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# LOCAL RESULTS FOR AN ITERATIVE METHOD OF CONVERGENCE ORDER SIX AND EFFICIENCY INDEX 1.8171

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**Abstract.** We present a local convergence analysis of an iterative method of convergence order six and efficiency index 1.8171 in order to approximate a locally unique solution of a nonlinear equation. In earlier studies such as [16] the convergence order of these methods was given under hypotheses reaching up to the fourth derivative of the function although only the first derivative appears in these methods. In this paper, we expand the applicability of these methods by showing convergence using only the first and second derivatives. Moreover, we compare the convergence radii and provide computable error estimates for these methods using Lipschitz constants.

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*Key words and phrases:* Halley's method; Jarratt method; King-Werner method; local convergence; efficiency index; sixth order of convergence

## 1. Introduction

The problem of approximating a locally unique solution $x^*$ of equation

$$F(x) = 0, \tag{1.1}$$

where $F : D \subseteq \mathbb{R} \to \mathbb{R}$ is a nonlinear function, $D$ is a convex subset of $\mathbb{R}$ has many applications in mathematics and engineering. Newton-like methods are famous for finding solution of (1.1). These methods are usually studied based on: semi-local (that is based on the information around an initial point, to give conditions ensuring the convergence of the iterative procedure) and local convergence (that is based on the information around a solution, to find estimates of the radii of convergence balls [1–25]).

Many authors (see [1–25]) have used higher order multi-point methods for approximating a locally unique solution $x^*$ of (1.2). Higher order methods such as Euler's, Halley's, super Halley's, Chebyshev's [1–25] require the evaluation of the higher order derivative of $F$ at each step, which in general is very expensive.

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In this paper we present the local convergence analysis of method defined for each $n = 0, 1, 2, \cdots$ by

$$\begin{aligned} y_0 &= x_0, \\ x_1 &= x_0 - A_0^{-1} F(x_0), \\ y_n &= x_n - A_{n-1}^{-1} F(x_n), \\ x_{n+1} &= x_n - A_n^{-1} F(x_n), \end{aligned} \tag{1.2}$$

where $x_0$ is an initial point and

$$A_n = F'(\frac{1}{2}(x_n + y_n)) - \frac{1}{2}\frac{F(x_n)F''(\frac{1}{2}(x_n + y_n))}{F'(\frac{1}{2}(x_n + y_n))}.$$

Method (1.2) was introduced and studied in [15]. The motivation and favorable comparisons were also given in [15]. The sixth order of convergence was shown in [16] using Taylor expansions, Maple software and hypotheses reaching up to the fourth derivative. The efficiency index is $6^{\frac{1}{3}} = 1.8171$ which is larger than the efficiency indices of other methods (see Table 1).

| Method | Number of function or derivative evaluations | Efficiency index |
|---|---|---|
| Newton, quadratic | 2 | $2^{\frac{1}{2}} \approx 1.4142$ |
| Cubic methods | 3 | $3^{\frac{1}{3}} \approx 1.4422$ |
| Kou's $5^{th}$ order [25] | 4 | $5^{\frac{1}{4}} \approx 1.4953$ |
| Kou's $6^{th}$ order [25] | 4 | $6^{\frac{1}{4}} \approx 1.5651$ |
| Jarratt's $4^{th}$ order | 3 | $4^{\frac{1}{3}} \approx 1.5874$ |
| Secant | 1 | $0.5(1+\sqrt{5}) \approx 1.6180$ |
| Modified Halley's method | 43 | $6^{\frac{1}{3}} \approx 1.8171$ |

Table 1: table 1. Comparison of efficiencies of various methods

However, the hypotheses up to the fourth derivative of function $F$ limit the applicability of these methods. As a motivational example, let us define the function $f$ on $D = [-\frac{1}{2}, \frac{5}{2}]$ by

$$f(x) = \begin{cases} x^3 \ln x^2 + x^5 - x^4, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Choose $x^* = 1$. We have that

$$\begin{aligned} f'(x) &= 3x^2 \ln x^2 + 5x^4 - 4x^3 + 2x^2, \ f'(1) = 3, \\ f''(x) &= 6x \ln x^2 + 20x^3 - 12x^2 + 10x \\ f'''(x) &= 6 \ln x^2 + 60x^2 - 24x + 22. \end{aligned}$$

Then, obviously, function $f'''$ is unbounded on $D$. Notice that, in particular there is a plethora of iterative methods for approximating solutions of nonlinear equations defined on $\mathbb{R}$ [1–25]. These results show that if the initial point $x_0$ is sufficiently close to the solution $x^*$, then the sequence $\{x_n\}$ converges to $x^*$. But how close to the solution $x^*$ should the initial guess $x_0$ be? These local results give no information on the radius of the convergence ball for the corresponding method. We address this question for method (1.2) in Section 2. The same technique can be used to other methods [1–25].

In the present paper we only use hypotheses up to the second derivative. This way we expand the applicability of these methods.

The rest of the paper is organized as follows: Section 2 contains the local convergence analysis of the method. The numerical examples are presented in the concluding Section 3.

## 2. Local convergence analysis

We present the local convergence analysis of method (1.2) in this section. Let $L_0 > 0, L > 0, N > 0$ and $M \geq 1$ be parameters. It is convenient for the local convergence analysis of method (1.2) that follows to introduce some scalar functions and parameters. Define functions $p, q, h_p$ and $h_q$ on the interval $[0, \frac{1}{L_0})$ by

$$\begin{aligned} p(t) &= (L_0 + \frac{MN}{2(1-L_0t)})t, \\ q(t) &= \frac{1}{2}(4L_0 + \frac{MN}{1-L_0t})t, \\ h_p(t) &= p(t) - 1, \end{aligned}$$

and

$$h_q(t) = q(t) - 1.$$

Notice that the functions $p, q$ and $h_q$ are increasing on the interval $[0, \frac{1}{L_0})$. We have that $h_p(0) = -1 < 0$ and $h_p(t) \to +\infty$ as $t \to \frac{1}{L_0}^-$. It follows from the intermediate value theorem that the function $h_p$ has zeros in the interval $(0, \frac{1}{L_0})$. Denote by $r_p$ the smallest such zero. Similarly, denote the smallest zero of the function $h_q$ on the interval $(0, \frac{1}{L_0})$ by $r_q$. Notice that $h_q(t) = h_p(t) + \frac{L_0}{2}t$. In particular $h_q(r_p) = h_p(r_p) + \frac{L_0}{2}r_p = \frac{L_0}{2}r_p > 0$, since $h_p(r_p) = 0$ and $r_p > 0$. Hence, we deduce that $r_q < r_p$. Moreover, define functions $g_1$ and $h_1$ on the interval $[0, r_p)$ by

$$g_1(t) = \frac{1}{2(1-L_0t)}(Lt + \frac{2Mq(t)}{1-p(t)})$$

and

$$h_1(t) = g_1(t) - 1.$$

The functions $g_1$ and $h_1$ are increasing on $[0, r_p)$. We have that $h_1(0) = -1 < 0$ and $h_1(t) \to +\infty$ as $t \to r_p^-$. Denote by $r_1$ the smallest zero of the function $h_1$ in the interval $(0, r_p)$. Set

$$r = \min\{r_q, r_1\}. \tag{2.1}$$

Then, we have for each $t \in [0, r)$ that

$$0 \leq p(t) < 1 \tag{2.2}$$

$$0 \leq q(t) < 1, \tag{2.3}$$

and

$$0 \leq g_1(t) < 1. \tag{2.4}$$

Let $U(v, \rho)$ denote an interval in $\mathbb{R}$, with center $v \in \mathbb{R}$ and of radius $\rho > 0$. Then, by $\bar{U}(v, \rho)$ we denote the closure of the interval $U(v, \rho)$. Next, we present the local convergence analysis of method (1.2) using the preceding notation.

**Theorem 2.1.** *Let $F : D \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function. Suppose that there exist $x^* \in D$, $L_0 > 0, L > 0, N > 0$ and $M \geq 1$ such that for each $x$, $y \in D$*

$$F(x^*) = 0, \;\; F'(x^*) \neq 0, \tag{2.5}$$

$$|F'(x^*)^{-1}(F'(x) - F'(x^*))| \leq L_0|x - x^*|, \tag{2.6}$$

$$|F'(x^*)^{-1}(F'(x) - F'(y))| \leq L|x - y|, \tag{2.7}$$

$$|F'(x^*)^{-1}F'(x)| \leq M, \tag{2.8}$$

$$|F'(x^*)^{-1}F''(x)| \leq N, \tag{2.9}$$

*and*

$$\bar{U}(x^*, r) \subseteq D, \tag{2.10}$$

*hold, where the radius $r$ is given by (2.1). Then, the sequence $\{x_n\}$ generated for $x_0 \in U(x^*, r) - \{x^*\}$ by method (1.2) is well defined, remains in $U(x^*, r)$ for each $n = 0, 1, 2, \cdots$ and converges linearly to $x^*$. Moreover, the following estimates hold*

$$|y_n - x^*| \leq c|x_n - x^*| \leq |x_n - x^*| < r \text{ for each } n = 1, 2, \ldots \tag{2.11}$$

*and*

$$|x_{n+1} - x^*| \leq c|x_n - x^*| \leq |x_n - x^*| \text{ for each } n = 1, 2, \ldots, \tag{2.12}$$

*where,*

$$c = g_1(|x_0 - x^*|) \in [0,1) \tag{2.13}$$

*and the function $g_1$ is as defined previously. Furthermore, for $T \in [r, \frac{2}{L_0})$ the solution $x^*$ is unique in $D_0 := \bar{U}(x^*, T) \cap D$.*

*Proof.* We shall show estimates (2.11) and (2.12) using mathematical induction. By hypothesis $x_0 \in U(x^*, r) - \{x^*\}$, (2.1) and (2.6), we get since $|\frac{1}{2}(x_0 + y_0) - x^*| \le \frac{1}{2}(|x_0 - x^*| + |y_0 - x^*|) < r$ that

$$\begin{aligned} |F'(x^*)^{-1}(F'(\frac{x_0+y_0}{2}) - F'(x^*))| &\le L_0|\frac{x_0+y_0}{2} - x^*| \\ &\le \frac{L_0}{2}(|x_0 - x^*| + |y_0 - x^*|) \\ &= L_0|x_0 - x^*| < L_0 r < 1. \end{aligned} \tag{2.14}$$

It follows from (2.14) that $F'(\frac{x_0+y_0}{2}) \neq 0$ and by the Banach Lemma on invertible functions [3, 4, 24]

$$|F'(\frac{x_0+y_0}{2})^{-1}F'(x^*)| \le \frac{1}{1 - \frac{L_0}{2}(|x_0 - x^*| + |y_0 - x^*|)}. \tag{2.15}$$

We can write by (2.5) that

$$F(x_0) = F(x_0) - F(x^*) = \int_0^1 F'(x^* + \theta(x_0 - x^*))(x_0 - x^*)d\theta. \tag{2.16}$$

Notice that $|x^* + \theta(y_0 - x^*) - x^*| = \theta|y_0 - x^*| < r$. That is $x^* + \theta(y_0 - x^*) \in U(x^*, r)$. Then, by (2.8) and (2.16), we get that

$$|F'(x^*)^{-1}F(x_0)| \le M|x_0 - x^*|. \tag{2.17}$$

Next, we show that $A_0 \neq 0$. We have by (2.1), (2.2), (2.9), (2.15) and (2.17) that

$$\begin{aligned} &|F'(x^*)^{-1}(A_0 - F'(x^*))| \\ &\le |F'(x^*)^{-1}(F'(\frac{1}{2}(x_0+y_0)) - F'(x^*))| \\ &\quad + \frac{1}{2}|F'(x^*)^{-1}F(x_0)||F'(x^*)^{-1}F'(\frac{1}{2}(x_0+y_0))||F'(\frac{1}{2}(x_0+y_0))^{-1}F'(x^*)| \\ &\le L_0|\frac{1}{2}(x_0+y_0) - x^*| + \frac{MN|x_0 - x^*|}{2(1 - \frac{L_0}{2}(|x_0 - x^*| + |y_0 - x^*|))} \\ &\le \frac{L_0}{2}(|x_0 - x^*| + |y_0 - x^*|) + \frac{MN|x_0 - x^*|}{2(1 - \frac{L_0}{2}(|x_0 - x^*| + |y_0 - x^*|)} \\ &\le [L_0 + \frac{MN}{2(1 - \frac{L_0}{2}(|x_0 - x^*| + |y_0 - x^*|))}]|x_0 - x^*| \\ &\le p(|x_0 - x^*|) < 1. \end{aligned} \tag{2.18}$$

That is, $A_0 \neq 0$ and

$$|A_0^{-1}F'(x^*)| \leq \frac{1}{1-p(|x_0-x^*|)}. \tag{2.19}$$

Hence, $x_1$ is well defined. Notice that we can write in turn that

$$\begin{aligned} x_1 - x^* &\\ = &\ (x_0 - x^* - F'(x_0)^{-1}F(x_0)) + (F'(x_0)^{-1} - A_0^{-1})F(x_0) \\ = &\ (x_0 - x^* - F'(x_0)^{-1}F(x_0)) + (F'(x_0)^{-1}F'(x^*))(F'(x^*)^{-1}(A_0 - F'(x_0)) \\ &\ \times(A_0^{-1}F'(x^*))(F'(x^*)^{-1}F(x_0)), \end{aligned}$$

so

$$\begin{aligned} |x_1 - x^*| &\\ \leq &\ |x_0 - x^* - F'(x_0)^{-1}F(x_0)| + |F'(x_0)^{-1}F'(x^*)| \\ &\ \times|F'(x^*)^{-1}(A_0 - F'(x_0))| \\ &\ \times|A_0^{-1}F'(x^*)||F'(x^*)^{-1}F(x_0)|. \end{aligned} \tag{2.20}$$

Moreover, we also have that

$$\begin{aligned} |F'(x^*)^{-1}(A_0 - F'(x_0))| \leq &\ L_0|\frac{x_0+y_0}{2} - x^*| + L_0|x_0 - x^*| \\ &\ +\frac{1}{2}\frac{|F'(x^*)^{-1}F(x_0)||F'(x^*)^{-1}F''(\frac{1}{2}(x_0+y_0))|}{|F'(x^*)^{-1}F'(\frac{1}{2}(x_0+y_0))|} \\ \leq &\ 2L_0|x_0 - x^*| \\ &\ +\frac{1}{2}\frac{MN|x_0 - x^*|}{1 - \frac{L_0}{2}(|x_0 - x^*| + |y_0 - x^*|)} \\ \leq &\ q(|x_0 - x^*|). \end{aligned} \tag{2.21}$$

Then, in view of (2.1), (2.4), (2.7), (2.19), (2.20) and (2.21) we get in turn that

$$\begin{aligned} &|x_1 - x^*| \\ \leq &\ |F'(x_0)^{-1}F'(x^*)||\int_0^1 F'(x^*)^{-1}(F'(x^* + \theta(x_0 - x^*)) - F'(x_0))(x_0 - x^*)d\theta| \\ &\ +|F'(x_0)^{-1}F'(x^*)||F'(x^*)^{-1}(A_0 - F'(x_0))||A_0^{-1}F'(x^*)||F'(x^*)^{-1}F(x_0)| \\ \leq &\ \frac{L|x_0 - x^*|^2}{2(1 - L_0|x_0 - x^*|)} + \frac{Mq(|x_0 - x^*|)|x_0 - x^*|}{(1 - L_0|x_0 - x^*|)(1 - p(|x_0 - x^*|))} \\ \leq &\ g_1(|x_0 - x^*|)|x_0 - x^*| \leq |x_0 - x^*| < r, \end{aligned} \tag{2.22}$$

which shows (2.12) for $n = 0$ and $x_1 \in U(x^*, r)$, where we also used the estimates $|F'(x^*)^{-1}(F'(x_0) - F'(x^*))| \leq L_0|x_0 - x^*| < L_0 r < 1$, so as in (2.15)

$|F'(x_0)^{-1}F'(x^*)| \leq \frac{1}{1-L_0|x_0-x^*|}$. Furthermore, we also get by (2.3), (2.6), (2.9), (2.15) and (2.17) that

$$\begin{aligned}
&|F'(x^*)^{-1}(A_0 - F'(x_1))| \\
&\leq |F'(x^*)^{-1}(F'(\frac{1}{2}(x_0+y_0)) - F'(x^*))| \\
&\quad +|F'(x^*)^{-1}(F'(x_1) - F'(x^*))| \\
&\quad +\frac{1}{2}\frac{|F'(x^*)^{-1}F(x_0)||F'(x^*)^{-1}F''(\frac{1}{2}(x_0+y_0))|}{|F'(x^*)^{-1}F'(\frac{1}{2}(x_0+y_0))|} \\
&\leq \frac{L_0}{2}(|x_0-x^*| + |y_0-x^*|) \\
&\quad +L_0|x_1-x^*| + \frac{MN|x_0-x^*|}{2(1-\frac{L_0}{2}(|x_0-x^*|+|y_0-x^*|))} \\
&\leq 2L_0|x_0-x^*| + \frac{MN|x_0-x^*|}{2(1-L_0|x_0-x^*|)} \\
&= q(|x_0-x^*|). \qquad (2.23)
\end{aligned}$$

We have by the third sub-step of method (1.2) for $n=0$ that

$$\begin{aligned}
|y_1 - x^*| \leq\ & |x_1 - x^* - F'(x_1)^{-1}F(x_1)| \\
&+|F'(x_1)^{-1}F'(x^*)||F'(x^*)^{-1}(A_0 - F'(x_1))| \\
&\times|A_0^{-1}F'(x^*)||F'(x^*)^{-1}F(x_1)|. \qquad (2.24)
\end{aligned}$$

Then, we also get by (2.1), (2.20) and (2.22) that

$$\begin{aligned}
|y_1 - x^*| &\leq \frac{L|x_1-x^*|^2}{2(1-L_0|x_1-x^*|)} + \frac{Mq(|x_0-x^*|)|x_1-x^*|}{(1-L_0|x_1-x^*|)(1-p(|x_0-x^*|))} \\
&\leq g_1(|x_0-x^*|)|x_1-x^*| \leq |x_1-x^*| < r, \qquad (2.25)
\end{aligned}$$

which shows (2.11) for $n=1$ and $y_1 \in U(x^*,r)$ where we also used the estimates $|F'(x^*)^{-1}(F'(x_1)-F'(x^*))| \leq L_0|x_1-x^*| < L_0 r < 1$, so $|F'(x_1)^{-1}F'(x^*)| \leq \frac{1}{1-L_0|x_0-x^*|}$. Then, as in (2.19) and (2.21), respectively, we get that

$$|A_1^{-1}F'(x^*)| \leq \frac{1}{1-p(|x_0-x^*|)} \qquad (2.26)$$

and

$$|F'(x^*)^{-1}(A_1 - F'(x_1))| \leq q(|x_1-x^*|). \qquad (2.27)$$

It then follows from (2.1), (2.4), (2.7), (2.20), (2.26) and (2.27) and the last substep of method (1.2) for $n=1$ since

$$x_2 - x^* = x_1 - x^* - F'(x_1)^{-1}F(x_1) + F'(x_1)^{-1}(A_1 - F'(x_1))A_1^{-1}F(x_1), \qquad (2.28)$$

that

$$
\begin{aligned}
&|x_2 - x^*| \\
&\leq |F'(x_1)^{-1}F(x^*)|\left|\int_0^1 F'(x^* + \theta(x_1 - x^*))(x_1 - x^*)d\theta\right| \\
&\quad + |F'(x_1)^{-1}F'(x^*)||F'(x^*)^{-1}(A_1 - F'(x_1))||A_1^{-1}F'(x^*)||F'(x^*)^{-1}F(x_1)| \\
&\leq \frac{L|x_1 - x^*|^2}{2(1 - L_0|x_1 - x^*|)} + \frac{Mq(|x_1 - x^*|)|x_1 - x^*|}{(1 - L_0|x_1 - x^*|)(1 - p(|x_1 - x^*|))} \\
&\leq g_1(|x_1 - x^*|)|x_1 - x^*| \\
&\leq g_1(|x_0 - x^*|)|x_1 - x^*| \leq |x_1 - x^*| < r,
\end{aligned} \tag{2.29}
$$

which shows (2.12) for $n = 1$ and $x_2 \in U(x^*, r)$, since function $g_1$ is increasing on $[0, r)$ and $|x_1 - x^*| \leq |x_0 - x^*|$. By simply replacing $x_0, y_0, x_1$ by $x_k, y_k, x_{k+1}$ in the preceding estimates we arrive at estimates (2.11) and (2.12). Using the estimate

$$
|x_{k+1} - x^*| \leq g_1(|x_0 - x^*|)|x_k - x^*| \leq c|x_k - x^*| < r, \tag{2.30}
$$

we deduce that $x_{k+1} \in U(x^*, r)$, $\lim_{k\to\infty} x_k = x^*$. To show the uniqueness part, let $Q = \int_0^1 F'(y^* + \theta(x^* - y^*)d\theta$ for some $y^* \in D_0$ with $F(y^*) = 0$. Using (2.6) we get that

$$
\begin{aligned}
|F'(x^*)^{-1}(Q - F'(x^*))| &\leq \int_0^1 L_0|y^* + \theta(x^* - y^*) - x^*|d\theta \\
&\leq \int_0^1 (1 - \theta)|x^* - y^*|d\theta \leq \frac{L_0}{2}T < 1.
\end{aligned} \tag{2.31}
$$

It follows from (2.31) and the Banach Lemma on invertible functions that $Q$ is invertible. Finally, from the identity $0 = F(x^*) - F(y^*) = Q(x^* - y^*)$, we conclude that $x^* = y^*$. □

*REMARK* 2.2. 1. In view of (2.6) and the estimate

$$
\begin{aligned}
|F'(x^*)^{-1}F'(x)| &= |F'(x^*)^{-1}(F'(x) - F'(x^*)) + I| \\
&\leq 1 + |F'(x^*)^{-1}(F'(x) - F'(x^*))| \leq 1 + L_0|x - x^*|
\end{aligned}
$$

condition (2.8) can be dropped and $M$ can be replaced by

$$
M(t) = 1 + L_0 t
$$

or by

$$
M = M(t) = 2,
$$

since $t \in [0, \frac{1}{L_0})$. In view of (2.7) and (2.9), we can also choose $L = N$.

2. The results obtained here can be used for operators $F$ satisfying autonomous differential equations [3] of the form
$$F'(x) = P(F(x))$$
where $P$ is a continuous operator. Then, since $F'(x^*) = P(F(x^*)) = P(0)$, we can apply the results without actually knowing $x^*$. For example, let $F(x) = e^x - 1$. Then we can choose $P(x) = x + 1$.

3. The radius $r_A = \frac{2}{2L_0+L}$ was shown by us to be the convergence radius of Newton's method [3], [4]
$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \text{ for each } n = 0, 1, 2, \cdots$$
under the conditions (2.6)– (2.7). It follows from the definition that the convergence radius $r$ of the method (1.2) cannot be larger than the convergence radius $r_A$ of the second order Newton's method. As already noted in [3, 4] $r_A$ is at least as large as the convergence ball given by Rheinboldt [22]
$$r_R = \frac{2}{3L}.$$
In particular, for $L_0 < L$ we have that
$$r_R < r$$
and
$$\frac{r_R}{r_A} \to \frac{1}{3} \;\; as \;\; \frac{L_0}{L} \to 0.$$
That is, our convergence ball $r_A$ is at most three times larger than Rheinboldt's [22]. The same value for $r_R$ was given by Traub [24].

4. It is worth noticing that method (1.2) is not changing when we use the conditions of Theorem 2.1 instead of the stronger conditions used in [16]. Moreover, we can compute the computational order of convergence (COC) defined by
$$\xi = \ln\left(\frac{|x_{n+1} - x^*|}{|x_n - x^*|}\right) / \ln\left(\frac{|x_n - x^*|}{|x_{n-1} - x^*|}\right)$$
or the approximate computational order of convergence
$$\xi_1 = \ln\left(\frac{|x_{n+1} - x_n|}{|x_n - x_{n-1}|}\right) / \ln\left(\frac{|x_n - x_{n-1}|}{|x_{n-1} - x_{n-2}|}\right).$$
This way we obtain in practice the order of convergence in a way that avoids the bounds involving estimates using estimates higher than the second derivative of operator $F$.

## 3. Numerical Examples

We present numerical examples in this section.

*EXAMPLE* 3.1. Let $D = (-\infty, +\infty)$. Define the function $f$ of $D$ by

$$f(x) = \sin(x). \tag{3.1}$$

Then we have for $x^* = 0$ that $L_0 = L = M = N = 1$. The parameters are

$$r_p = 0.5000,\ r_q = 0.5000 \text{ and } r_1 = 0.1896 = r.$$

*EXAMPLE* 3.2. Let $D = [-1, 1]$. Define the function $f$ of $D$ by

$$f(x) = e^x - 1. \tag{3.2}$$

Using (3.2) and $x^* = 0$ we get that $L_0 = e - 1 < L = N = e, M = 2$. The parameters are

$$r_p = 0.1776,\ r_q = 0.1657 \text{ and } r_1 = 0.0503 = r.$$

*EXAMPLE* 3.3. Returning back to the motivational example at the introduction of this study, we have $L_0 = L = N = 146.6629073$, $M = 2$. The parameters are

$$r_p = 0.0026,\ r_q = 0.0025 \text{ and } r_1 = 0.0007 = r.$$

**Conflict of interest:** The author(s) declare(s) that there is no conflict of interest regarding the publication of this manuscript.

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