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SYMMETRIC PROPERTIES OF ORTHOGONALITY OF LINEAR OPERATORS ON $(\mathbb{R}^n, \|.\|_1)$

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Abstract. In this paper we study the orthogonality in the sense of Birkhoff-James of bounded linear operators on $(\mathbb{R}^n, \|.\|_1)$. We prove that $T \perp_B A \Rightarrow A \perp_B T$ for all operators A on $(\mathbb{R}^n, \|.\|_1)$ if and only if T attains norm at only one extreme point, image of which is a left symmetric point of $(\mathbb{R}^n, \|.\|_1)$ and images of other extreme points are zero. We also prove that $A \perp_B T \Rightarrow T \perp_B A$ for all operators A on $(\mathbb{R}^n, \|.\|_1)$ if and only if T attains norm at all extreme points and images of the extreme points are scalar multiples of extreme points.

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1. Introduction

Let $(\mathbb{X}, \|.\|)$ be a normed linear space. For any two elements x, y in X, x is said to be orthogonal to y in the sense of Birkhoff-James[1, 2, 3], written as $x \perp_B y$ iff $\|x\| \leq \|x+\lambda y\|$ for all $\lambda \in K(= \mathbb{R}\ or\ \mathbb{C})$. In [2, 3] James studied many important properties related to the notion of orthogonality in the sense of Birkhoff-James. Orthogonality is related to many important geometric properties of normed linear spaces, including strict convexity, uniform convexity and smoothness of the space. We studied the notion of Birkhoff-James orthogonality in [8, 9, 5, 7, 6]. Let $B(\mathbb{X})$ denote the Banach algebra of all bounded linear operators on $\mathbb{X}$. The notion of Birkhoff-James orthogonality [1] plays a very important role in the geometry of Banach spaces. For any two elements $T, A \in B(\mathbb{X})$, T is said to be orthogonal to A, written as $T \perp_B A$, iff

$$\|T\| \leq \|T + \lambda A\| \ \forall \lambda \in \mathbb{R}.$$

James [2] proved that Birkhoff-James orthogonality is symmetric in a normed linear space $\mathbb{X}$ of three or more dimensions if and only if inner product can be

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defined on $\mathbb{X}$. Since $B(\mathbb{X})$ is not an inner product space, it is interesting to study the symmetry of orthogonality of operators on $\mathbb{X}$.

In [4] we proved that if $\mathbb{H}$ is a real finite dimensional Hilbert space and $T \in B(\mathbb{H})$, then for all $A \in B(\mathbb{H}), A \perp_B T \Rightarrow T \perp_B A$ if and only if $M_T = S_{\mathbb{H}}$, where $M_T = \{x \in S_{\mathbb{H}} : \|Tx\| = \|T\|\}$. We also proved that $T \perp_B A \Rightarrow A \perp_B T$ for all $A \in B(\mathbb{H})$ if and only if T is the zero operator. These results are not true in general if we consider the operators on Banach spaces. In this paper we study the symmetric properties of orthogonality of linear operators on $(\mathbb{R}^n, \|.\|_1)$ in the sense of Birkhoff-James. In [5] Sain and Paul studied orthogonality of linear operators on $(\mathbb{R}^n, \|.\|_\infty)$. We here find a necessary and sufficient condition for a linear operator T on $(\mathbb{R}^n, \|.\|_1)$ to be such that $T \perp_B A$ implies $A \perp_B T$ for all linear operators A on $(\mathbb{R}^n, \|.\|_1)$, we also characterize operators T for which $A \perp_B T$ implies $T \perp_B A$ for all operators A on $(\mathbb{R}^n, \|.\|_1)$. We prove that $T \perp_B A$ implies $A \perp_B T$ for all operators A on $(\mathbb{R}^n, \|.\|_1)$ if and only if T attains norm at only one extreme point, image of which is a left symmetric point of $(\mathbb{R}^n, \|.\|_1)$ and images of other extreme points are zero. We also prove that $A \perp_B T$ implies $T \perp_B A$ for all operators A on $(\mathbb{R}^n, \|.\|_1)$ if and only if T attains norm at all extreme points and images of the extreme points are scalar multiples of extreme points.

From now onwards by $\mathbb{R}^n$ we will mean the normed linear space $\mathbb{R}^n$ equipped with the ℓ_1 norm, which will be denoted by $\|.\|$. We also write e_i for the element $(0, 0, \dots, 0, 1, 0, \dots, 0)$ where the i-th coordinate is 1 and all other coordinates are zero, for any $i \in \{1, 2, \dots, n\}$.

2. Main results

In a normed linear space $\mathbb{X}, x \perp_B y$ may not imply $y \perp_B x$. Motivated by this fact we define the notion of left symmetric and right symmetric points.
Left symmetric point. An element $x \in \mathbb{X}$ is called left symmetric if $x \perp_B y \Rightarrow y \perp_B x$ for all $y \in \mathbb{X}$. In $(\mathbb{R}^2, \|.\|_1)$ $(1,1)$ is a left symmetric point.
Right-symmetric point. An element $x \in \mathbb{X}$ is called right symmetric if $y \perp_B x \Rightarrow x \perp_B y$ for all $y \in \mathbb{X}$.

We first prove the following lemma.

Lemma 2.1. *Extreme points (or their scalar multiples) of the closed unit ball are the only right symmetric points of* $\mathbb{R}^n$.

Proof. We first show that extreme points are right symmetric. Let $(t_1, t_2, \dots, t_n) \perp_B e_i$ for $i \in \{1, 2, \dots, n\}$. Then it follows that $t_i = 0$ i.e. $(t_1, t_2, \dots, t_{i-1}, 0, t_{i+1}, \dots, t_n) \perp_B e_i$. Also $e_i \perp_B (t_1, t_2, \dots, t_{i-1}, 0, t_{i+1}, \dots, t_n)$. So e_i is right symmetric.

Let $(t_1, t_2, ..., t_n)$ be a right symmetric point such that $|t_1|+|t_2|+...+|t_n| = 1$.
Claim: $(t_1, t_2, ..., t_n)$ is an extreme point.

If not, then at least two of $t_1, t_2, ..., t_n$ are non-zero. Without any loss of generality we can assume that $t_1, t_2 \neq 0$. Also at least one of $|t_1|, |t_2|$ is less than or equal to $\frac{1}{2}$. Without any loss of generality we assume that $|t_1| \leq \frac{1}{2}$. We show that $(1, 0, \ldots, 0) \perp_B (t_1, t_2, \ldots, t_n)$. Let $|\lambda| < \frac{1}{|t_1|}$. Then $\|(1, 0, \ldots, 0) + \lambda(t_1, t_2, \ldots, t_n)\| = |1+\lambda t_1|+|\lambda t_2|+\ldots+|\lambda t_n| \geq |1-|\lambda t_1||+|\lambda||t_2|+\ldots+|\lambda||t_n| = 1-|\lambda t_1|+|\lambda|(|t_2|+|t_3|+\ldots+|t_n|) = 1-|\lambda||t_1|+|\lambda|(1-|t_1|) = 1+|\lambda|(1-2|t_1|) \geq 1$. So $(1, 0, \ldots, 0) \perp_B (t_1, t_2, \ldots, t_n)$. But $(t_1, t_2, \ldots, t_n) \not\perp_B (1, 0, \ldots, 0)$, as $\|(t_1, t_2, \ldots, t_n) - t_1(1, 0, \ldots, 0)\| < \|(t_1, t_2, \ldots, t_n)\|$. Hence the proof. $\square$

In the next theorem we characterize the class of operators T which are right symmetric i.e., $A \perp_B T \Rightarrow T \perp_B A$ for all linear operators A on $\mathbb{R}^n$.

Theorem 2.2. *Suppose $T = (t_{ij})$ is a linear operator on $\mathbb{R}^n$. Then for any linear operator A on $\mathbb{R}^n, A \perp_B T \Rightarrow T \perp_B A$ if and only if T attains norm at all extreme points and images of the extreme points are scalar multiples of extreme points.*

Proof. Without any loss of generality we may assume that $\|T\| = 1$. Let $A \perp_B T \Rightarrow T \perp_B A$ for all linear operators A on $\mathbb{R}^n$. Claim: T attains norm at all extreme points.

Suppose $\|T\| > \|Te_1\|$. Define a linear operator A as $Ae_1 = w_0,\ Ae_i = -Te_i,\ i \neq 1$ where $w_0 = (w_1, w_2, \ldots, w_n)$ is such that $\|w_0\| = 1$ and $w_0 \perp_B Te_1$. It is easy to see that A attains norm at e_1 and $Ae_1 \perp_B Te_1$. So we have $A \perp_B T$. Choose $0 < \lambda < 1 - \|Te_1\|$. Then

$$(T + \lambda A) = \begin{pmatrix} t_{11} + \lambda w_1 & (1-\lambda)t_{12} & . & . & (1-\lambda)t_{1n} \\ t_{21} + \lambda w_2 & (1-\lambda)t_{22} & . & . & (1-\lambda)t_{2n} \\ . & . & . & . & . \\ . & . & . & . & . \\ t_{n1} + \lambda w_n & (1-\lambda)t_{n2} & . & . & (1-\lambda)t_{nn} \end{pmatrix}$$

and so we have, $\|T + \lambda A\| = \max\{(|t_{11} + \lambda w_1| + |t_{21} + \lambda w_2| + ... + |t_{n1} + \lambda w_n|), (|1-\lambda||t_{12}| + ... + |1-\lambda||t_{n2}|), ..., (|1-\lambda||t_{1n}| + ... + |1-\lambda)|t_{nn}|)\}$. Now $|t_{11} + \lambda w_1| + |t_{21} + \lambda w_2| + ... + |t_{n1} + \lambda w_n| = \|Te_1 + \lambda w_0\| \leq \|Te_1\| + |\lambda|\|w_0\| < 1 = \|T\|$ (By the choice of λ). Also $|1-\lambda||t_{1j}| + ... + |1-\lambda||t_{nj}| = (1-\lambda)(|t_{1j}| + ... + |t_{nj}|) < (|t_{1j}| + ... + |t_{nj}|) \leq \|T\|$ for all $j \in \{2, ..., n\}$. So $\|T + \lambda A\| < \|T\|$, which shows that $T \not\perp_B A$.

Next claim : Image of any extreme point under T is a right symmetric point.

Suppose Te_1 is not right symmetric i.e., there exists a vector w_0 such that $w_0 \perp_B Te_1$ but $Te_1 \not\perp_B w_0$ and $\|w_0\| > 1$. For the two vectors Te_1, w_0 either $\|Te_1 + \lambda w_0\| \geq 1$ for all $\lambda \geq 0$, or $\|Te_1 + \lambda w_0\| \geq 1$ for all $\lambda \leq 0$. Without any loss of generality we can assume that $\|Te_1 + \lambda w_0\| \geq 1$ for all $\lambda \geq 0$. As $Te_1 \not\perp_B w_0$, there exists $-1 < \lambda_0 < 0$ such that $\|Te_1 + \lambda_0 w_0\| < \|Te_1\| = 1$. Define a linear operator A as $Ae_1 = w_0,\ Ae_i = Te_i,\ \ i \neq 1$. It is easy to see that A attains norm at e_1 and $Ae_1 \perp_B Te_1$. So we have $A \perp_B T$. Then

$$(T + \lambda_0 A) = \begin{pmatrix} t_{11} + \lambda_0 w_1 & (1+\lambda_0)t_{12} & . & . & (1+\lambda_0)t_{1n} \\ t_{21} + \lambda_0 w_2 & (1+\lambda_0)t_{22} & . & . & (1+\lambda_0)t_{2n} \\ . & . & . & . & . \\ . & . & . & . & . \\ t_{n1} + \lambda_0 w_n & (1+\lambda_0)t_{n2} & . & . & (1+\lambda_0)t_{nn} \end{pmatrix}$$

and so we have, $\|T+\lambda_0 A\| = \max\{(|t_{11}+\lambda_0 w_1| + |t_{21}+\lambda_0 w_2| + ... + |t_{n1} + \lambda_0 w_n|), (|1+\lambda_0||t_{12}|+...+|1+\lambda_0||t_{n2}|), ..., (|1+\lambda_0||t_{1n}|+...+|1+\lambda_0||t_{nn}|)\}$. Now $|t_{11}+\lambda_0 w_1|+|t_{21}+\lambda_0 w_2|+...+|t_{n1}+\lambda_0 w_n| = \|Te_1+\lambda_0 w_0\| < 1 = \|T\|$ (by the choice of λ_0). Also $|1+\lambda_0||t_{1j}|+...+|1+\lambda_0||t_{nj}| = (1+\lambda_0)(|t_{1j}|+...+|t_{nj}|) < (|t_{1j}|+...+|t_{nj}|) = \|T\|$ for all $j \in \{2,...,n\}$. So $\|T+\lambda_0 A\| < \|T\|$, which shows that $T \not\perp_B A$. Hence by Lemma 2.1, the image of any extreme point under T is an extreme point.

Conversely, suppose that a linear operator T attains norm at all extreme points and images of the extreme points are scalar multiples of extreme points. So we can assume that T is of the form : $Te_1 = c_1 e_{i_1},\ Te_2 = c_2 e_{i_2}, \ldots, Te_n = c_n e_{i_n}$ where $c_i = \pm 1$ for all $i \in \{1,2,...,n\}$. Let $A = (a_{ij})$ be a linear operator such that $A \perp_B T$.

Case 1.

Let $sgn(c_k) = sgn(a_{i_k k})$ for all $k \in \{1,2,...,n\}$. We first claim that at least one of $a_{i_1 1}, a_{i_2 2}, ..., a_{i_n n}$ is zero. If not, suppose $\lambda = -\min\{|a_{i_1 1}|, |a_{i_2 2}|, ..., |a_{i_k k}|\}$. Then

$$(A+\lambda T) = \begin{pmatrix} a_{11}+\lambda t_{11} & a_{12}+\lambda t_{12} & . & . & a_{1n}+\lambda t_{1n} \\ a_{21}+\lambda t_{21} & a_{22}+\lambda t_{22} & . & . & a_{2n}+\lambda t_{2n} \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{n1}+\lambda t_{n1} & a_{n2}+\lambda t_{n2} & . & . & a_{nn}+\lambda t_{nn} \end{pmatrix}$$

Also $|a_{1j}+\lambda t_{1j}| + |a_{2j}+\lambda t_{2j}| + ... + |a_{nj}+\lambda t_{nj}| = |a_{1j}| + ... + |a_{i_j j}+\lambda c_j| + |a_{(i_j+1)j}| + ... + |a_{nj}| < |a_{1j}| + |a_{2j}| + ... + |a_{nj}|$ for all $j \in \{1,2,...,n\}$. So $\|A+\lambda T\| < \|A\|$, which is a contradiction. Hence at least one of $a_{i_1 1}, a_{i_2 2}, ..., a_{i_n n}$ is zero. Without loss of generality we can assume that $a_{i_1 1} = 0$. Then $\|T+\lambda A\| \geq |t_{11}+\lambda a_{11}| + |t_{21}+\lambda a_{21}| + ... + |t_{i_1 1}+\lambda a_{i_1 1}| + ... + |t_{n1}+\lambda a_{n1}| = |\lambda a_{11}| + |\lambda a_{21}| + ... + |c_1| + |\lambda a_{i_1+1 j}| + ... + |\lambda a_{n1}| \geq |c_1| = 1 = \|T\|$. So $T \perp_B A$.

Case 2. Let $sgn(c_k) = -sgn(a_{i_k k})$ for all $k \in \{1,2,...,n\}$. As in Case 1 we can show that $A \perp_B T \Rightarrow T \perp_B A$.

Case 3. There exist k, l such that $sgn(c_k) = sgn(a_{i_k k})$ and $sgn(c_l) = -sgn(a_{i_l l})$. Without loss of generality we can assume that $k = 1, l = 2$. Let $\lambda > 0$. Then $\|T+\lambda A\| \geq |t_{11}+\lambda a_{11}| + |t_{21}+\lambda a_{21}| + ... + |t_{i_1 1}+\lambda a_{i_1 1}| + ... + |t_{n1}+\lambda a_{n1}| = |\lambda a_{11}| + |\lambda a_{21}| + ... + |c_1+\lambda a_{i_1 1}| + |\lambda a_{i_1+1 j}| + ... + |\lambda a_{n1}| \geq |c_1+\lambda a_{i_1 1}| = |sgn(c_1)|c_1| + \lambda sgn(a_{i_1 1})|a_{i_1 1}|| = ||c_1| + \lambda|a_{i_1 1}|| = |c_1| + \lambda|a_{i_1 1}| \geq |c_1| = 1 = \|T\|$. Also $\|T-\lambda A\| \geq |t_{12}-\lambda a_{12}| + |t_{22}-\lambda a_{22}| + ... + |t_{i_2 2}-\lambda a_{i_2 2}| + ... + |t_{n2}-\lambda a_{n2}| = |\lambda a_{12}| + |\lambda a_{22}| + ... + |c_2-\lambda a_{i_2 2}| + |\lambda a_{i_2+12}| + ... + |\lambda a_{n2}| \geq |c_2-\lambda a_{i_2 2}| = |sgn(c_2)|c_2| - \lambda sgn(a_{i_2 2})|a_{i_2 2}|| = ||c_2| + \lambda|a_{i_2 2}|| = |c_2| + \lambda|a_{i_2 2}| \geq |c_2| = 1 = \|T\|$. Hence $T \perp_B A$. This completes the proof. $\square$

For any two linear operators T, A on $\mathbb{R}^n$, $T \perp_B A$ may not imply $A \perp_B T$. We next prove a theorem which characterizes those T for which $A \perp_B T \Rightarrow T \perp_B A$ for all A on $\mathbb{R}^n$.

Theorem 2.3. *Suppose $T = (t_{ij})$ is a linear operator on $\mathbb{R}^n$. Then for any linear operator A on $\mathbb{R}^n$, $T \perp_B A \Rightarrow A \perp_B T$ if and only if T attains norm at only one extreme point, image of which is a left symmetric point of $\mathbb{R}^n$ and images of other extreme points are zero.*

Proof. Suppose T attains norm at only one extreme point and images of other extreme points are zero. Without any loss of generality we can assume that T attains norm only at $\pm e_1$. Let A be an operator such that $T \perp_B A$. Then by Theorem 2.1 of Sain and Paul [8] $Te_1 \perp_B Ae_1$. As Te_1 is a left symmetric point, it follows that $Ae_1 \perp_B Te_1$. Also $Ae_i \perp_B Te_i = 0$ for all $i \in \{2, 3, .., n\}$. Clearly, A attains norm at an extreme point e_j (say) i.e. $\|Ae_j\| = \|A\|$. Thus A attains norm at e_j such that $Ae_j \perp_B Te_j$ and so we get $A \perp_B T$.

Conversely, let $T \perp_B A \Rightarrow A \perp_B T$ for all A on $\mathbb{R}^n$. Clearly, T attains norm at an extreme point, say at e_{i_0}.

Claim: $Te_i = 0$ for all $i \in \{1, 2, ..., n\} - \{i_0\}$.

Suppose $Te_j \neq 0$ for some $j \neq i_0$. Define a linear operator A on $\mathbb{R}^n$ as $Ae_j = Te_j,\ Ae_i = 0,\ \ i \neq j$. It is easy to verify that A attains norm only at $\pm e_j$. Also $T \perp_B A$, as $Te_{i_0} \perp_B Ae_{i_0}$ and $\|Te_{i_0}\| = \|T\|$. By Theorem 2.1 of [8] we get $A \not\perp_B T$ as $Ae_j \not\perp_B Te_j$ and $M_A = \{\pm e_j\}$. So $Te_i = 0$ for all $i \neq i_0$. Our next claim is that Te_{i_0} is a left symmetric point. Suppose Te_{i_0} is not a left symmetric point, i.e. there exists w such that $Te_{i_0} \perp_B w$, but $w \not\perp_B Te_{i_0}$. Define a linear operator A on $\mathbb{R}^n$ as $Ae_{i_0} = w,\ Ae_i = 0,\ i \neq i_0$. It is easy to verify that A attains norm only at $\pm e_{i_0}$. Also $T \perp_B A$, as $Te_{i_0} \perp_B Ae_{i_0}$ and $\|Te_{i_0}\| = \|T\|$. But $A \not\perp_B T$ as $Ae_{i_0} \not\perp_B Te_{i_0}$. Thus we get $T \perp_B A$, but $A \not\perp_B T$. This contradiction completes the proof. □

Remark 2.4. Any operator T on $\mathbb{R}^2$ satisfying the property as in the Theorem 2.3 is of the form $\begin{pmatrix} 1 & 0 \\ \pm 1 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ 0 & \pm 1 \end{pmatrix}$ or their scalar multiples.

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