t A(u)du} r_j(s) x(s-\tau_j(s))\,ds. \end{aligned}\tag{3.1}$$

Here B, $\hat{D}_j$, r_j, $\hat{D}$ are defined respectively by

$$B(t) := \left(\dot{x}(t_0) + \sum_{j=1}^{N}\omega_j(t_0)x(t_0 - \tau_j(t_0))\right)e^{-\int_{t_0}^t A(w)dw},\tag{3.2}$$

$$\hat{D}_j(t) := D_j(p_j(t)) \;\; \text{with } D_j(t) := \frac{b_j(t)}{\left(1 - \tau_j'(t)\right)^2},\tag{3.3}$$

$$r_j(t) = \left[\left(b_j(t)A(t) + b_j'(t)\right)\left(1 - \tau_j'(t)\right) + b_j(t)\tau_j''(t)\right] / \left(1 - \tau_j'(t)\right)^2,\tag{3.4}$$

$$\hat{D}(t) := \sum_{j=1}^{N}\hat{D}_j(t),\tag{3.5}$$

for $t \in [m(t_0), \infty)$ and $m(t_0) := \inf\{m_j(t_0), 1 \le j \le N\}$.

Proof. Indeed, applying the variation of parameters formula by multiplying both sides of the second equation of (1.9) by the factor $e^{\int_{t_0}^t A(u)du}$ and integrating from t_0 to any $t \in [t_0, T]$, we obtain

$$\begin{aligned}y(t) &= y(t_0)e^{-\int_{t_0}^t A(w)dw}\\&\quad - \sum_{j=1}^{N}\int_{t_0}^{t} e^{-\int_s^t A(w)dw}\int_{s-\tau_j(s)}^{s} a_j(s,v)g_j(v, x(v))\,dvds\\&\quad - \sum_{j=1}^{N}\int_{t_0}^{t} e^{-\int_s^t A(v)dv}\omega_j(s)\frac{d}{ds}x(s-\tau_j(s))\,ds.\end{aligned}\tag{3.6}$$

Substituting $\dot{x}(\cdot)$ into (3.6) and performing an integration by parts to the last right hand term we obtain

$$\begin{aligned}\dot{x}(t) &= \dot{x}(t_0)e^{-\int_{t_0}^t A(v)dv} \\ &\quad - \sum_{j=1}^{N}\int_{t_0}^{t} e^{-\int_s^t A(v)dv}\int_{s-\tau_j(s)}^{s} a_j(s,v)g_j\left(v,x\left(v\right)\right)dvds \\ &\quad - \sum_{j=1}^{N}\omega_j(t)x\left(t-\tau_j\left(t\right)\right) + e^{-\int_{t_0}^t A(v)dv}\sum_{j=1}^{N}\omega_j(t_0)x\left(t_0-\tau_j\left(t_0\right)\right) \\ &\quad + \sum_{j=1}^{N}\int_{t_0}^{t} e^{-\int_s^t A(v)dv} r_j(s)x\left(s-\tau_j\left(s\right)\right)ds. \end{aligned} \tag{3.7}$$

Having in mind the fact that $p_j\left(t-\tau_j\left(t\right)\right)=t$, then, by rewriting the third term on the right had side of (3.7), we reduce (1.5) (retaining its initial condition) to a first-order integro-differential equation as follows

$$\begin{aligned}\dot{x}(t) &= -\hat{D}\left(t\right)x\left(t\right) + \left(\dot{x}(t_0)+\sum_{j=1}^{N}\omega_j(t_0)x\left(t_0-\tau_j\left(t_0\right)\right)\right)e^{-\int_{t_0}^t A(v)dv} \\ &\quad - \sum_{j=1}^{N}\int_{t_0}^{t} e^{-\int_s^t A(v)dv}\int_{s-\tau_j(s)}^{s} a_j(s,v)g_j\left(v,x\left(v\right)\right)dvds \\ &\quad + \sum_{j=1}^{N}\frac{d}{dt}\int_{t-\tau_j(t)}^{t}\hat{D}_j\left(s\right)x\left(s\right)ds + \sum_{j=1}^{N}\int_{t_0}^{t} e^{-\int_s^t A(v)dv} r_j(s)x\left(s-\tau_j\left(s\right)\right)ds.\end{aligned}$$

Making use of (3.2), we see that this last equation is exactly (3.1). □

Lemma 3.2. *The equation*

$$\sigma(t) = -\sum_{j=1}^{N}\int_{t-\tau_j(t)}^{t} a_j(t,s)g_j\left(s,x\left(s\right)\right)ds, \tag{3.8}$$

is equivalent to

$$\begin{aligned}\sigma\left(t\right) &= \sum_{j=1}^{N}\frac{d}{dt}\int_{t-\tau_j(t)}^{t} C_j(t,s)g_j\left(s,x(s)\right)ds \\ &\quad + \sum_{j=1}^{N} C_j(t,t-\tau_j(t))(1-\tau_j'(t))g_j\left(t-\tau_j(t),x(t-\tau_j(t))\right),\end{aligned} \tag{3.9}$$

where

(3.10)

$$C_j(t,s) = \int_t^s a_j(u,s)du \text{ and } C_j(t,t-\tau_j(t)) = \int_t^{t-\tau_j(t)} a_j(u,t-\tau_j(t))du.$$

Proof. Differentiating the integral term in (3.8), we have

$$\begin{aligned}
&\frac{d}{dt}\int_{t-\tau_j(t)}^{t} C_j(t,s)g_j\left(s,x(s)\right)ds \\
&= \int_{t-\tau_j(t)}^{t} \frac{\partial}{\partial t}C_j(t,s)g_j\left(s,x(s)\right)ds + C_j(t,t)g_j\left(t,x(t)\right) \\
&- C_j(t,t-\tau_j(t))(1-\tau_j'(t))g_j\left(t-\tau_j(t),x(t-\tau_j(t))\right).
\end{aligned}$$

It follows that if $C_j(t,t)=0$, $\dfrac{\partial C_j(t,s)}{\partial t} = -a_j(t,s)$, then (3.8) is equivalent to (3.9). The calculation shows that the previous conditions on C_j yields (3.11)

$$C_j(t,s) = \int_t^s a_j(u,s)du \text{ and } C_j(t,t-\tau_j(t)) = \int_t^{t-\tau_j(t)} a_j(u,t-\tau_j(t))du.$$

□

Lemma 3.3. *Suppose that condition (1.6) is fulfilled and $\tau_j(\cdot)$ is twice differentiable for all $j=1,...,N$. If $x(t)$ is a solution of equation (1.9) and hence solution of (1.5) on an interval $[t_0,T)$ satisfying the initial condition $x(t)=\psi(t)$ on $[m(t_0),t_0]$ and $y(t_0)=\dot{x}(t_0)$, then $x(t)$ is the solution of the following integral equation*

$$\begin{aligned}
x(t) &= \left[\psi(t_0) - \sum_{j=1}^{N}\int_{t_0-\tau_j(t_0)}^{t_0}\hat{D}_j(s)\psi(s)ds\right]e^{-\int_{t_0}^{t}\hat{D}(v)dv} \\
&+ \left[\dot{x}(t_0) + \sum_{j=1}^{N}\left(\frac{b_j(t_0)}{1-\tau_j'(t_0)}\psi\left(t_0-\tau_j(t_0)\right)\right.\right. \\
&\left.\left.- \int_{t_0-\tau_j(t_0)}^{t_0} C_j(t_0,v)g_j\left(v,\psi(v)\right)dv\right)\right] \\
&\times \int_{t_0}^{t} e^{-\int_u^t \hat{D}(v)dv}e^{-\int_{t_0}^{u}A(v)dv}du + \sum_{j=1}^{N}\int_{t-\tau_j(t)}^{t}\hat{D}_j(u)x(u)du \\
&+ \sum_{j=1}^{N}\int_{t_0}^{t} e^{-\int_u^t \hat{D}(v)dv}\int_{u-\tau_j(u)}^{u} C_j(u,v)g_j\left(v,x(v)\right)dvdu \\
&- \sum_{j=1}^{N}\int_{t_0}^{t} e^{-\int_u^t \hat{D}(v)dv}\int_{t_0}^{u} A(s)e^{-\int_s^u A(v)dv}\int_{s-\tau_j(s)}^{s} C_j(s,v)g_j\left(v,x(v)\right)dvdsdu \\
&+ \sum_{j=1}^{N}\int_{t_0}^{t} e^{-\int_u^t \hat{D}(v)dv}\int_{t_0}^{u} e^{-\int_s^u A(v)dv}C_j(s,s-\tau_j(s))(1-\tau_j'(s)) \\
&\times g_j\left(s-\tau_j(s),x(s-\tau_j(s))\right)dsdu
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{j=1}^{N} \int_{t_0}^{t} \hat{D}(u)\, e^{-\int_u^t \hat{D}(v)dv} \int_{u-\tau_j(u)}^{u} \hat{D}_j(s)\, x(s)\, dsdu \\
&+ \sum_{j=1}^{N} \int_{t_0}^{t} e^{-\int_u^t \hat{D}(v)dv} \int_{t_0}^{u} e^{-\int_s^u A(v)dv} r_j(s) x(s-\tau_j(s))\, dsdu, \qquad (3.12)
\end{aligned}
$$

on $[t_0, T)$*, where* $r_j(\cdot)$ *and* $\hat{D}(\cdot)$ *are respectively given by (3.4),(3.5) in Lemma 3.1. Conversely, if a continuous function* $x(\cdot)$ *is equal to* $\psi(\cdot)$ *for* $t \in [m(t_0), t_0]$ *and is the solution of above integral equation on an interval* $[t_0, T_1]$*, then* $x(\cdot)$ *is a solution of (1.9) on* $[t_0, T_1]$*.*

Proof. By Lemma (3.2), equation (3.1) can be written as

$$
\begin{aligned}
\dot{x}(t) = {}& -\hat{D}(t)\, x(t) + \left(\dot{x}(t_0) + \sum_{j=1}^{N} \omega_j(t_0) x(t_0 - \tau_j(t_0)) \right) e^{-\int_{t_0}^t A(v)dv} \\
&+ \sum_{j=1}^{N} \int_{t_0}^{t} e^{-\int_s^t A(v)dv} \frac{d}{ds} \int_{s-\tau_j(s)}^{s} C_j(s,v) g_j(v, x(v))\, dvds \\
&+ \sum_{j=1}^{N} \int_{t_0}^{t} e^{-\int_s^t A(v)dv} C_j(s, s-\tau_j(s))(1-\tau_j'(s)) g_j(s-\tau_j(s), x(s-\tau_j(s)))\, ds \\
&+ \sum_{j=1}^{N} \frac{d}{dt} \int_{t-\tau_j(t)}^{t} \hat{D}_j(s)\, x(s)\, ds + \sum_{j=1}^{N} \int_{t_0}^{t} e^{-\int_s^t A(v)dv} r_j(s) x(s-\tau_j(s))\, ds. \qquad (3.13)
\end{aligned}
$$

Multiplying both sides of the above equation by $e^{\int_{t_0}^t \hat{D}(v)dv}$ and integrating with respect to u from t_0 to t, we obtain

$$
\begin{aligned}
x(t) = {}& x(t_0)\, e^{-\int_{t_0}^t \hat{D}(v)dv} \\
&+ \left(\dot{x}(t_0) + \sum_{j=1}^{N} \omega_j(t_0) x(t_0 - \tau_j(t_0)) \right) \int_{t_0}^{t} e^{-\int_u^t \hat{D}(v)dv} e^{-\int_{t_0}^u A(v)dv} du \\
&+ \sum_{j=1}^{N} \int_{t_0}^{t} e^{-\int_u^t \hat{D}(v)dv} \int_{t_0}^{u} e^{-\int_s^u A(v)dv} \frac{d}{ds} \int_{s-\tau_j(s)}^{s} C_j(s,v) g_j(v, x(v))\, dvdsdu \\
&+ \sum_{j=1}^{N} \int_{t_0}^{t} e^{-\int_u^t \hat{D}(v)dv} \int_{t_0}^{u} e^{-\int_s^u A(v)dv} C_j(s, s-\tau_j(s))(1-\tau_j'(s)) \\
&\times g_j(s-\tau_j(s), x(s-\tau_j(s)))\, dsdu
\end{aligned}
$$

$$+\sum_{j=1}^{N}\int_{t_0}^{t} e^{-\int_u^t \hat{D}(v)dv}\frac{d}{du}\int_{u-\tau_j(u)}^{u}\hat{D}_j(s)\,x(s)\,dsdu$$
$$+\sum_{j=1}^{N}\int_{t_0}^{t} e^{-\int_u^t \hat{D}(v)dv}\int_{t_0}^{u} e^{-\int_s^u A(v)dv} r_j(s)x(s-\tau_j(s))\,dsdu.$$

Performing an integration by parts and using definitions (1.10) and (3.2), we obtain

$$\begin{aligned}
x(t) &= \left[\psi(t_0) - \sum_{j=1}^{N}\int_{t_0-\tau_j(t_0)}^{t_0}\hat{D}_j(s)\,\psi(s)\,ds\right] e^{-\int_{t_0}^{t}\hat{D}(v)dv} \\
&+ \left[\dot{x}(t_0) + \sum_{j=1}^{N}\left(\frac{b_j(t_0)}{1-\tau_j'(t_0)}\psi(t_0-\tau_j(t_0))\right.\right. \\
&\left.\left. - \int_{t_0-\tau_j(t_0)}^{t_0} C_j(t_0,v)g_j(v,\psi(v))\,dv\right)\right] \\
&\times \int_{t_0}^{t} e^{-\int_u^t \hat{D}(v)dv} e^{-\int_{t_0}^{u} A(v)dv} du \\
&+ \sum_{j=1}^{N}\int_{t-\tau_j(t)}^{t}\hat{D}_j(u)\,x(u)\,du \\
&+ \sum_{j=1}^{N}\int_{t_0}^{t} e^{-\int_u^t \hat{D}(v)dv}\int_{u-\tau_j(u)}^{u} C_j(u,v)g_j(v,x(v))\,dvdu \\
&- \sum_{j=1}^{N}\int_{t_0}^{t} e^{-\int_u^t \hat{D}(v)dv}\int_{t_0}^{u} A(s)\,e^{-\int_s^u A(v)dv}\int_{s-\tau_j(s)}^{s} C_j(s,v)g_j(v,x(v))\,dvdsdu \\
&+ \sum_{j=1}^{N}\int_{t_0}^{t} e^{-\int_u^t \hat{D}(v)dv}\int_{t_0}^{u} e^{-\int_s^u A(v)dv} C_j(s,s-\tau_j(s))(1-\tau_j'(s)) \\
&\times g_j(s-\tau_j(s),x(s-\tau_j(s)))\,dsdu \\
&- \sum_{j=1}^{N}\int_{t_0}^{t}\hat{D}(u)\,e^{-\int_u^t \hat{D}(v)dv}\int_{u-\tau_j(u)}^{u}\hat{D}_j(s)\,x(s)\,dsdu \\
&+ \sum_{j=1}^{N}\int_{t_0}^{t} e^{-\int_u^t \hat{D}(v)dv}\int_{t_0}^{u} e^{-\int_s^u A(v)dv} r_j(s)x(s-\tau_j(s))\,dsdu,
\end{aligned}$$

where $r_j(t)$ is defined in (3.4). This leads exactly to (3.12).

Conversely, suppose that a continuous function $x(\cdot)$ is equal to $\psi(\cdot)$ on $[m(t_0), t_0]$ and satisfies (3.12) on an interval $[t_0, T_1)$. Then it is twice differentiable on $[t_0, T_1)$. Differentiating (3.12) with the aid of Leibniz's rule, we obtain (1.5). □

Next, we will define a mapping directly from (3.12). Remember that, by Lemma 3.3 a fixed point of that map will be a solution of equation (1.5). To obtain stability of the zero solution of (1.5), we need the mapping defined by (3.12) to map bounded functions into bounded functions. For that, we let $(\mathcal{C}, \|\cdot\|)$ to be the Banach space of real-valued bounded continuous functions on $[m(t_0), \infty)$ with the supremum norm $\|\cdot\|$, that is for $\varphi \in \mathcal{C}$

$$\|\varphi\| := \sup\{|\varphi(t)| \ ; t \in [m(t_0), \infty)\}.$$

Our investigations will be carried out on the complete metric space $(\mathcal{C}, \rho)$, where ρ is the uniform metric. That is, for $\varphi, \phi \in \mathcal{C}$ we set $\rho(\varphi, \phi) = \|\varphi - \phi\|$.

Let $\psi \in \mathcal{C}([m(t_0), t_0], \mathbb{R})$ be fixed and define

$$S_\psi := \{\varphi : [m(t_0), \infty) \to \mathbb{R} \mid \varphi \in \mathcal{C},\ \varphi(t) = \psi(t) \text{ for } t \in [m(t_0), t_0]\}.$$

Being closed in $\mathcal{C}$, (S_ψ, ρ) is itself complete. There is no confusion if we use the norm $\|\cdot\|$ on $[m(t_0), t_0]$ or on $[m(t_0), \infty)$.

Below we want to force the mapping suggested by (3.12) and explicitly defined in the next lemma to map S_ψ into itself. For that reason we assume that the followings conditions hold.

i.

$$\liminf_{t \longrightarrow \infty} \int_{t_0}^{t} \hat{D}(s)ds > -\infty. \tag{3.14}$$

ii. There exists some functions $R_j(\cdot) \in \mathcal{C}(\mathbb{R}, \mathbb{R}^+)$ such that, for $x_1, x_2 \in \mathbb{R}$,

$$|g_j(t, x_1) - g_j(t, x_2)| \leq R_j(t)|x_1 - x_2|,\ j = 1, ..., N \text{ for all } t \in \mathbb{R}, \tag{3.15}$$

$$g_j(t, 0) = 0,\ j = 1, ..., N \text{ for } t \in \mathbb{R}^+. \tag{3.16}$$

iii. For $t \geq t_0$, there is a constant $\alpha > 0$ satisfying

$$\begin{aligned}
&\sum_{j=1}^{N} \int_{t-\tau_j(t)}^{t} \hat{D}_j(u)\,du + \sum_{j=1}^{N} \int_{t_0}^{t} \hat{D}(u)\, e^{-\int_u^t \hat{D}(v)dv} \int_{u-\tau_j(u)}^{u} \hat{D}_j(s)\, x(s)\, dsdu \\
&+ \sum_{j=1}^{N} \int_{t_0}^{t} e^{-\int_u^t \hat{D}(v)dv} \int_{u-\tau_j(u)}^{u} |C_j(u, v)|\, R_j(v)\, dvdu \\
&+ \sum_{j=1}^{N} \int_{t_0}^{t} e^{-\int_u^t \hat{D}(v)dv} \int_{t_0}^{u} A(s)\, e^{-\int_s^u A(v)dv} \int_{s-\tau_j(s)}^{s} |C_j(s, v)|\, R_j(v)\, dvdsdu \\
&+ \sum_{j=1}^{N} \int_{t_0}^{t} e^{-\int_u^t \hat{D}(v)dv} \int_{t_0}^{u} e^{-\int_s^u A(v)dv} \left|C_j(s, s - \tau_j(s))(1 - \tau_j'(s))\right| \\
&\times R_j(s - \tau_j(s))dsdu \\
&+ \sum_{j=1}^{N} \int_{t_0}^{t} e^{-\int_u^t \hat{D}(v)dv} \int_{t_0}^{u} e^{-\int_s^u A(v)dv} |r_j(s)|\, dsdu
\end{aligned}$$

(3.17)

$$\leq \alpha.$$

iv. There exist constants $a_0 > 0$, $\gamma > 0$, $Q_0 > 0$ and a continuous function $A_1 \in \mathcal{C}(\mathbb{R}^+, \mathbb{R}^+)$ such that, for $t \geq t_0$

$$f(t, x, y) \geq A_1(t) \geq 0 \text{ for all } x, y \in \mathbb{R}, \tag{3.18}$$

and for each $t \geq u \geq Q_0$ we have

$$\int_u^t \hat{D}(v)\, dv + \int_{t_0}^u A_1(v)\, dv \geq a_0 u^\gamma + b,\ b \in \mathbb{R}. \tag{3.19}$$

v. There exists a constant $\beta > 0$ that satisfies the following inequality for $t \geq t_0$

$$\begin{aligned} &\sum_{j=1}^N \frac{|b_j(t)|}{|1 - \tau_j'(t)|} \\ &+ \sum_{j=1}^N \int_{t_0}^t e^{-\int_s^t A(u)du} \left(\int_{s-\tau_j(s)}^s |a_j(s,v)|\, R_j(v)\, dv + |r_j(s)| \right) ds \ \leq \ \beta. \end{aligned} \tag{3.20}$$

Lemma 3.4. *Define the mapping P on S_ψ as follows, for $\varphi \in S_\psi$,*

$$(P\varphi)(t) = \psi(t) \text{ if } t \in [m(t_0), t_0],$$

while for $t > t_0$

$$\begin{aligned} P\varphi(t) = &\left[\psi(t_0) - \sum_{j=1}^N \int_{t_0-\tau_j(t_0)}^{t_0} \hat{D}_j(s)\, \psi(s)\, ds \right] e^{-\int_{t_0}^t \hat{D}(v)dv} \\ &+ \left[\dot{x}(t_0) + \sum_{j=1}^N \left(\frac{b_j(t_0)}{1 - \tau_j'(t_0)} \psi(t_0 - \tau_j(t_0)) \right. \right. \\ &\left. \left. - \int_{t_0-\tau_j(t_0)}^{t_0} C_j(t_0, v) g_j(v, \psi(v))\, dv \right) \right] \\ &\times \int_{t_0}^t e^{-\int_u^t \hat{D}(v)dv} e^{-\int_{t_0}^u A(v)dv} du \\ &+ \sum_{j=1}^N \int_{t-\tau_j(t)}^t \hat{D}_j(u)\, \varphi(u)\, du \\ &- \sum_{j=1}^N \int_{t_0}^t \hat{D}(u)\, e^{-\int_u^t \hat{D}(v)dv} \int_{u-\tau_j(u)}^u \hat{D}_j(s)\, \varphi(s)\, ds du \\ &+ \sum_{j=1}^N \int_{t_0}^t e^{-\int_u^t \hat{D}(v)dv} \int_{u-\tau_j(u)}^u C_j(u, v) g_j(v, \varphi(v))\, dv du \end{aligned}$$

$$
\begin{aligned}
&-\sum_{j=1}^{N}\int_{t_0}^{t} e^{-\int_u^t \hat{D}(v)dv}\int_{t_0}^{u} A\left(s\right)e^{-\int_s^u A(v)dv} \\
&\times\int_{s-\tau_j(s)}^{s} C_j(s,v)g_j\left(v,\varphi\left(v\right)\right)dvdsdu \\
&+\sum_{j=1}^{N}\int_{t_0}^{t} e^{-\int_u^t \hat{D}(v)dv}\int_{t_0}^{u} e^{-\int_s^u A(v)dv}C_j(s,s-\tau_j(s))(1-\tau_j'(s)) \\
&\times g_j\left(s-\tau_j(s),\varphi(s-\tau_j(s))\right)dsdu \\
&+\sum_{j=1}^{N}\int_{t_0}^{t} e^{-\int_u^t \hat{D}(v)dv}\int_{t_0}^{u} e^{-\int_s^u A(v)dv}r_j(s)\varphi\left(s-\tau_j\left(s\right)\right)dsdu.
\end{aligned}
\tag{3.21}
$$

where $r_j\left(\cdot\right)$ is the expression (3.4). Suppose that the conditions (3.14), (3.15), (3.16), (3.17), (3.18) and (3.19) hold true. Then $P : S_\psi \to S_\psi$.

Proof. First, due to condition (3.14) one can define

$$M=\sup_{t\geq t_0}\left\{e^{-\int_{t_0}^t \hat{D}(v)dv}\right\}. \tag{3.22}$$

Obviously, if φ is continuous then $P\varphi$ and agrees with ψ on $[m(t_0),t_0]$ due to the definition of P. For $t>t_0$, note that from (3.14), (3.17), (3.15) and (3.16) it follows

$$
\begin{aligned}
\left|P\varphi\left(t\right)\right| &= \left\|\psi\right\|\left[1+\sum_{j=1}^{N}\int_{t_0-\tau_j(t_0)}^{t_0}\left|\hat{D}_j\left(v\right)\right|dv\right]M \\
&+\left[\left|\dot{x}\left(t_0\right)\right|+\left\|\psi\right\|\sum_{j=1}^{N}\left(\left|\frac{b_j(t_0)}{1-\tau_j'\left(t_0\right)}\right|+\int_{t_0-\tau_j(t_0)}^{t_0}\left|C_j(t_0,v)\right|R_j\left(t\right)dv\right)\right] \\
&\times\int_{t_0}^{t} e^{-\int_u^t \hat{D}(v)dv}e^{-\int_{t_0}^u A(v)dv}du+\alpha\left\|\varphi\right\|.
\end{aligned}
$$

To prove that $P : S_\psi \to S_\psi$ it is necessary to show that the term

$$\int_{t_0}^{t} e^{-\int_u^t \hat{D}(v)dv}e^{-\int_{t_0}^u A(v)dv}du,$$

is bounded. To do that, remember that 3.18 implies that $A(t)\geq A_1(t)\geq 0$ for $t\geq t_0$, so

$$\int_{t_0}^{t} e^{-\int_u^t \hat{D}(v)dv}e^{-\int_{t_0}^u A(v)dv}du\leq\int_{t_0}^{t} e^{-\int_u^t \hat{D}(v)dv}e^{-\int_{t_0}^u A_1(v)dv}du.$$

We decompose the last integral term in the following manner

$$
\begin{aligned}
\int_{t_0}^{t} e^{-\int_u^t \hat{D}(v)dv}e^{-\int_{t_0}^u A_1(v)dv}du &= \int_{t_0}^{J} e^{-\int_u^t \hat{D}(v)dv}e^{-\int_{t_0}^u A_1(v)dv}du \\
&+\int_{J}^{t} e^{-\int_u^t \hat{D}(v)dv}e^{-\int_{t_0}^u A_1(v)dv}du,
\end{aligned}
\tag{3.23}
$$

for some $J \geq Q_0$. The first term on the right hand side of (3.23) is obviously bounded. For the second term on the right hand side of (3.23), we use (3.19) to obtain

$$\int_J^t e^{-\int_u^t \hat{D}(v)dv} e^{-\int_{t_0}^u A_1(v)dv} du \leq e^{-b} \int_J^t e^{-a_0 u^\gamma} du. \tag{3.24}$$

Now, we define

$$F(J) := \int_J^\infty e^{-a_0 u^\gamma} du. \tag{3.25}$$

Performing the change of variables $u = \theta^{\frac{1}{\gamma}}$, we obtain

$$F(J) = \frac{1}{\gamma} \int_{J^\gamma}^\infty e^{-a_0\theta} \theta^{\frac{1}{\gamma}-1} d\theta \leq \frac{1}{\gamma} \int_0^\infty e^{-a_0\theta} \theta^{\frac{1}{\gamma}-1} d\theta = \frac{\Gamma(1/\gamma)}{\gamma a_0^{1/\gamma}}. \tag{3.26}$$

Then $F(J)$ is bounded for $\gamma > 0$. Consequently, $|P\varphi(t)| < +\infty$ and thus $P\varphi \in S_\psi$. □

Seizing upon Lemma (3.3) and Lemma (3.4) we built an existence and uniqueness result. Under the conditions of the next theorem, we prove that for a given continuous function $\psi : [m(t_0), t_0] \longrightarrow \mathbb{R}$ there exists a unique continuous function x which is solution of (1.5) on $[m(t_0), \infty)$ and coincides with ψ on $[m(t_0), t_0]$. We also prove that the zero solution of (1.5) have the property of Definition 2.2.

Theorem 3.5. *Suppose the condition (3.20) and all hypotheses of Lemma (3.4) hold with $\alpha \in (0,1)$ in (3.17). Then, for each initial continuous function $\psi : [m(t_0), t_0] \longrightarrow \mathbb{R}$, there is a unique continuous function with $x(t) = \psi(t)$ on $[m(t_0), t_0]$ that satisfies (1.5) on $[t_0, \infty)$. Moreover, $x(\cdot)$ is bounded on $[m(t_0), \infty)$. Furthermore, the zero solution of (1.5) is stable at $t = t_0$.*

Proof. Consider S_ψ the space defined by the initial continuous function $\psi : [m(t_0), t_0] \to \mathbb{R}$. By Lemma 3.4 we know that $P : S_\psi \to S_\psi$. In fact, P is a contraction with constant $\alpha < 1$, too. To see this, let φ, $\phi \in S_\psi$. Making use of condition (3.17) we obtain

$$\begin{aligned}
&\|P\varphi - P\phi\| \\
&\leq \Bigg[\sum_{j=1}^N \int_{t-\tau_j(t)}^t \hat{D}_j(u)\, du + \sum_{j=1}^N \int_{t_0}^t e^{-\int_u^t \hat{D}(v)dv} \int_{u-\tau_j(u)}^u |C_j(u,v)|\, R_j(v)\, dvdu \\
&+ \sum_{j=1}^N \int_{t_0}^t e^{-\int_u^t \hat{D}(v)dv} \int_{t_0}^u A(s)\, e^{-\int_s^u A(v)dv} \int_{s-\tau_j(s)}^s |C_j(s,v)|\, R_j(v)\, dvdsdu
\end{aligned}$$

$$+\sum_{j=1}^{N}\int_{t_0}^{t}e^{-\int_u^t \hat{D}(v)dv}\int_{t_0}^{u}e^{-\int_s^u A(v)dv}\left|C_j(s,s-\tau_j(s))(1-\tau_j'(s))\right|$$
$$\times R_j\left(s-\tau_j(s)\right)dsdu$$
$$+\sum_{j=1}^{N}\int_{t_0}^{t}\hat{D}\left(u\right)e^{-\int_u^t \hat{D}(v)dv}\int_{u-\tau_j(u)}^{u}\hat{D}_j\left(s\right)x\left(s\right)dsdu$$
$$\left.+\sum_{j=1}^{N}\int_{t_0}^{t}e^{-\int_u^t \hat{D}(v)dv}\int_{t_0}^{u}e^{-\int_s^u A(v)dv}\left|r_j(s)\right|dsdu\right]\|\varphi-\phi\|$$
$$\leq\alpha\left\|\varphi-\phi\right\|,$$

for $t>t_0$. Trivially, this inequality also holds on $[m(t_0),t_0]$. Therefore, P is a contraction mapping on the complete metric space (S_ψ,ρ) since we have supposed $\alpha<1$. By the contraction mapping principle, P possesses a unique fixed point x in S_ψ which is a bounded continuous function. Due to Lemma 3.3, this is a solution of (1.9) and hence a solution of (1.5) on $[m(t_0),\infty)$. It follows that x is the only bounded function satisfying (1.5) on $[m(t_0),\infty)$ and the initial function. It remains to show that the zero solution of (1.5) is stable. Toward this, let first

$$L:=\sup_{t\geq t_0}\int_{t_0}^{t}e^{-\int_u^t \hat{D}(v)dv}e^{-\int_{t_0}^u A_1(v)dv}du. \tag{3.27}$$

Let $\epsilon>0$ be given. Choose $|\dot{x}(t_0)|$ and $\psi:[m\left(t_0\right),t_0]\longrightarrow\mathbb{R}$ satisfying $\|\psi\|<\delta$ ($\delta\leq\varepsilon$), with δ such that

$$\delta\left[1+\sum_{j=1}^{N}\int_{t_0-\tau_j(t_0)}^{t_0}\left|\hat{D}_j\left(v\right)\right|dv\right]M$$
$$+\left[\left|\dot{x}\left(t_0\right)\right|+\delta\sum_{j=1}^{N}\left(\left|\frac{b_j(t_0)}{1-\tau_j'\left(t_0\right)}\right|+\int_{t_0-\tau_j(t_0)}^{t_0}\left|C_j(t_0,v)\right|R_j\left(t\right)dv\right)\right]L$$
$$\leq\left(1-\alpha\right)\epsilon. \tag{3.28}$$

If $(x\left(t\right),y\left(t\right))$ is a solution of (1.9) with $y=\dot{x}$ on $[t_0,\infty)$ and, $y\left(t_0\right)=\dot{x}\left(t_0\right)$ then, $x\left(\cdot\right)=\left(Px\right)\left(\cdot\right)$ defined in (3.21). Notice that with such a δ, $\left|x\left(s\right)\right|=\left|\psi\left(s\right)\right|<\epsilon$ on $[m\left(t_0\right),t_0]$. We claim that $\left|x\left(t\right)\right|<\epsilon$ for all $t\geq t_0$. If x is a solution with initial function ψ then, as a consequence of (3.21), we have

$$\begin{aligned}|x(t)| &\leq \left[\delta + \delta \sum_{j=1}^{N} \int_{t_0-\tau_j(t_0)}^{t_0} \left|\hat{D}_j(v)\right| dv\right] M \\ &+ \left[|\dot{x}(t_0)| + \delta \sum_{j=1}^{N} \frac{|b_j(t_0)|}{|1-\tau_j'(t_0)|} + \delta \int_{t_0-\tau_j(t_0)}^{t_0} |C_j(t_0,v)| R_j(v) dv\right] L + \varepsilon\alpha \\ &\leq (1-\alpha)\varepsilon + \varepsilon\alpha \leq \varepsilon. \end{aligned} \tag{3.29}$$

Now, recalling (3.1) of Lemma 3.1, we have

$$\begin{aligned}\dot{x}(t) &= \dot{x}(t_0) e^{-\int_{t_0}^{t} A(v)dv} \\ &- \sum_{j=1}^{N} \int_{t_0}^{t} e^{-\int_s^t A(v)dv} \int_{s-\tau_j(s)}^{s} a_j(s,v) g_j(v, x(v)) dvds \\ &- \sum_{j=1}^{N} \int_{t_0}^{t} e^{-\int_s^t A(v)dv} \omega_j(s) \frac{d}{ds} x(s-\tau_j(s)) ds.\end{aligned}$$

Integrating the last term on right hand side by parts we obtain

$$\begin{aligned}\dot{x}(t) &= e^{-\int_{t_0}^{t} A(v)dv} \left(\dot{x}(t_0) + \sum_{j=1}^{N} \frac{b_j(t_0)}{1-\tau_j'(t_0)} x(t_0 - \tau_j(t_0))\right) \\ &- \sum_{j=1}^{N} \frac{b_j(t)}{1-\tau_j'(t)} x(t-\tau_j(t)) \\ &+ \int_{t_0}^{t} e^{-\int_s^t A(v)dv} \sum_{j=1}^{N} \left(r_j(s) x(s-\tau_j(s)) - \int_{s-\tau_j(s)}^{s} a_j(s,v) g_j(v, x(v)) dv\right) ds.\end{aligned}$$

By conditions (3.28) and (3.20) we get the estimation

$$\begin{aligned}|\dot{x}(t)| &\leq |\dot{x}(t_0)| + \delta \sum_{j=1}^{N} \frac{|b_j(t_0)|}{|1-\tau_j'(t_0)|} \\ &+ \varepsilon \sum_{j=1}^{N} \left[\frac{|b_j(t)|}{|1-\tau_j'(t)|}\right. \\ &\left. + \int_{t_0}^{t} e^{-\int_s^t A(u)du} \left(|r_j(s)| + \int_{s-\tau_j(s)}^{s} |a_j(s,v)| R_j(v) dv\right) ds\right] \\ &\leq \frac{(1-\alpha)\varepsilon}{L} + \varepsilon\beta \leq \varepsilon\left(\frac{1}{L} + \beta\right).\end{aligned}$$

Therefore, the zero solution is stable at $t = t_0$. □

Theorem 3.6. *Under the hypotheses of Theorem 3.5, the zero solution of (1.5) is asymptotically stable if and only if*

$$\int_{t_0}^{t} \hat{D}(s)ds \longrightarrow \infty, \text{ as } t \longrightarrow \infty. \tag{3.30}$$

Proof. First, suppose that (3.30) holds. We wish the solutions of (1.5) to tend to zero whenever condition (3.30) holds. For this, we will modify S_ψ in order to functions in S_ψ^0 tend to zero as $t \longrightarrow \infty$. So, we let

$$S_\psi^0 := \{\varphi \in [m(t_0), \infty) \to \mathbb{R} \mid \varphi \in \mathcal{C},$$
$$\varphi(t) = \psi(t) \text{ for } t \in [m(t_0), t_0] \text{ and } \varphi(t) \to 0 \text{ as } t \to \infty\}.$$

Since S_ψ^0 is closed in S_ψ and (S_ψ, ρ) is complete, then the metric space $\left(S_\psi^0, \rho\right)$ is also complete. We begin by proving that $P\varphi(t) \to 0$ as $t \to \infty$ for $\varphi \in S_\psi^0$. To this end, denote the eight terms on the right hand side of (3.21) by I_1,I_2,...,I_8, respectively and let $\varphi \in S_\psi^0$ be fixed. Since $\int_0^t \hat{D}(s)ds \longrightarrow \infty$, as $t \longrightarrow \infty$, by condition (3.30), we see obviously that the first term I_1 of (3.21) tends to zero as $t \longrightarrow \infty$. For a given $\epsilon > 0$, choose $T_0 > 0$ large enough so that $t - \tau_j(t) \geq T_0$ for $j = \overline{1, N}$ implies $|\varphi(s)| < \epsilon$ if $s \geq t - \tau_j(t)$. Therefore, the third term I_3 in (3.21) satisfies

$$\begin{aligned} |I_3| &\leq \sum_{j=1}^{N} \int_{t-\tau_j(t)}^{t} |\varphi(v)| \left|\hat{D}_j(v)\right| dv \leq \varepsilon \sum_{j=1}^{N} \int_{t-\tau_j(t)}^{t} \left|\hat{D}_j(v)\right| dv \\ &\leq \alpha\epsilon < \epsilon. \end{aligned}$$

Thus, $I_3 \longrightarrow 0$ as $t \longrightarrow \infty$. We check that $I_2 \longrightarrow 0$ as $t \longrightarrow \infty$. So we have to prove that the two right hand side terms of the decomposition expression (3.23) go to zero at infinity. But the first term of that decomposition is as

$$\int_{t_0}^{J} e^{-\int_u^t \hat{D}(v)dv} e^{-\int_{t_0}^u A_1(v)dv} du = e^{-\int_J^t \hat{D}(v)dv} \int_{t_0}^{J} e^{-\int_u^J \hat{D}(v)dv} e^{-\int_{t_0}^u A_1(v)dv} du,$$

which tends to 0 as $t \longrightarrow \infty$ by condition (3.30). Nevertheless, the second term of on the right had side of (3.23) needs some more details for its convergence to zero. To overcome the difficulties, remember that from (3.19) we have obtained (3.25). Upon replacing of u by $J\theta$ in (3.25) we get

$$F(J) = J \int_1^\infty e^{-(a_0 J^\gamma)\theta^\gamma} d\theta. \tag{3.31}$$

The function $G(\lambda) := \int_1^\infty e^{-\lambda\theta^\gamma} d\theta$ satisfies the conditions of Theorem 2.1 where (3.32)

$$\lambda = a_0 J^\gamma, \quad \alpha = 1, \quad \varphi(\theta) = \theta^\gamma, \quad f \equiv 1, \quad \varphi'(\alpha) = \gamma\alpha^{\gamma-1} = \gamma, \quad f(\alpha) = 1.$$

It follows that

$$G(\lambda) \sim \frac{f(\alpha)}{\varphi'(\alpha)} \frac{1}{\lambda} e^{-\lambda\varphi(\alpha)} = \frac{1}{\gamma} \frac{1}{\lambda} e^{-\lambda}, \quad (\lambda \longrightarrow +\infty). \tag{3.33}$$

Thus we can write

$$F(J) \sim \frac{1}{\gamma a_0} J^{1-\gamma} e^{-a_0 J^\gamma}, \quad (J \longrightarrow +\infty). \tag{3.34}$$

It is enough to make $z = a_0 J^\gamma$ and a straightforward computation gives

$$\frac{1}{\gamma a_0} J^{1-\gamma} e^{-a_0 J^\gamma} = \frac{1}{\gamma a_0^{1/\gamma}} z^{\frac{1}{\gamma}-1} e^{-z} \leq \frac{1}{\gamma a_0^{1/\gamma}} z^m e^{-z} \longrightarrow 0 \text{ as } z \longrightarrow \infty. \tag{3.35}$$

where $m := [1/\gamma] + 1$. Thus, for every $\epsilon > 0$ we can find a $J^* \gg Q_0$ large enough such that for every $J \geq J^*$

$$\frac{e^{-b}}{\gamma a_0} J^{1-\gamma} e^{-a_0 J^\gamma} \leq \epsilon. \tag{3.36}$$

Clearly, the expansion (3.23) is valid if J is replaced by J^*. So, the last term tends towards zero when $t \longrightarrow \infty$. Hence the second term I_2 in (3.21) tends to zero as $t \longrightarrow \infty$. Now consider $I = |I_4| + ... + |I_8|$. To simplify our expressions, we define

$$\begin{aligned} V(u) &:= \sum_{j=1}^{N} \left(\hat{D}(u) \int_{u-\tau_j(u)}^{u} \hat{D}_j(v)\, dv + \int_{u-\tau_j(u)}^{u} |C_j(u,v)|\, R_j(v)\, dv \right) \\ &+ \sum_{j=1}^{N} \int_{t_0}^{u} e^{-\int_s^u A(v)dv} \left| C_j(s, s-\tau_j(s))(1-\tau_j'(s)) \right| R_j(s-\tau_j(s))\, ds \\ &+ \sum_{j=1}^{N} \int_{t_0}^{u} e^{-\int_s^u A(v)dv} \left(A(s) \int_{s-\tau_j(s)}^{s} |C_j(s,v)|\, R_j(v)\, dv + |r_j(s)| \right) ds. \end{aligned} \tag{3.37}$$

So, for the given $\epsilon > 0$, there exists a $T^* > t_0$ such that $s \geq T^*$ implies $|\varphi(s - \tau_j(s))| < \epsilon$ for $j = \overline{1, N}$. It is clear that $|\varphi(s)| < \varepsilon$ (because $s > s - \tau_j(s)$, $j = \overline{1,\ N}$). Thus, for $t \geq T^*$, by making use conditions (3.15) and (3.16) the term I satisfies

$$\begin{aligned} I &\leq \epsilon \sum_{j=1}^{N} \int_{t-\tau_j(t)}^{t} \left| \hat{D}_j(v) \right| dv + \varepsilon \int_{T^*}^{t} V(u)\, e^{-\int_u^t \hat{D}(v)dv} du \\ &+ \sup_{\zeta \geq m(t_0)} |\varphi(\zeta)| \int_{t_0}^{T^*} V(u)\, e^{-\int_u^t \hat{D}(v)dv} du \\ &\leq 2\alpha\epsilon + \sup_{\zeta \geq m(t_0)} |\varphi(\zeta)| \int_{t_0}^{T^*} V(u)\, e^{-\int_u^t \hat{D}(v)dv} du. \end{aligned} \tag{3.38}$$

Also, the condition (3.30) implies that there exists $T^{**} > T^*$ such that for $t \geq T^{**}$ we have

$$e^{-\int_{T^{**}}^t \hat{D}(v)dv} \sup_{\zeta \geq m(t_0)} |\varphi(\zeta)| \int_{T^*}^{t} V(u)\, e^{-\int_u^{T^{**}} \hat{D}(v)dv} du \leq \epsilon. \tag{3.39}$$

So, $I \longrightarrow 0$ as $t \longrightarrow \infty$ and consequently, $(P\varphi)(t) \longrightarrow 0$ as $t \longrightarrow \infty$. Thus, P maps S_ψ^0 into itself. Also, P is still a contraction on S_ψ^0 with a unique fixed point x. By Lemma 3.3, x is a solution of (1.5) on $[t_0, \infty)$. We conclude that $x(t)$ is the only continuous solution of (1.5) agreeing with the initial function ψ. As $x \in S_\psi^0$, $x(t) \to 0$ as $t \to \infty$. Therefore, the zero solution is asymptotically stable, since it is stable by Theorem 3.5 and we have just concluded that $|x(t)| + |y(t)| \longrightarrow 0$ as $t \longrightarrow \infty$ if condition (3.30) holds.

Conversely, we shall prove that $\int_{t_0}^{\infty} \hat{D}(v)dv = \infty$. Contrary to this, there exists a sequence $\{t_n\}_{n\geq 1}$ with $t_n \longrightarrow \infty$ as $n \longrightarrow \infty$ and such that $\int_{t_0}^{t_n} \hat{D}(v)dv = l$ for a certain finite number $l \in \mathbb{R}^+$. By condition (3.14), we may also choose $\mu > 0$ that satisfies the inequality, $-\mu \leq \int_{t_0}^{t_n} \hat{D}(v)dv \leq \mu$, for all $n \geq 1$. For convenience of notation we set

$$C_0 := \sum_{j=1}^{N} \left(\frac{|b_j(t_0)|}{\left|1 - \tau_j'(t_0)\right|} |\psi(t_0 - \tau_j(t_0))| + \int_{t_0 - \tau_j(t_0)}^{t_0} |C_j(t_0, v)| R_j(v) |\psi(v)| dv \right).$$

Recalling (3.37), we define the function $W(\cdot)$ by

$$W(u) := V(u) + C_0 e^{-\int_{t_0}^{u} A(v)dv}.$$

By conditions (3.17), (3.18) and (3.19), we have

$$\int_{t_0}^{t_n} e^{-\int_{t_0}^{t_n} \hat{D}(v)dv} W(u)\, du \leq (\alpha + C_0 L). \tag{3.40}$$

This yields

$$e^{-\int_{t_0}^{t_n} \hat{D}(v)dv} \int_{t_0}^{t_n} e^{\int_{t_0}^{u} \hat{D}(v)dv} W(u)\, du \leq (\alpha + C_0 L). \tag{3.41}$$

Then,

$$\int_{t_0}^{t_n} e^{\int_{t_0}^{u} \hat{D}(v)dv} W(u)\, du \leq (\alpha + C_0 L) e^{\mu}. \tag{3.42}$$

The inequality (3.42) leads to the fact that the sequence

$$\int_{t_0}^{t_n} e^{\int_{t_0}^{u} \hat{D}(v)dv} W(u)\, du, \tag{3.43}$$

is bounded, so there exists a convergent subsequence. For brevity, we assume that

$$\lim_{t \longrightarrow \infty} \int_{0}^{t_n} e^{\int_{0}^{u} \hat{D}(v)dv} W(u)\, du = \sigma > 0. \tag{3.44}$$

Then, we can choose a positive integer n_0 large enough such that

$$\int_{t_{n_0}}^{t_n} e^{\int_{t_0}^{u} \hat{D}(v)dv} W(u)\, du < \frac{\delta_0}{8M}. \tag{3.45}$$

for $n \geq n_0$, where $\epsilon > \delta_0 > 0$ satisfies

$$\begin{aligned}
&\left[|\psi(t_{n_0})| + \delta_0 \sum_{j=1}^{N} \int_{t_{n_0}-\tau_j(t_{n_0})}^{t_{n_0}} \left|\hat{D}_j(v)\right| dv\right] M \\
&+ \left(|\dot{x}(t_{n_0})| + \delta_0 \sum_{j=1}^{N} \frac{|b_j(t_{n_0})|}{|1-\tau_j'(t_{n_0})|} + \delta_0 \int_{t_{n_0}-\tau_j(t_{n_0})}^{t_{n_0}} |C_j(t_{n_0}, v)| R_j(v)\, dv\right) L \\
&\leq (1-\alpha). \qquad (3.46)
\end{aligned}$$

Now, we consider the solution $x(t) = x(t, \psi, \dot{x}(t_{n_0}))$ of equation (1.5), for the initial values ψ and $\dot{x}(t_{n_0})$ such that

$$\begin{gathered}
\psi(t_{n_0}) = \frac{3\delta_0}{4}, \ \ \dot{x}(t_{n_0}) = \frac{\delta_0}{4}, \\
|\psi(s)| + |\dot{x}(s)| \leq \delta_0, \qquad s \leq t_{n_0}.
\end{gathered} \tag{3.47}$$

We may choose ψ such that

$$\psi(t_{n_0}) - \sum_{j=1}^{N} \int_{t_{n_0}-\tau_j(t_{n_0})}^{t_{n_0}} \hat{D}_j(v)\, \psi(v)\, dv \geq \frac{\delta_0}{4}. \tag{3.48}$$

By a similar argument as in (3.29) and by replacing ϵ by 1, this implies that $|x(t)| \leq 1$. Having in mind the fact that x is a fixed point of P, we have for $n \geq n_0$

$$\begin{aligned}
&\left|x(t_n) - \sum_{j=1}^{N} \int_{t_n-\tau_j(t_n)}^{t_n} \hat{D}_j(v)\, x(v)\, dv\right| \\
&\geq \left|e^{-\int_{t_{n_0}}^{t_n} \hat{D}(v)dv} \left(\psi(t_{n_0}) - \sum_{j=1}^{N} \int_{t_{n_0}-\tau_j(t_{n_0})}^{t_{n_0}} \hat{D}_j(v)\, \psi(v)\, dv\right)\right. \\
&\left. + \dot{x}(t_{n_0}) \int_{t_{n_0}}^{t_n} e^{-\int_u^{t_n} \hat{D}(v)dv} e^{-\int_{t_{n_0}}^{u} A(v)dv} du\right| - \left|\int_{t_{n_0}}^{t_n} e^{-\int_u^{t_n} \hat{D}(v)dv} W(u)du\right| \\
&\geq e^{-\int_{t_{n_0}}^{t_n} \hat{D}(v)dv} \frac{\delta_0}{4} - \int_{t_{n_0}}^{t_n} e^{-\int_u^{t_n} \hat{D}(v)dv} W(u)du \\
&\geq e^{-\int_{t_{n_0}}^{t_n} \hat{D}(v)dv} \left[\frac{\delta_0}{4} - e^{-\int_0^{t_{n_0}} \hat{D}(v)dv} \int_{t_{n_0}}^{t_n} e^{\int_0^u \hat{D}(v)dv} W(u)du\right] \geq \frac{\delta_0}{8} e^{-2\mu} > 0. \qquad (3.49)
\end{aligned}$$

On the other hand, if the zero solution is asymptotically stable, then $x(t) = x(t, \psi, \dot{x}(t_{n_0})) \longrightarrow 0$, as $t \longrightarrow \infty$. It remains only to check that the term

$\sum\limits_{j=1}^{N}\int_{t_n-\tau_j(t_n)}^{t_n}\hat{D}_j(v)\,x(v)\,dv$ decays to zero at infinity to obtain the contradiction . By the mean value theorem and condition (3.17), we have

$$\begin{aligned}\left|\sum_{j=1}^{N}\int_{t_n-\tau_j(t_n)}^{t_n}\hat{D}_j(v)\,x(v)\,dv\right| &= |x(\eta_{t_n})|\left|\sum_{j=1}^{N}\int_{t_n-\tau_j(t_n)}^{t_n}\hat{D}_j(v)\,dv\right| \\ &\leq \alpha\,|x(\eta_{t_n})| \leq |x(\eta_{t_n})|\,. \end{aligned}\tag{3.50}$$

Since t_n and $t_n-\tau_j(t_n)\to\infty$ as $n\to\infty$, then also $\eta_{t_n}\longrightarrow\infty$. It follows that

$$\lim_{t_n\longrightarrow\infty}\left(x(t_n)-\sum_{j=1}^{N}\int_{t_n-\tau_j(t_n)}^{t_n}\hat{D}_j(v)\,x(v)\,dv\right)=0,$$

which leads to a contradiction. This completes the proof of our claim. □

In this section, we will give an example to apply our results

Example 3.7. Consider the following nonlinear neutral integro-differential equation with variable delay

$$\ddot{x}+f(t,x,\dot{x})\,\dot{x}+\int_{t-\tau(t)}^{t}a(t,s)g(s,x(s))\,ds+b(t)x'(t-\tau(t))=0,\tag{3.51}$$

for $t\geq 0$. We let $A(t):=f(t,x(t),\dot{x}(t))=\dfrac{1-0.5\cos(\dot{x}(t)x(t))}{5(t+1)^{\frac{1}{5}}}+2\tanh t$, $g(t,x(t)):=t\sin x(t)$, $a(t,s):=e^{-3(t+s)}$, $\tau(t):=0.5t$ and $b(t):=\alpha_0\dfrac{t}{t^2+1}$, where $0<\alpha_0\leq 1/7$. Then the zero solution of (3.51) is asymptotically stable.

Proof. We prove that all the hypotheses of Theorem (3.6) hold for equation (3.51). Observe that the conditions (3.15), (3.16) and (3.18) are satisfied, with $R(t):=t$ and $A_1(t):=\dfrac{0.5}{5(t+1)^{\frac{1}{5}}}+2\tanh t$, for $t\geq 0$. Now, clearly, the conditions (3.14) and (3.30) hold. Furthermore, for $t\geq u\geq 0$ we have

$$\begin{aligned}\int_u^t\hat{D}(v)\,dv+\int_0^u A_1(t)\,dv &\geq \int_0^u A_1(t)\,dv\geq 0.5\int_0^u\frac{1}{5(v+1)^{\frac{1}{5}}}dv \\ &\geq \frac{1}{8}u^{\frac{4}{5}}-\frac{1}{8}.\end{aligned}\tag{3.52}$$

Consequently, condition (3.19) is satisfied with $a_0=1/8$, $b_0=-1/8$ and $\gamma=4/5$.

It remains to prove that the condition (3.17) is also satisfied. There are six terms on the left hand side of (3.17). Performing the substitution $u=s-\tau(s)$ we obtain $s=p(u)=2u$, $(1-\tau'(s))\,ds=du$ and

$$\int_{t-\tau(t)}^{t}\hat{D}(u)\,du=\int_t^{p(t)}D(s)(1-\tau'(s))\,ds=2\int_t^{2t}b(s)\,ds,$$

and by integration we see that,

$$\int_{t-\tau(t)}^{t} \hat{D}(u)\,du = \alpha_0\left(\ln\frac{4t^2+1}{t^2+1}\right) \leq \alpha_0 1.39. \tag{3.53}$$

Since $\hat{D}(t) \geq 0$ for $t \geq 0$, then making use of (3.53) we can derive the estimation

$$\begin{aligned}
&\int_0^t e^{-\int_u^t \hat{D}(v)dv}\hat{D}(u)\int_{u-\tau(u)}^{u}\hat{D}(v)\,dvdu \\
&\leq \left(\sup_{u\geq 0}\int_{0.5u}^{u}\hat{D}(u)\,dv\right)\int_0^t e^{-\int_u^t \hat{D}(v)dv}\hat{D}(u)\,du \\
&\leq \alpha_0 1.39.
\end{aligned} \tag{3.54}$$

For the third term of (3.17) we observe that

$$\begin{aligned}
|C(u,v)| &= \left|\int_u^v a(w,v)\,dw\right| = \left|e^{-3v}\int_u^v e^{-3w}dw\right| \\
&= \frac{1}{3}\left|e^{-3v}\left(e^{-3u}-e^{-3v}\right)\right|.
\end{aligned} \tag{3.55}$$

But,

$$\left(e^{-3v}-e^{-3u}\right) \geq 0, \text{ for } 0 \leq v \leq u.$$

So,

$$\begin{aligned}
&\int_0^t e^{-\int_u^t \hat{D}(v)dv}\int_{u-\tau(u)}^{u}|C(u,v)|\,R(v)\,dvdu \\
&\leq \int_0^t\int_{0.5u}^{u}|C(u,v)|\,|R(v)|\,dvdu \\
&\leq \frac{1}{3}\int_0^t\int_{0.5u}^{u}\left(e^{-6v}-e^{-3v-3u}\right)vdvdu := N(t) \\
&\leq 0.0013.
\end{aligned} \tag{3.56}$$

This is because, the function $N(\cdot)$ is a strictly positive and increasing on $[0,\infty)$ satisfying $N(t) \to 0.0013717$ as $t \longrightarrow \infty$. To estimate the fourth term of (3.17), note that

$$A(t) \leq \frac{1}{5(t+1)^{\frac{1}{5}}} + 2\tanh t \leq 2.3,$$

and

$$e^{-\int_s^u A(v)dv} \leq e^{-\int_s^u A_1(v)dv} \leq e^{-2\int_s^u\left(\frac{\sinh v}{\cosh v}\right)dv} = \frac{\cosh^2 s}{\cosh^2 u}.$$

Therefore,

$$\int_0^t e^{-\int_u^t \hat{D}(v)dv} \int_0^u A(s)e^{-\int_s^u A(v)dv} \int_{s-\tau(s)}^s |C(s,v)|\,|R(v)|\, dvdsdu$$

$$\leq 2.3 \int_0^t \int_0^u e^{-2\int_s^u \tanh v dv} \frac{1}{3} \int_{0.5s}^s \left(e^{-6v} - e^{-3v-3s}\right) vdsdvdu$$

$$\leq 2.3 \int_0^t \frac{1}{\cosh^2 u} \int_0^u \frac{1}{3}\frac{1}{4}\left(e^{2s}+2+e^{-2s}\right) \int_{0.5s}^s \left(e^{-6v} - e^{-3v-3s}\right) vdsdvdu$$

$$\leq \frac{2.3}{12} \sup_{u\geq 0} \left|\bar{N}(u)\right| \int_0^t \frac{1}{\cosh^2 u} du$$

(3.57)

$$\leq \frac{2.3}{12} 0.09 \int_0^t \frac{1}{\cosh^2 u} du = \frac{2.3}{12} 0.09 \tanh t \leq 2.3 \frac{0.09}{12} = 0.01725.$$

This follows from the fact that,

$$\bar{N}(u) := \int_0^u \left(e^{2s}+2+e^{-2s}\right) \int_{0.5s}^s \left(e^{-6v} - e^{-3v-3s}\right) vdvds,$$

is a positive function, increasing on $[0,\infty)$ and $\bar{N}(u) \leq 0.09$ for any $u \geq 0$. Moreover, similar arguments as above show that one can estimate the fifth term of (3.17) as

$$\int_{t_0}^t e^{-\int_u^t \hat{D}(v)dv} \int_0^u e^{-\int_s^u A(v)dv} \left|C(s, s-\tau_j(s))\right| \left|(1-\tau'(s)\right| \left|R(s-\tau(s))\right| dsdu$$

$$\leq \int_0^t \frac{1}{\cosh^2 u} \int_0^u \frac{1}{4}\left(e^{2s}+2+e^{-2s}\right) \frac{0.25}{3} \left(e^{-3s} - e^{-4.5s}\right) sdsdu$$

(3.58)

$$\leq \frac{0.25}{12} \int_0^t \frac{1}{\cosh^2 u} \int_0^u \left(e^{2s}+2+e^{-2s}\right)\left(e^{-3s} - e^{-4.5s}\right) sdsdu \leq 0.020833.$$

For the sixth we have, for $t \geq 0$,

$$\begin{aligned} r(t) &= \alpha_0 2\left(\frac{t}{t^2+1}A(t) + \frac{1-t^2}{\left(t^2+1\right)^2}\right) \\ &\geq \alpha_0 2\frac{1}{t^2+1}\left(\frac{0.5t}{5(t+1)^{\frac{1}{5}}} + t2\tanh t + \frac{1-t^2}{(t^2+1)}\right) \\ &\geq \alpha_0 2\frac{1}{t^2+1} > 0. \end{aligned} \tag{3.59}$$

It is clear that the function $r(\cdot)$ is positive, because $2t\tanh t + \frac{1-t^2}{(t^2+1)} > 0$, and

$$\begin{aligned} D(t) &= \frac{b(t)}{(0.5)^2},\ \hat{D}(t) = D(p(t)) = \frac{b(p(t))}{(0.5)^2} = \frac{b(2t)}{(0.5)^2} \\ &= 4\alpha_0\frac{2t}{(4t^2+1)} \geq 2\alpha_0\frac{t}{t^2+1} \geq 2b(t) \ \text{ for } t \geq 0. \end{aligned}$$

Which implies that

$$\begin{aligned}
&\int_0^t e^{-\int_u^t \hat{D}(v)dv} \int_0^u e^{-\int_s^u A(v)dv} |r(s)|\, dsdu \\
&= \int_0^t e^{-\int_u^t \hat{D}(v)dv} b(u)\, du \leq \frac{1}{2}\int_0^t e^{-\int_u^t 2b(v)dv} 2b(u)\, du \\
&\leq 0.5. \qquad (3.60)
\end{aligned}$$

The summation yields

$$\begin{aligned}
\alpha &:= 2\alpha_0 1.39 + 0.0013 + 0.01725 + 0.020833 + 0.5 \\
&\leq 2.78\frac{1}{7} + 0.53938 = 0.93652.
\end{aligned}$$

Also, we remark that the condition (3.20) holds, because $0 \leq 2b(t) + \frac{1}{12}\bar{N}(t) < +\infty$. Consequently, the zero solution of equation (3.51) is asymptotically stable. □

References

[1] Ardjouni, A., Djoudi, A., Stability in nonlinear neutral integro-differential equations with variable delay using fixed point theory. J. Appl. Math. Comput. 44 (2014), 317–336.

[2] Ardjouni, A., Djoudi, A., Fixed points and stability in linear neutral differential equations with variable delays. Nonlinear Analysis 74 (2011), 2062–2070.

[3] Ardjouni, A., Djoudi, A., Stability in nonlinear neutral differential with variable delays using fixed point theory. Electronic Journal of Qualitative Theory of Differential Equations 43 (2011), 1–11.

[4] Ardjouni, A., Djoudi, A., Fixed point and stability in neutral nonlinear differential equations with variable delays. Opuscula Mathematica 32(1) (2012), 5–19.

[5] Ardjouni, A., Djoudi, A., Soualhia, I., Stability for linear neutral integro-differential equations with variable delays. Electron. J. Differ. Equ. 2012(172) (2012), 1–14.

[6] Becker, L. C., Burton, T. A., Stability, fixed points and inverse of delays. Proc. Roy. Soc. Edinburgh 136A (2006), 245–275.

[7] Burton, T. A., Stability by Fixed Point Theory for Functional Differential Equations, New York: Dover Publications, 2006.

[8] Burton, T. A., Stability by fixed point theory or Liapunov's theory: A comparison. Fixed Point Theory 4 (2003), 15–32.

[9] Burton, T. A., Stability and periodic solutions of ordinary functional differential equations. New York: Academic Press, 1985.

[10] Burton, T. A., Fixed points, stability, and exact linearization. Nonlinear Analysis: Theory, Methods & Applications 61(5) (2005), 857–870.

[11] Djoudi, A., Khemis, R., Fixed point techniques and stability for neutral nonlinear differential equations with unbounded delays. Georgian Mathematical Journal 13(1) (2006), 25–34.

[12] Faraut, J., Calcul Intégral. Collection Enseignement SUP-Maths de EDP Sciences, 2006.

[13] Hale, J. K., Theory of Functional Differential Equations. New York: Springer Verlag, 1977.

[14] Jin, C. H., Luo, J. W., Stability of an integro-differential equation. Computers and Mathematics with Applications 57 (2009), 1080–1088.

[15] Levin, J. J., Nohel, J. A., Global asymptotic stability for nonlinear systems of differential equations and applications to reactor dynamics. Archive for Rational Mechanics and Analysis 5 (1960), 194–211.

[16] Pi, D., Study the stability of solutions of functional differential equations via fixed points. Nonlinear Analysis: Theory, Methods & Applications 74(2) (2011), 1639–651.

[17] Pi, D., Stability conditions of second order integrodifferential equations with variable delay. Abstract and Applied Analysis2014 (2014), Article ID 371639, 1–11.

[18] Pi, D., Fixed Points and Stability of a Class of Integrodifferential Equations. Mathematical Problems in Engineering 2014 (2014), Article ID 286214, 1–10.

[19] Schiff, J. L., The Laplace Transform: Theory and Applications. New York: Springer-Verlag, 1999.

Received by the editors June 21, 2016
First published online April 18, 2017