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REGULAR SEMIGROUPS OF PARTIAL TRANSFORMATIONS PRESERVING A FENCE $\mathbb{N}$

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Abstract. Semigroups of order-preserving transformations have been extensively studied for finite chains. We study the monoid $OP_{\mathbb{N}}$ of all order-preserving partial transformations on the set $\mathbb{N}$ of natural numbers, where the partial order is a fence (also called zigzag poset). The monoid $OP_{\mathbb{N}}$ is not regular. In this paper, we determine particular maximal regular subsemigroups of $OP_{\mathbb{N}}$ and show that $OP_{\mathbb{N}}$ has infinitely many maximal regular subsemigroups.

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1. Introduction

Semigroups of transformations on a set X preserving a partially order set $(X;\preceq)$ have long been considered in the literature, starting with the works by Aĭzenštat [1] and Popova [15], in 1962. In the case that $(X;\preceq)$ is a linearly ordered set, semigroups of order-preserving transformations have been the object of study by several authors and several papers (e.g. [6, 7, 8, 9, 10, 11]). In particular, the semigroup of all transformations preserving a linear order is regular. Semigroups of order-preserving transformations on $(X;\preceq)$, where $(X;\preceq)$ is any infinite partially ordered set, have not been extensively studied. For example in [12], the authors investigated rank properties of endomorphisms of infinite partially ordered sets. We consider semigroups of transformations preserving a particular non-linear order on a countable infinite set, where the partial order is a fence. Let us recall that a fence (also called a zigzag poset) is a partially ordered set $(X;\preceq)$, where the order on X is

$$a_1 \prec a_2 \succ a_3 \prec a_4 \succ a_5 \prec \cdots \succ a_{2m-1} \prec a_{2m} \succ a_{2m+1} \prec \cdots$$

or

$$a_1 \succ a_2 \prec a_3 \succ a_4 \prec a_5 \succ \cdots \prec a_{2m-1} \succ a_{2m} \prec a_{2m+1} \succ \cdots ,$$

whenever $X = \{a_1, a_2, \ldots\}$ is finite or countable infinite. The definition of the partial order $\preceq$ is self-explanatory. Every element of X is either minimal or

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maximal. The number of order-preserving maps of fences and of crowns as well as of maps between fences and crowns are calculated in [3, 5, 16]. Recently, regular semigroups of transformations preserving a fence were characterized in [2, 13].

Let $\mathbb{N}$ be the set of natural numbers and let $\leq$ be the natural order on $\mathbb{N}$. Throughout the paper, we consider the following binary relation $\preceq$ on $\mathbb{N}$:

$$n \prec n+1 \quad \text{if} \quad n \text{ is odd;}$$
$$n \succ n+1 \quad \text{if} \quad n \text{ is even.}$$

It is easy to verify that $(\mathbb{N};\preceq)$ is a fence. Where there is no possibility of confusion, we will write $\mathbb{N}$ instead of $(\mathbb{N};\preceq)$. Let Σ be any subset of $\mathbb{N}$. We say that Σ is a fence if $(\Sigma;\preceq \ \cap \ (\Sigma\times\Sigma))$ is a fence, or in other words, $x\in\Sigma$ implies either $x+1\in\Sigma$ or $x+k\notin\Sigma$ for all $k\in\mathbb{N}$. Note that $x\perp y$ $(x,y\in\mathbb{N})$ means that x and y are comparable in the fence $\mathbb{N}$, while $x\|y$ means that x and y are incomparable in the fence $\mathbb{N}$. If Σ_1 and Σ_2 are subsets of $\mathbb{N}$, we write $\Sigma_1\perp\Sigma_2$ (and $\Sigma_1\|\Sigma_2$, respectively) if $x_1\perp x_2$ (and $x_1\|x_2$, respectively) for all $x_1\in\Sigma_1$ and all $x_2\in\Sigma_2$.

In this paper, we study regular semigroups of partial transformations preserving the fence $\mathbb{N}$. Let $Y\subseteq\mathbb{N}$. A mapping $\alpha: Y\to\mathbb{N}$ is called a *partial transformation* on $\mathbb{N}$ with domain Y (in symbols: dom $\alpha=Y$). If $Y=\mathbb{N}$, then α is called a *full transformation* on $\mathbb{N}$. We write transformations to the right of their argument and compose from left to right. The *range* of a partial transformation α is denoted by ran $\alpha=\{y:$ there exists $x\in$ dom α such that $x\alpha=y\}$ and the *kernel* of α is denoted by ker $\alpha=\{(x,y): x,y\in$ dom α and $x\alpha=y\alpha\}$. The *rank* of α is the size of its range, denoted by rank $\alpha:=\mid$ ran $\alpha\mid$. For $\Sigma\subseteq\mathbb{N}$, the set $\Sigma\alpha^{-1}:=\{x\in$ dom $\alpha: x\alpha\in\Sigma\}$ denotes the pre-image of Σ under α. If Σ is a singleton set, say $\Sigma=\{x\}$, then we write $x\alpha^{-1}$ rather than $\{x\}\alpha^{-1}$. We denote by $\alpha|_\Sigma$ the *restriction* of α to $\Sigma\cap$ dom α and by id_Σ the identity mapping on Σ. A subset T of dom α is said to be a *transversal* of ker α if the intersection of T and any class of the partition corresponding to the equivalence relation ker α is a singleton set.

We say that a partial transformation α on $\mathbb{N}$ is *order-preserving* (or preserving the fence) if $x\preceq y$ implies $x\alpha\preceq y\alpha$, for all $x,y\in$ dom α. Note that ran α is a fence, whenever α is order-preserving and dom α is a fence [2]. Of course, the empty transformation $\varnothing$ and the identity mapping $\mathrm{id}_\mathbb{N}$ on $\mathbb{N}$ are order-preserving partial transformations on $\mathbb{N}$. If α is order-preserving, then $\alpha|_\Sigma$ is also order-preserving for any set $\Sigma\subseteq$ dom α. We denote by $OP_\mathbb{N}$ the set of all order-preserving partial transformations on $\mathbb{N}$ and remark that $OP_\mathbb{N}$ is a semigroup under composition. For a subset A of $OP_\mathbb{N}$, the least subsemigroup of $OP_\mathbb{N}$ containing A is denoted by $\langle A\rangle$.

A classical topic in the study of semigroups is the characterization of the regularity. An element x of a semigroup is said to be *regular* if there exists y in this semigroup such that $xyx=x$. A semigroup is called *regular* if all its elements are regular. Groups are of course regular semigroup, but the class of regular semigroups is vastly more extensive than the class of groups. In [14], regular elements of some order-preserving transformation semigroups are investigated. A characterization of the maximal regular subsemigroups of ideals of order-preserving or order-reversing transformations was given in [4]. Note that the semigroup $OP_{\mathbb{N}}$ of all partial transformations preserving the fence $\mathbb{N}$ is not regular.

Already in 1972, B. Schein asked for maximal regular subsemigroups of particular semigroups of binary relations. After that many authors studied maximal regular subsemigroups of transformations preserving an order (e.g. [4, 17, 18]). The purpose of this paper is to study (maximal) regular subsemigroups of $OP_{\mathbb{N}}$. A subsemigroup C of $OP_{\mathbb{N}}$ is called a maximal regular subsemigroup if C is regular and any subsemigroup of $OP_{\mathbb{N}}$ having C as proper subsemigroup is not regular. In the next section, we describe the regular elements in $OP_{\mathbb{N}}$ and the surjective transformations in $OP_{\mathbb{N}}$. In Section 3, we determine two maximal regular subsemigroups of $OP_{\mathbb{N}}$ whose union covers all the regular transformations with rank ≤ 2. The main result of this paper is given in Section 4. It seems almost impossible to determine all maximal regular subsemigroups of $OP_{\mathbb{N}}$ but we classify a countable infinite set of maximal regular subsemigroups of $OP_{\mathbb{N}}$.

2. The Regular Elements in $OP_{\mathbb{N}}$

This section is devoted the set $Reg(OP_{\mathbb{N}})$ of all regular elements in $OP_{\mathbb{N}}$. First of all, we give a straightforward description of the elements in $Reg(OP_{\mathbb{N}})$. For the sake of completeness, we give a proof. Since the proof uses only the properties of a partially ordered set, the statement is true for any semigroup of order-preserving partial transformations.

Proposition 2.1. *Let $\alpha \in OP_{\mathbb{N}} \setminus \{\varnothing\}$. Then α is regular if and only if there exists a subset Y of dom α such that $\alpha|_Y$ is a bijection from Y to ran α and $\big(\alpha|_Y\big)^{-1} \in OP_{\mathbb{N}}$.*

Proof. Let α be regular. Then there exists $\beta \in OP_{\mathbb{N}}$ such that $\alpha\beta\alpha = \alpha$. It is clear that ran $\alpha \subseteq$ dom β and ran $\alpha\beta \subseteq$ dom α. We show that $Y :=$ ran $\alpha\beta$ satisfies the required properties. First, we observe that $\alpha|_{\text{ran }\alpha\beta} : \text{ran }\alpha\beta \to \text{ran }\alpha$ is a bijection from ran $\alpha\beta$ to ran α. It remains to show that $(\alpha|_{\text{ran }\alpha\beta})^{-1} \in OP_{\mathbb{N}}$. Let $a, b \in$ ran α with $a \prec b$. Since $\alpha|_{\text{ran }\alpha\beta}$ is a bijection from ran $\alpha\beta$ to ran α, there exist a unique $a_1 \in$ ran $\alpha\beta$ and a unique $b_1 \in$ ran $\alpha\beta$ such that $a_1\alpha|_{\text{ran }\alpha\beta} = a$ and $b_1\alpha|_{\text{ran }\alpha\beta} = b$. Since $a_1, b_1 \in$ ran $\alpha\beta$, there are $a_2, b_2 \in$ ran $\alpha \subseteq$ dom β such that $a_2\beta = a_1$ and $b_2\beta = b_1$. Since $a_2, b_2 \in$ ran α, there are $a_3, b_3 \in$ dom α such that $a_3\alpha = a_2$ and $b_3\alpha = b_2$. Thus, we obtain $a = a_3\alpha\beta\alpha$ as well as $b = b_3\alpha\beta\alpha$ and so

$$a_2 = a_3\alpha = a_3\alpha\beta\alpha = a \prec b = b_3\alpha\beta\alpha = b_3\alpha = b_2.$$

Since $a_2, b_2 \in$ ran $\alpha \subseteq$ dom β and $\beta \in OP_{\mathbb{N}}$, we obtain $a_1 = a_2\beta \prec b_2\beta = b_1$. Hence,

$$a(\alpha|_{\text{ran }\alpha\beta})^{-1} = a_1 \prec b_1 = b(\alpha|_{\text{ran }\alpha\beta})^{-1}.$$

Conversely, suppose that there exists a subset Y of dom α such that $\alpha|_Y$ is a bijection from Y to ran α and $\left(\alpha|_Y\right)^{-1} \in OP_{\mathbb{N}}$. Then, we have $\alpha(\alpha|_Y)^{-1}\alpha = \alpha\mathrm{id}_{\mathrm{ran}\ \alpha} = \alpha$, i.e. α is regular. $\square$

Note that the set Y in Proposition 2.1 is nothing but a transversal of ker α. We prefer the setting in Proposition 2.1 since it is more practical for our proofs. We will frequently use this proposition in our proofs without mentioning it.

It is clear that any regular subsemigroup of $OP_{\mathbb{N}}$ is a closed set (under composition) of regular elements in $OP_{\mathbb{N}}$ but not conversely, i.e. a closed set (under composition) of regular elements in $OP_{\mathbb{N}}$ does not need to be a regular subsemigroup of $OP_{\mathbb{N}}$. For example, let us consider the set

$$\mathrm{Sur} := \{\alpha \in Reg(OP_{\mathbb{N}}) : \alpha \text{ is surjective}\}.$$

Proposition 2.2. *Sur is a semigroup which is not regular.*

Proof. First, we show that Sur is closed under composition. Let $\gamma, \beta \in$ Sur. Then $\gamma\beta$ is surjective. Since $\gamma, \beta \in Reg(OP_{\mathbb{N}})$, there exist subsets $Y_1 \subseteq$ dom γ and $Y_2 \subseteq$ dom β such that $\gamma|_{Y_1} : Y_1 \to$ ran $\gamma = \mathbb{N}$ as well as $\beta|_{Y_2} : Y_2 \to$ ran $\beta = \mathbb{N}$ are bijective and $(\gamma|_{Y_1})^{-1}, (\beta|_{Y_2})^{-1} \in OP_{\mathbb{N}}$. It is easy to verify that

$$\gamma\beta|_{Y_2\gamma^{-1}\cap Y_1} : Y_2\gamma^{-1} \cap Y_1 \to \mathrm{ran}\ \gamma\beta$$

is bijective and $(\gamma\beta|_{Y_2\gamma^{-1}\cap Y_1})^{-1} \in OP_{\mathbb{N}}$. This shows that $\gamma\beta \in Reg(OP_{\mathbb{N}})$ and hence Sur is a semigroup. We now consider the partial transformation $\alpha : \mathbb{N} \setminus \{1, 2\} \to \mathbb{N}$ defined by $x\alpha = x - 2$ for all $x \in \mathbb{N} \setminus \{1, 2\}$. It is clear that $\alpha \in$ Sur. Assume that there exists $\delta \in$ Sur with $\alpha\delta\alpha = \alpha$. Then ran $\left(\delta|_{\mathrm{ran}\ \alpha}\right) =$ dom α, where $\mathbb{N} =$ ran $\alpha \subseteq$ dom δ and we conclude that dom $\delta = \mathbb{N}$ and so ran $\delta = \mathbb{N}\delta = \mathbb{N} \setminus \{1, 2\}$ (otherwise, δ is not a function). It contradicts with the assumption that δ is surjective. Therefore, there is no $\delta \in$ Sur such that $\alpha\delta\alpha = \alpha$ and so Sur is not regular. $\square$

Now, we give a kind of constructive description of the transformations in Sur.

Proposition 2.3. *Sur is the set of all partial transformations α on $\mathbb{N}$ such that there are a natural number k and a transformation $\beta \in OP_{\mathbb{N}}$ with the following properties:*

(*i*) *dom* $\beta = \{i \in \mathbb{N} : 1 \leq i \leq 2(k-1)\} \cap$ *dom* α;

(*ii*) $(2(k-1))\beta \in \{1, 2\}$, *whenever* $2(k-1) \in$ *dom* β;

(*iii*) $\alpha|_{dom\ \beta} = \beta$;

(*iv*) $(2(k-1)+i)\alpha = i$ *for all* $i \in \mathbb{N}$.

Proof. Clearly, a partial transformation α satisfying the given properties is order-preserving and surjective. Let $Y := \{2(k-1)+i : i \in \mathbb{N}\} \subseteq$ dom α.

It is easy to verify that $\alpha|_Y : Y \to \mathbb{N}$ is bijective and $(\alpha|_Y)^{-1} \in OP_{\mathbb{N}}$, i.e. $\alpha \in Reg(OP_{\mathbb{N}})$.

Conversely, let $\alpha \in$ Sur. Then $\alpha \in Reg(OP_{\mathbb{N}})$ and ran $\alpha = \mathbb{N}$. Since $\alpha \in Reg(OP_{\mathbb{N}})$, there exists a subset Y of dom α such that $\alpha|_Y : Y \to \mathbb{N}$ is bijective and $(\alpha|_Y)^{-1} \in OP_{\mathbb{N}}$. Since ran $\alpha = \mathbb{N}$ is a fence and $(\alpha|_Y)^{-1} \in OP_{\mathbb{N}}$, we obtain that Y is a fence. Let a be the least natural number in Y with respect to the natural order of $\mathbb{N}$. Assume that $a\alpha = x$ for some $x \in \mathbb{N}\setminus\{1\}$. Since $x-1, x+1 \in$ ran $\alpha = \mathbb{N}$ and $\alpha|_Y : Y \to \mathbb{N}$ is bijective, there exist $(x-1)', (x+1)' \in Y$ such that $(x-1)'\alpha = x-1$ and $(x+1)'\alpha = x+1$. Since $(\alpha|_Y)^{-1} \in OP_{\mathbb{N}}$ and either $x-1 \prec x \succ x+1$ or $x-1 \succ x \prec x+1$, we obtain that either $(x-1)' \prec a \succ (x+1)'$ or $(x-1)' \succ a \prec (x+1)'$. Because $(x-1)' \neq (x+1)'$, this provides that either $(x-1)' = a-1$ or $(x+1)' = a-1$. It contradicts with a is the least natural number in Y. So, we have $a\alpha = 1$. Since $1 \prec 2$ in ran $\alpha = \mathbb{N}$, from $\alpha|_Y : Y \to \mathbb{N}$ is bijective and $(\alpha|_Y)^{-1} \in OP_{\mathbb{N}}$, we conclude that $a = 1(\alpha|_Y)^{-1} \prec 2(\alpha|_Y)^{-1}$. This shows that a is odd. We choose $k \in \mathbb{N}$ such that $2(k-1)+1 = a$. Using the facts that $a\alpha = 1$ and that the image of $\{a, \dots, a+i\}$ under $\alpha|_Y$ is a fence of size $i+1$ for all $i \in \mathbb{N}$, we obtain recursively that $(a+i)\alpha = i+1$ for all $i \in \mathbb{N}$. So, $(2(k-1)+i)\alpha = i$ for all $i \in \mathbb{N}$.

Let us put $\beta := \alpha|_{\{1,\dots,a-1\}\cap \text{dom } \alpha}$. Clearly, $\beta \in OP_{\mathbb{N}}$ and dom $\beta = \{i \in \mathbb{N} : 1 \leq i \leq 2(k-1)\} \cap$ dom α. Suppose $2(k-1) \in$ dom β. Since $2(k-1) \succ 2(k-1)+1 = a$ in dom α, we have that $(2(k-1))\alpha \succeq (2(k-1)+1)\alpha = a\alpha = 1$. This implies that either $(2(k-1))\alpha = 1$ or $(2(k-1))\alpha = 2$. □

Note that Inj $:= \{\alpha \in OP_{\mathbb{N}} : \alpha$ is injective and $\alpha^{-1} \in OP_{\mathbb{N}}\} = \{\alpha \in Reg(OP_{\mathbb{N}}) : \alpha$ is injective$\}$ is an inverse subsemigroup of $OP_{\mathbb{N}}$. It is well known that a partial transformation α on $\mathbb{N}$ is idempotent if $\alpha|_{\text{ran } \alpha}$ is the identity mapping on ran α. Any idempotent element in $OP_{\mathbb{N}}$ is regular in $OP_{\mathbb{N}}$. But $OP_{\mathbb{N}}$ is not orthodox, i.e. the idempotent elements in $OP_{\mathbb{N}}$ do not form a regular semigroup.

3. Regular Transformations with rank ≤ 2

In this section, we study regular subsemigroups of $OP_{\mathbb{N}}$ containing regular transformations with rank ≤ 2. The set

$$I_1 := \{\alpha \in OP_{\mathbb{N}} : \text{rank } \alpha \leq 1\}$$

is a regular subsemigroup of $OP_{\mathbb{N}}$ since I_1 is an ideal consisting entirely of regular elements in $OP_{\mathbb{N}}$.

Remark 3.1. *I_1 is contained in all maximal regular subsemigroups of $OP_{\mathbb{N}}$.*

Proof. Let C be any maximal regular subsemigroup of $OP_{\mathbb{N}}$. Since I_1 and C are regular subsemigroups of $OP_{\mathbb{N}}$ and I_1 is an ideal of $OP_{\mathbb{N}}$, we obtain that $I_1 \cup C$ is a regular subsemigroup of $OP_{\mathbb{N}}$. Since C is a maximal regular subsemigroup of $OP_{\mathbb{N}}$, we have that $C = I_1 \cup C$ and so $I_1 \subseteq C$. □

Not all order-preserving partial transformations with rank = 2 are regular, for example the partial transformation $\alpha \in OP_{\mathbb{N}}$ with dom $\alpha = \mathbb{N} \setminus \{2\}$ defined by $1\alpha = 1$ and $x\alpha = 2$ for all $x \in \mathbb{N} \setminus \{1, 2\}$ is not regular in $OP_{\mathbb{N}}$.

Proposition 3.2. *Let $\alpha \in OP_{\mathbb{N}}$ with rank $\alpha = 2$, say ran $\alpha = \{a, b\}$. Then the following statements are equivalent:*

(*i*) $\alpha \in Reg(OP_{\mathbb{N}})$;

(*ii*) $a \| b$ *if and only if* $a\alpha^{-1} \| b\alpha^{-1}$.

Proof. Suppose that $\alpha \in Reg(OP_{\mathbb{N}})$. Then there exists a subset Y of dom α such that $\alpha|_Y$ is a bijection from Y to ran $\alpha = \{a, b\}$ and $(\alpha|_Y)^{-1} \in OP_{\mathbb{N}}$. If $a \| b$, then $\alpha \in OP_{\mathbb{N}}$ implies $a\alpha^{-1} \| b\alpha^{-1}$. If $a\alpha^{-1} \| b\alpha^{-1}$, then $(\alpha|_Y)^{-1} \in OP_{\mathbb{N}}$ implies $a \| b$.

Conversely, suppose that $a \| b$ if and only if $a\alpha^{-1} \| b\alpha^{-1}$. If $a \| b$, then we let Y be a transversal of ker α. If $a \perp b$, then there are $a' \in a\alpha^{-1}, b' \in b\alpha^{-1}$ such that $a' \perp b'$ and we set $Y := \{a', b'\}$. It is easy to verify that in both cases $\alpha|_Y$ is a bijection from Y to ran α and $(\alpha|_Y)^{-1} \in OP_{\mathbb{N}}$, i.e. $\alpha \in Reg(OP_{\mathbb{N}})$. □

In the remainder of this section, we will frequently use the following well known fact.

Remark 3.3. *Let $\alpha, \beta \in OP_{\mathbb{N}}$. Then rank $\alpha\beta \leq \min\{$rank α, rank $\beta\}$, where $\min\{$rank α, rank $\beta\}$ means the least one of both cardinals rank α and rank β.*

Let now $K_1 := \{\alpha \in Reg(OP_{\mathbb{N}}) :$ ran α is a fence and rank $\alpha = 2\} \cup I_1$.

Proposition 3.4. *K_1 is a regular subsemigroup of $OP_{\mathbb{N}}$.*

Proof. First, we show that K_1 is a subsemigroup of $OP_{\mathbb{N}}$. Let $\alpha, \beta \in K_1$. By Remark 3.3, we have that rank $\alpha\beta \leq 2$. If rank $\alpha\beta \leq 1$, then $\alpha\beta \in I_1 \subseteq K_1$. Suppose that rank $\alpha\beta = 2$. Then rank α = rank $\beta = 2$ and ran $\alpha\beta$ = ran β as well as dom $\alpha\beta$ = dom α. Suppose that ran $\alpha = \{a_1, a_2\}$ with $a_1 \prec a_2$. There are $x_1, x_2 \in$ dom α = dom $\alpha\beta$ with $x_1\alpha = a_1$ and $x_2\alpha = a_2$ such that $x_1 \prec x_2$ by Proposition 3.2. Since $\alpha, \beta \in OP_{\mathbb{N}}$, we obtain $x_1\alpha\beta \prec x_2\alpha\beta$, where ran $\alpha\beta$ = ran $\beta = \{a_1\beta, a_2\beta\} = \{x_1\alpha\beta, x_2\alpha\beta\}$. This provides that $\alpha\beta \in Reg(OP_{\mathbb{N}})$ by Proposition 3.2 and so $\alpha\beta \in K_1$.

Now, we show that K_1 is a regular subsemigroup of $OP_{\mathbb{N}}$. Since I_1 is a regular semigroup and $I_1 \subset K_1$, we have only to consider the elements in $K_1 \setminus I_1$. Let $\alpha \in K_1 \setminus I_1$. Assume now that ran $\alpha = \{b_1, b_2\}$ with $b_1 \prec b_2$. Since $\alpha \in Reg(OP_{\mathbb{N}})$, there are $a_1, a_2 \in$ dom α with $a_1\alpha = b_1, a_2\alpha = b_2$, and $a_1 \prec a_2$ by Proposition 3.2. Let us consider the partial transformation $\beta : \{b_1, b_2\} \to \{a_1, a_2\}$ defined by $b_1\beta = a_1$ and $b_2\beta = a_2$. It is easy to verify that $\beta \in K_1$ and $\alpha\beta\alpha = \alpha \mathrm{id}_{\mathrm{ran}\ \alpha} = \alpha$. This shows that α is regular in K_1. Altogether, we have that K_1 is a regular subsemigroup of $OP_{\mathbb{N}}$. □

As an immediately consequence, we obtain the following:

Corollary 3.5. *$K_1 \cup \{id_{\mathbb{N}}\}$ is a regular subsemigroup of $OP_{\mathbb{N}}$.*

Proof. The monoid $K_1 \cup \{id_{\mathbb{N}}\}$ corresponding to the regular semigroup K_1 is regular. □

In order to prove that $K_1 \cup \{\mathrm{id}_{\mathbb{N}}\}$ is a maximal regular subsemigroup of $OP_{\mathbb{N}}$, we need two technical lemmas. The first one states that the union of the semigroups K_1 and Sur is also a semigroup. The second one shows that $\mathrm{id}_{\mathbb{N}}$ is the only bijection from $\mathbb{N}$ to $\mathbb{N}$ which preserves the fence $\mathbb{N}$.

Lemma 3.6. *$K_1 \cup Sur$ is a subsemigroup of $OP_{\mathbb{N}}$.*

Proof. Proposition 2.2 and Theorem 3.4 show that both Sur and K_1 are closed. Let $\alpha \in K_1$ and $\beta \in$ Sur. By Remark 3.3, we have that rank $\alpha\beta \leq 2$ and rank $\beta\alpha \leq 2$. If rank $\alpha\beta$, rank $\beta\alpha \leq 1$, then $\alpha\beta, \beta\alpha \in I_1 \subseteq K_1$.

Assume now that rank $\alpha\beta = 2$. Then rank $\alpha = 2$ and we have that ran $\alpha \subseteq$ dom β and dom $\alpha\beta =$ dom α. Since ran $\alpha \subseteq$ dom β and ran α is a fence, we have that ran $\alpha\beta = (\text{ran } \alpha)\beta$ is a fence. It remains to show that $\alpha\beta \in Reg(OP_{\mathbb{N}})$. Suppose that ran $\alpha = \{b_1, b_2\}$ and ran $\alpha\beta = \{x_1, x_2\}$ with $b_1 \prec b_2$ and $x_1 \prec x_2$. Then there exist $a_1 \in b_1\alpha^{-1}$ and $a_2 \in b_2\alpha^{-1}$ such that $a_1 \prec a_2$ by Proposition 3.2. We conclude that $a_1\alpha\beta = x_1$ and $a_2\alpha\beta = x_2$, i.e. $a_1 \in x_1(\alpha\beta)^{-1}$ and $a_2 \in x_2(\alpha\beta)^{-1}$, since $x_1 \prec x_2, b_1 \prec b_2$, and ran $\alpha\beta = \{x_1, x_2\}$. We obtain $\alpha\beta \in Reg(OP_{\mathbb{N}})$ by Proposition 3.2.

Next, we assume that rank $\beta\alpha = 2$. Since ran $\beta = \mathbb{N}$, we have that dom $\alpha \subseteq$ ran β and ran $\beta\alpha =$ ran α, say ran $\alpha = \{x_1, x_2\}$ with $x_1 \prec x_2$. It remains to show that $\beta\alpha \in Reg(OP_{\mathbb{N}})$. Since $\alpha \in K_1$, there exist $b_1 \in x_1\alpha^{-1}$ and $b_2 \in x_1\alpha^{-1}$ such that $b_1 \prec b_2$ by Proposition 3.2. Because $\beta \in Reg(OP_{\mathbb{N}})$, there exists a subset Y of dom β such that $\beta|_Y$ is a bijection from Y to ran $\beta = \mathbb{N}$ and $(\beta|_Y)^{-1} \in OP_{\mathbb{N}}$. Since $b_1 \prec b_2$ in dom $\alpha \subseteq$ ran β, we obtain $b_1(\beta|_Y)^{-1} \prec b_2(\beta|_Y)^{-1}$, where $b_1(\beta|_Y)^{-1} \in x_1(\beta\alpha)^{-1} \subseteq$ dom $\beta\alpha$ and $b_2(\beta|_Y)^{-1} \in x_2(\beta\alpha)^{-1} \subseteq$ dom $\beta\alpha$. Together with ran $\beta\alpha = \{x_1, x_2\}$ and $x_1 \prec x_2$, we conclude that $\beta\alpha \in Reg(OP_{\mathbb{N}})$ by Proposition 3.2. □

Lemma 3.7. *Let $\alpha \in OP_{\mathbb{N}}$ be a bijection from $\mathbb{N}$ to $\mathbb{N}$. Then $\alpha = id_{\mathbb{N}}$.*

Proof. Assume that $\alpha \neq \mathrm{id}_{\mathbb{N}}$. Then there exists the least natural number $n \in \mathbb{N}$ with respect to the natural order on $\mathbb{N}$ such that $n\alpha \neq n$, i.e. $n\alpha > n$. If $n \geq 2$, then $n - 1 \in$ dom α. Since $n - 1 \perp n$ in dom α but $(n-1)\alpha = n - 1 \| n\alpha$, we obtain $\alpha \notin OP_{\mathbb{N}}$, a contradiction. Suppose now that $n = 1$. Then $a\alpha > 1$ and there are $a_1, \ldots, a_{1\alpha-1} \in$ dom α with $\{a_1\alpha, \ldots, (a_{1\alpha-1})\alpha\} = \{1, \ldots, 1\alpha - 1\}$. Let $b := \max\{a_1, \ldots, a_{1\alpha-1}\}$ (the maximal element with respect to the natural order on $\mathbb{N}$). Because of $b\alpha < 1\alpha < (b+1)\alpha$, we get that $b\alpha \| (b+1)\alpha$, a contradiction with $\alpha \in OP_{\mathbb{N}}$. □

Now, we are able to prove that the regular subsemigroup $K_1 \cup \{id_{\mathbb{N}}\}$ of $OP_{\mathbb{N}}$ is a maximal one.

Theorem 3.8. *$K_1 \cup \{id_{\mathbb{N}}\}$ is a maximal regular subsemigroup of $OP_{\mathbb{N}}$.*

Proof. $K_1 \cup \{id_{\mathbb{N}}\}$ is a regular subsemigroup of $OP_{\mathbb{N}}$ by Corollary 3.5. It remains to show the maximality. Assume that there exists a regular semigroup S such that

$$K_1 \cup \{id_{\mathbb{N}}\} \subsetneq S.$$

Then there is $\beta \in S \setminus (K_1 \cup \{id_{\mathbb{N}}\})$.

We claim that there exists an element in $S \setminus (K_1 \cup \{id_{\mathbb{N}}\})$ which is not in Sur. If $\beta \notin Sur$, then β is the required element. Assume now that $\beta \in Sur$. Since S is regular and $\beta \in S$, there exists $\alpha \in S$ such that $\beta\alpha\beta = \beta$. Assume that ran $\alpha = \mathbb{N}$. Since $\beta\alpha\beta = \beta$, we have $\mathbb{N} = \text{ran } \beta \subseteq \text{dom } \alpha$ and so dom $\alpha = \mathbb{N}$. Since ran $\alpha = \mathbb{N} = \text{dom } \alpha$ and $\beta\alpha\beta = \beta$, we obtain that $\alpha : \mathbb{N} \to \mathbb{N}$ is a bijection from $\mathbb{N}$ to $\mathbb{N}$. Therefore, $\alpha = id_{\mathbb{N}}$ by Lemma 3.7. Then $\beta\alpha\beta = \beta$ implies that $\beta^2 = \beta$. But β is idempotent means that $\beta|_{\mathbb{N}} = \text{id}_{\mathbb{N}}$, i.e. $\beta = \text{id}_{\mathbb{N}}$, a contradiction. Thus, ran $\alpha \neq \mathbb{N}$ and so $\alpha \notin$ Sur. Since $\beta\alpha\beta = \beta$, we obtain rank $\alpha > 2$ by Remark 3.3, i.e. $\alpha \notin K_{1.}$ Consequently, we have shown that $\alpha \in S \setminus (K_1 \cup \{id_{\mathbb{N}}\})$ and $\alpha \notin Sur$. Then α is the required element and so we have the claim.

So, there is $\gamma \in S \setminus (K_1 \cup \{id_{\mathbb{N}}\})$ with $\gamma \notin Sur$. If ran $\gamma = \mathbb{N} \setminus \{1, 2, \ldots, i\}$ for some $i \in \mathbb{N}$, then we put $b_1 := i + 1, b_2 := i + 3$, and $b_3 := i$. If ran $\gamma = \{i, i + 1, i + 2, \ldots, j\}$ with $2 \leq i + 1 < j \in \mathbb{N}$, then we put $b_1 := j, b_2 := j - 2$, and $b_3 := j + 1$. If rank $\gamma = 2$ and ran γ is not a fence, say ran $\gamma = \{c, d\}$, then we put $b_1 := c, b_2 := d$, and $b_3 := c + 1$. If ran γ is not a fence with rank $\gamma \geq 3$, then there exist $c, d \in \text{ran } \gamma$ with $c < d$ and $c + 1 \notin \text{ran } \gamma$. In this case we put $b_1 := c, b_2 := d$, and $b_3 := c + 1$. It is easy to verify that we have covered all the possibilities for ran γ. Now, we define a partial transformation $\alpha : \{b_1, b_2, b_3\} \to \{b_1, b_3\}$ by

$$x\alpha := \begin{cases} b_1 & \text{if } x = b_1 \\ b_3 & \text{if } x = b_2, b_3. \end{cases}$$

It is clear that $\alpha \in K_1$. Then $\gamma\alpha \in S$. Now, we have ran $\gamma\alpha = \{b_1, b_3\}$. Since $b_1 || b_2$ in ran γ, we obtain $b_1\gamma^{-1} \| b_2\gamma^{-1}$. Since $b_1(\gamma\alpha)^{-1} = b_1\alpha^{-1}\gamma^{-1} = b_1\gamma^{-1}$ and $b_3(\gamma\alpha)^{-1} = b_3\alpha^{-1}\gamma^{-1} = \{b_2, b_3\}\gamma^{-1} = b_2\gamma^{-1}$, we get $b_1(\gamma\alpha)^{-1} \| b_3(\gamma\alpha)^{-1}$. By Proposition 3.2, we obtain $\gamma\alpha \notin Reg(OP_{\mathbb{N}})$. It contradicts that S is regular.

Altogether, we obtain that $K_1 \cup \{id_{\mathbb{N}}\}$ is a maximal regular subsemigroup of $OP_{\mathbb{N}}$. □

Let $K_2 := \{\alpha \in Reg(OP_{\mathbb{N}}) : a \perp b \Rightarrow a\alpha^{-1} \perp b\alpha^{-1} \text{for all } a \neq b \in \text{ran } \alpha\}$.

It is easy to verify that any $\alpha \in Reg(OP_{\mathbb{N}})$ with rank $\alpha = 2$ such that ran α is not a fence is contained in K_2. Moreover, we can observe that $x_1 \prec x_2$ for all $x_1 \in a_1\alpha^{-1}$ and all $x_2 \in a_2\alpha^{-1}$, whenever $\alpha \in K_2$ and $a_1, a_2 \in \text{ran } \alpha$ with $a_1 \prec a_2$. We will show that K_2 is a maximal regular subsemigroup of $OP_{\mathbb{N}}$. Since we will need it in the proof of the next theorem, let us note that K_2 contains both the inverse semigroups Inj and I_1.

Theorem 3.9. *K_2 is a maximal regular subsemigroup of $OP_{\mathbb{N}}$.*

Proof. First, we show that K_2 is a subsemigroup of $OP_{\mathbb{N}}$. Let $\alpha, \beta \in K_2$. Since $\alpha \in Reg(OP_{\mathbb{N}})$, there exists a subset Y of dom α such that $\alpha|_Y$ is a bijection from Y to ran α and $(\alpha|_Y)^{-1} \in OP_{\mathbb{N}}$. Since $\beta \in K_2$, it is easy to verify that $\beta|_{\text{ran }\alpha} \in K_2$. Since $\beta|_{\text{ran }\alpha} \in Reg(OP_{\mathbb{N}})$, there exists $Z \subset \text{dom}\beta|_{\text{ran }\alpha}$ such that $\beta|_{\text{ran }\alpha}$ is a bijection from Z to $\text{ran}\beta|_{\text{ran }\alpha} = \text{ran }\alpha\beta$. Let

$$Y' := Z(\alpha|_Y)^{-1}.$$

It is easy to check that $\alpha\beta|_{Y'}$ is a bijection from Y' to ran $\alpha\beta$. Let $a_1, a_2 \in$ ran $(\alpha\beta|_{Y'})$ with $a_1 \prec a_2$. Then there are $x_1, x_2 \in Y'$ with $x_1\alpha\beta = a_1$ and $x_2\alpha\beta = a_2$ and thus $x_1\alpha \in a_1\beta^{-1}$ and $x_2\alpha \in a_2\beta^{-1}$. This provides $x_1\alpha \prec x_2\alpha$ and so $x_1 \prec x_2$ since $\beta \in K_2$ and $\alpha \in K_2$, respectively. Consequently, $(\alpha\beta|_{Y'})^{-1} \in OP_{\mathbb{N}}$, i.e. $\alpha\beta \in Reg(OP_{\mathbb{N}})$. Now, we show that $\alpha\beta \in K_2$. Let $b_1, b_2 \in$ ran $\alpha\beta$ with $b_1 \perp b_2$. Assume that there are $a_1 \in b_1(\alpha\beta)^{-1}$ and $a_2 \in b_2(\alpha\beta)^{-1}$ such that $a_1 \| a_2$. This implies $a_1\alpha \| a_2\alpha$ since $\alpha \in K_2$, where $a_1\alpha, a_2\alpha \in$ dom β. Thus, $\beta \in K_2$ implies $a_1\alpha\beta \| a_2\alpha\beta$. This means $b_1 \| b_2$, a contradiction. Hence, $b_1(\alpha\beta)^{-1} \perp b_2(\alpha\beta)^{-1}$. This shows $\alpha\beta \in K_2$.

Next, we show that K_2 is a regular subsemigroup of $OP_{\mathbb{N}}$. Let $\alpha \in K_2$ and let Y be a transversal of ker α. Then $(\alpha|_Y)^{-1}$ is a bijection from ran α to Y with $(\alpha|_Y)^{-1} \in OP_{\mathbb{N}}$, which is an immediate consequence of the fact that $\alpha \in K_2$. On the other hand, $\alpha \in OP_{\mathbb{N}}$ implies $((\alpha|_Y)^{-1})^{-1} = \alpha|_Y \in OP_{\mathbb{N}}$. Then $(\alpha|_Y)^{-1} \in \text{Inj} \subseteq K_2$, where $\alpha(\alpha|_Y)^{-1}\alpha = \alpha\text{id}_{\text{ran }\alpha} = \alpha$.

Finally, we show that K_2 is a maximal regular subsemigroup of $OP_{\mathbb{N}}$. Assume that there is a regular semigroup S such that $K_2 \subsetneq S$. Then there exists $\beta \in S \setminus K_2$. Since $\beta \notin K_2$, there are $a, b \in$ ran β with $a \perp b$ such that there are $x \in a\beta^{-1}$ and $y \in b\beta^{-1}$ with $x \| y$. Let α be the identity mapping on $\{x, y\}$. It is clear that $\alpha \in K_2$ and so $\alpha\beta \in S$. Now, we have that dom $\alpha\beta = \{x, y\}$ and ran $\alpha\beta = \{a, b\}$. Thus, $a \perp b$ and $x \| y$ implies $\alpha\beta \notin Reg(OP_{\mathbb{N}})$ by Proposition 3.2. It contradicts that S is a regular semigroup. Consequently, K_2 is a maximal regular subsemigroup of $OP_{\mathbb{N}}$. □

We finish this section with the remark that the union of both maximal regular subsemigroups $K_1 \cup \{id_{\mathbb{N}}\}$ and K_2 covers all regular partial transformations with rank ≤ 2. In the next section, we will show that there are countably infinitely many maximal regular subsemigroups of $OP_{\mathbb{N}}$.

4. Maximal Regular Subsemigroups

It seems almost impossible to classify all (maximal) regular subsemigroups of $OP_{\mathbb{N}}$. But we are able to determine countably infinitely many maximal regular subsemigroups of $OP_{\mathbb{N}}$.

Definition 4.1. Let $a \in \mathbb{N} \setminus \{1\}$. Then let C_a^* be the set of all $\alpha \in Reg(OP_{\mathbb{N}})$ with

(*i*) $a \notin$ ran α or ran $\alpha \subseteq \{a-1, a, a+1\}$;

(*ii*) $a \notin \text{dom } \alpha$ or $\mid a\alpha\alpha^{-1} \mid\geq 2$ or $\text{ran } \alpha = \text{ran } \big(\alpha|_{\{a-1,a,a+1\}}\big)$;

(*iii*) $\alpha|_{\text{dom } \alpha\setminus\{a\}} \in K_2$.

Corollary 4.2. *Let $a, b \in \mathbb{N} \setminus \{1\}$ with $a \neq b$. Then $C_a^* \neq C_b^*$.*

Proof. Let $a, b \in \mathbb{N}\setminus\{1\}$ with $a \neq b$. Without loss of generality, we can suppose that $a = b+i$ for some $i \in \mathbb{N}$. Let α be the identity mapping on $\{a, a+2\}$. It is clear that $\alpha \in C_b^*$. Since $a \in \text{dom } \alpha$ and $\text{ran } \alpha = \{a, a+2\} \nsubseteq \{a-1, a, a+1\}$, we conclude that $\alpha \notin C_a^*$ and so $C_a^* \neq C_b^*$. □

We show now that $C_a := C_a^* \cup \{id_{\mathbb{N}}\}$ is a maximal regular subsemigroup of $OP_{\mathbb{N}}$. First, we observe that any $\alpha \in C_a$ with $a \notin \text{dom } \alpha$ belongs to K_2. But also the remaining elements in C_a^* are related to the semigroup K_2.

Lemma 4.3. *Let $a \in \mathbb{N} \setminus \{1\}$. If $\alpha \in C_a^* \setminus \{\varnothing\}$, then there exists a subset Y of dom α such that $\alpha|_Y$ is a bijection from Y to ran α with*

(*a*) $\alpha|_Y \in K_2$:

(*b*) $a \notin Y$ *if* $a \notin$ *dom* α *or* $\mid a\alpha\alpha^{-1} \mid\geq 2$:

(*c*) $Y \subseteq \{a-1, a, a+1\}$ *if ran* $\alpha =$ *ran* $(\alpha|_{\{a-1,a,a+1\}})$ *and* $\mid a\alpha\alpha^{-1} \mid= 1$.

Proof. Let $\alpha \in C_a^* \setminus \{\varnothing\}$. If $a \notin \text{dom } \alpha$ or $\mid a\alpha\alpha^{-1} \mid\geq 2$, then let Y be a transversal of $\ker \alpha$ with $a \notin Y$. Since $\alpha|_{\text{dom } \alpha\setminus\{a\}} \in K_2$, we obtain that $\alpha|_Y \in K_2$. Assume now that $\text{ran } \alpha = \text{ran } \big(\alpha|_{\{a-1,a,a+1\}}\big)$ and $\mid a\alpha\alpha^{-1} \mid= 1$. Let $Y \subseteq \{a-1, a, a+1\}$ be a transversal of $\ker \alpha$. Then $\mid a\alpha\alpha^{-1} \mid= 1$ implies $a \in Y$ and so Y is a fence. Hence, $\alpha|_Y \in K_2$ is clear. □

In the next step, we verify that C_a is a semigroup which is regular.

Lemma 4.4. *Let $a \in \mathbb{N} \setminus \{1\}$. Then C_a is a regular subsemigroup of $OP_{\mathbb{N}}$.*

Proof. First, we show that C_a is a subsemigroup of $OP_{\mathbb{N}}$. Let $\alpha, \beta \in C_a$. If $\alpha\beta = \varnothing$, then $\alpha\beta \in C_a$. Moreover, $\alpha = \text{id}_{\mathbb{N}}$ or $\beta = \text{id}_{\mathbb{N}}$ gives immediately $\alpha\beta \in C_a$. Assume now that $\alpha, \beta \neq \text{id}_{\mathbb{N}}$ and $\alpha\beta \neq \varnothing$. By Lemma 4.3, there are sets $X \subseteq \text{dom } \alpha$ and $Y \subseteq \text{dom } \beta$ such that $\alpha|_X : X \to \text{ran } \alpha$ and $\beta|_Y : Y \to \text{ran } \beta$ are bijective and $\alpha|_X, \beta|_Y \in K_2$. Let

$$X' := (\text{ran } \alpha \cap Y)\alpha^{-1} \cap X \subseteq \text{dom } \alpha\beta.$$

It is clear that $\alpha\beta|_{X'} : X' \to \text{ran } \alpha\beta$ is bijective and $\big(\alpha\beta|_{X'}\big)^{-1} \in OP_{\mathbb{N}}$, i.e. $\alpha\beta \in Reg(OP_{\mathbb{N}})$. It remains to show that $\alpha\beta$ satisfies the three properties of C_a^*.

(i) Since $a \notin \text{ran } \beta$ or $\text{ran } \beta \subseteq \{a-1, a, a+1\}$, we obtain that $a \notin \text{ran } \alpha\beta$ or $\text{ran } \alpha\beta \subseteq \text{ran } \beta \subseteq \{a-1, a, a+1\}$.

(ii) If $a \notin \text{dom } \alpha$ or $\mid a\alpha\alpha^{-1} \mid\geq 2$, then $a \notin \text{dom } \alpha\beta$ or $\mid a(\alpha\beta)(\alpha\beta)^{-1} \mid\geq 2$. Suppose now

$$\text{ran } \alpha = \text{ran } \big(\alpha|_{\{a-1,a,a+1\}}\big).$$

This implies
$$(\text{ran }\alpha)\beta = (\text{ran }\alpha|_{\{a-1,a,a+1\}})\beta,$$
where
$$(\text{ran }\alpha)\beta = \text{ran }\alpha\beta$$
and
$$(\text{ran }\alpha|_{\{a-1,a,a+1\}})\beta = \text{ran }(\alpha|_{\{a-1,a,a+1\}}\beta) = \text{ran }(\alpha\beta|_{\{a-1,a,a+1\}}).$$
Altogether, we obtain that
$$\text{ran }\alpha\beta = \text{ran }\big(\alpha\beta|_{\{a-1,a,a+1\}}\big).$$

(iii) Let $b_1, b_2 \in \text{ran }\big(\alpha\beta|_{\text{dom }\alpha\beta\setminus\{a\}}\big)$ be such that $b_1 \neq b_2$ and $b_1 \perp b_2$. Assume that there are $a_1 \in b_1\big(\alpha\beta|_{\text{dom }\alpha\beta\setminus\{a\}}\big)^{-1}$ and $a_2 \in b_2\big(\alpha\beta|_{\text{dom }\alpha\beta\setminus\{a\}}\big)^{-1}$ with $a_1 \| a_2$. Since $a_1 \| a_2$ in dom $\alpha \setminus \{a\}$ and $\alpha|_{\text{dom }\alpha\setminus\{a\}} \in K_2$, we have $a_1\alpha \| a_2\alpha$. If $a \notin$ ran α, then $a_1\alpha \| a_2\alpha$ in dom $\beta \setminus \{a\}$. Assume now that ran $\alpha \subseteq \{a-1, a, a+1\}$. If $a_1\alpha = a$ or $a_2\alpha = a$, then $a_1\alpha \perp a_2\alpha$, a contradiction. Hence, $a_1\alpha, a_2\alpha \neq a$ and $a_1\alpha \| a_2\alpha$ in dom $\beta \setminus \{a\}$. Since $\beta|_{\text{dom }\beta\setminus\{a\}} \in K_2$, we obtain $a_1\alpha\beta \| a_2\alpha\beta$. This means that $b_1 \| b_2$, a contradiction. Thus,
$$b_1\big(\alpha\beta|_{\text{dom }\alpha\beta\setminus\{a\}}\big)^{-1} \perp b_2\big(\alpha\beta|_{\text{dom }\alpha\beta\setminus\{a\}}\big)^{-1}.$$
Altogether, we have shown that $\alpha\beta|_{\text{dom }\alpha\beta\setminus\{a\}} \in K_2$.

Next, we check that C_a is a regular subsemigroup of $OP_{\mathbb{N}}$. Let $\alpha \in C_a$. If $\alpha \in \{\varnothing, \text{id}_{\mathbb{N}}\}$, then α is regular in C_a. Assume now that $\alpha \in C_a \setminus \{\varnothing, \text{id}_{\mathbb{N}}\}$. By Lemma 4.3, there exists a subset Y of dom α such that $\alpha|_Y : Y \to \text{ran }\alpha$ is bijective and $\alpha|_Y \in K_2$ satisfying the properties (a), (b), and (c). Let $\beta := (\alpha|_Y)^{-1}$. It is clear that $\alpha\beta\alpha = \alpha\text{id}_{\text{ran }\alpha} = \alpha$. We still have to show that $\beta \in C_a^*$.
(i) If $a \notin$ dom α or $| a\alpha\alpha^{-1} | \geq 2$, then $a \notin Y = \text{ran }\beta$ by (b). If ran α = ran $\big(\alpha|_{\{a-1,a,a+1\}}\big)$ and $| a\alpha\alpha^{-1} | = 1$, then we obtain that ran $\beta = Y \subseteq \{a-1, a, a+1\}$ by (c).
(ii) If $a \notin$ ran α, then $a \notin \text{ran }\alpha = \text{dom }\beta$. If ran $\alpha \subseteq \{a-1, a, a+1\}$, then dom β = ran $\alpha \subseteq \{a-1, a, a+1\}$ and so $\beta = \beta|_{\{a-1,a,a+1\}}$, i.e. ran β = ran $(\beta|_{\{a-1,a,a+1\}})$.
(iii) Since $\beta^{-1} = ((\alpha|_Y)^{-1})^{-1} = \alpha|_Y \in K_2 \subseteq OP_{\mathbb{N}}$ is bijective, we conclude that $\beta \in \text{Inj} \subseteq K_2$. Hence, $\beta|_{\text{dom }\beta\setminus\{a\}} \in K_2$. □

Now we show that C_a is a maximal regular subsemigroup of $OP_{\mathbb{N}}$. In order to prove it we need three technical lemmas.

Lemma 4.5. *Let $\alpha \in Sur$. If there exists $x \in \mathbb{N}$ such that $x\alpha = x$ and $| x\alpha^{-1} | = 1$, then $\alpha = id_{\mathbb{N}}$.*

Proof. By Proposition 2.3, there is $k \in \mathbb{N}$ such that $(2(k-1)+i)\alpha = i$ for all $i \in \mathbb{N}$. Since $(2(k-1)+x)\alpha = x$ and $x\alpha^{-1} = \{x\}$, we obtain that $2(k-1)+x = x$ and so $k = 1$. Therefore, $(2(k-1)+i)\alpha = i\alpha = i$ for all $i \in \mathbb{N}$. This shows that $\alpha = \text{id}_{\mathbb{N}}$. □

Lemma 4.6. *Let $a \in \mathbb{N} \setminus \{1\}$, let C be a regular subsemigroup of $OP_{\mathbb{N}}$ with $C_a \subseteq C$ and let $\alpha \in Reg(OP_{\mathbb{N}})$. If $a \in ran\ \alpha$ and if there is $b \in ran\ \alpha$ with $a \| b$ and $b+1 \notin ran\ \alpha$ or $b-1 \in \mathbb{N} \setminus ran\ \alpha$, then $\alpha \notin C$.*

Proof. Let $a, b \in \text{ran } \alpha$ with $a \| b$ such that $b+1 \notin \text{ran } \alpha$ or $b-1 \in \mathbb{N} \setminus \text{ran } \alpha$. Assume that $\alpha \in C$. If $b+1 \notin \text{ran } \alpha$, then we put $d := b+1$. If $b+1 \in \text{ran } \alpha$, then $b-1 \in \mathbb{N} \setminus \text{ran } \alpha$ and we put $d := b-1$. We define a partial transformation $\beta : \{a, d, b\} \to \{d, b\}$ by

$$z\beta := \begin{cases} d & \text{if } z = a, d \\ b & \text{if } z = b. \end{cases}$$

It is clear that $\beta \in C_a$ and so $\alpha\beta \in C$. Now, we have that dom $\alpha\beta = a\alpha^{-1} \cup b\alpha^{-1}$, ran $\alpha\beta = \{d, b\}$, and $d \perp b$. Since $a \| b$ in ran α, we get $x \| y$ for all $x \in a\alpha^{-1} = d(\alpha\beta)^{-1}$ and for all $y \in b\alpha^{-1} = b(\alpha\beta)^{-1}$. By Proposition 3.2, we obtain that $\alpha\beta \notin Reg(OP_{\mathbb{N}})$ and so $\alpha\beta \notin C$, a contradiction. Hence, $\alpha \notin C$. □

Lemma 4.7. *Let $a \in \mathbb{N} \setminus \{1, 2, 3\}$ and let C be a regular subsemigroup of $OP_{\mathbb{N}}$ with $C_a \subseteq C$. If $\alpha \in C \setminus C_a$, then $|\ a\alpha\alpha^{-1}\ | \neq 1$ or $ran\ \alpha = ran\ \big(\alpha|_{\{a-1,a,a+1\}}\big)$.*

Proof. Let $\alpha \in C \setminus C_a$. Assume that $|\ a\alpha\alpha^{-1}\ | = 1$ and ran $\alpha \neq$ ran $\big(\alpha|_{\{a-1,a,a+1\}}\big)$. Then there exists $b \in \text{ran } \alpha$ with $a-1, a, a+1 \notin b\alpha^{-1}$ and so $a\alpha \| b$ since $\alpha \in Reg(OP_{\mathbb{N}})$. We will consider the both cases $a\alpha = a$ and $a\alpha \neq a$.

Case 1 : $a\alpha = a$. Assume that ran $\alpha = \mathbb{N}$. Then $\alpha \in$ Sur with $a\alpha = a$ and $|\ a\alpha^{-1}\ | = 1$. Hence, we obtain $\alpha = \text{id}_{\mathbb{N}} \in C_a$ by Lemma 4.5, a contradiction. Thus, ran $\alpha \neq \mathbb{N}$. If $b+1 \notin \text{ran } \alpha$ or $b-1 \in \mathbb{N} \setminus \text{ran } \alpha$, then we obtain $\alpha \notin C$ by Lemma 4.6, a contradiction. Suppose that $b+1 \in \text{ran } \alpha$ and either $b-1 \notin \mathbb{N}$ or $b-1 \in \text{ran } \alpha$. Since ran $\alpha \neq \mathbb{N}$, there exists $c \in \mathbb{N} \setminus \text{ran } \alpha$ such that $c \| b$. Then we define a partial transformation $\gamma : \{a, b, c\} \to \{b, c\}$ by

$$y\gamma := \begin{cases} c & \text{if } y = a, c \\ b & \text{if } y = b. \end{cases}$$

It is clear that $\gamma \in C_a$ and so $\alpha\gamma \in C$. Now, we have that dom $\alpha\gamma = b\alpha^{-1} \cup \{a\}$ and ran $\alpha\gamma = \{c, b\}$, where $a\alpha\gamma = c$ and $z\alpha\gamma = b$ for all $z \in b\alpha^{-1}$. Since $\alpha\gamma \in C$ and C is a regular subsemigroup of $OP_{\mathbb{N}}$, there exists $\beta \in C$ such that $(\alpha\gamma)\beta(\alpha\gamma) = \alpha\gamma$. Then $c\beta = a$ and $b\beta = x$ for some $x \in b\alpha^{-1}$. Note that $c \neq a$ (since $a \in \text{ran } \alpha$ and $c \notin \text{ran } \alpha$), $b \neq a$ (since $a\alpha = a$ and $a\alpha \| b$), and $a \| x$ (since $x \in b\alpha^{-1}$ and $a-1, a, a+1 \notin b\alpha^{-1}$). Let δ be the identity mapping on $\{b, c\}$. It is clear that $\delta \in C_a$ and so $\delta\beta \in C$. Now, we have ran $\delta\beta = \{x, a\}$ and $x \| a$. We obtain $\delta\beta \notin C$ by Lemma 4.6, a contradiction.

Case 2 : $a\alpha \neq a$. Since $\alpha \in C \subseteq Reg(OP_{\mathbb{N}})$, there exists $\beta \in C$ such that $\alpha\beta\alpha = \alpha$. Then $(a\alpha)\beta = a$ and $b\beta = x$ for some $x \in b\alpha^{-1}$. Note that $a \| x$ (since $x \in b\alpha^{-1}$ and $a-1, a, a+1 \notin b\alpha^{-1}$). If $x+1 \notin \text{ran } \beta$ or $x-1 \in \mathbb{N} \setminus \text{ran } \beta$, then we obtain $\beta \notin C$ by Lemma 4.6, a contradiction. Suppose that $x+1 \in \text{ran } \beta$ and either $x-1 \notin \mathbb{N}$ or $x-1 \in \text{ran } \beta$. If $a \neq b$, then we let γ be the identity mapping on $\{a\alpha, b\}$. It is clear that $\gamma \in C_a$ and so $\gamma\beta \in C$. Now, we have ran $\gamma\beta = \{a, x\}$ as well as $a \| x$ and obtain $\gamma\beta \notin C$ by Lemma 4.6, a contradiction. Suppose now that $a = b$. If $x - 1 \notin \mathbb{N}$, then $x = 1$. Since $x + 1 = 2 \in \text{ran } \beta$ and $a\beta = 1$, there exists $y \in 2\beta^{-1} \setminus \{a\}$. In this case, we put $c := y$ and $d := 2$, where $a \| d$ since $a \geq 4$. If $x - 1 \in \text{ran } \alpha$, then there exists $l \in \{x - 1, x + 1\}$ such that $a \| l$. Since $b\beta = x$ and $b = a$, there exists $y' \in l\beta^{-1}$ such that $y' \neq a$. In this case, we put $c := y'$ and $d := l$. Note that $a \| d$ and $a\alpha \| c$ (since $a\alpha \in a\beta^{-1}$ and $c \in d\beta^{-1}$). Let δ be the identity mapping on $\{a\alpha, c\}$. It is clear that $\delta \in C_a$ and so $\delta\beta \in C$. Now, we have ran $\delta\beta = \{a, d\}$ with $a \| d$ and obtain $\delta\beta \notin C$ by Lemma 4.6, a contradiction.

□

Now, we are able to prove the main result of this paper.

Theorem 4.8. *Let $a \in \mathbb{N} \setminus \{1, 2, 3\}$. Then C_a is a maximal regular subsemigroup of $OP_{\mathbb{N}}$.*

Proof. C_a is a regular subsemigroup of $OP_{\mathbb{N}}$ by Lemma 4.4. It remains to show that C_a is maximal. Let C be a regular subsemigroup of $OP_{\mathbb{N}}$ with $C_a \subseteq C$. Assume that C_a is a proper subsemigroup of C, i.e. $C \setminus C_a \neq \emptyset$. Then there exists $\alpha \in C \setminus C_a$.

By Lemma 4.7, we obtain that $| a\alpha\alpha^{-1} | \neq 1$ (i.e. $a \notin \text{dom } \alpha$ or $| a\alpha\alpha^{-1} | \geq 2$) or ran $\alpha = \text{ran } \left(\alpha|_{\{a-1,a,a+1\}}\right)$.

Next, we show that $a \notin \text{ran } \alpha$ or ran $\alpha \subseteq \{a-1, a, a+1\}$. Assume that $a \in \text{ran } \alpha$ and ran $\alpha \nsubseteq \{a-1, a, a+1\}$. Then there exists $b \in \text{ran } \alpha \setminus \{a-1, a, a+1\}$ and so $a \| b$. Assume that $| a\alpha\alpha^{-1} | = 1$. Then ran $\alpha = \text{ran } \left(\alpha|_{\{a-1,a,a+1\}}\right)$. Without loss of generality, we suppose that a is odd. Since $| a\alpha\alpha^{-1} | = 1$ and $a \| b$ in ran α, there exists $c \in \text{ran } \alpha \setminus \{a, b\}$ such that $\{a-1, a, a+1\}\alpha = \{a, b, c\}$. Since $\{a-1, a, a+1\}$ is a fence and $a \| b$ in ran α, we can conclude that $a\alpha = c$. If $(a-1)\alpha = a$, then $(a-1) \succ a$ implies $a = (a-1)\alpha \succ a\alpha = c$, a contradiction with a is odd. If $(a+1)\alpha = a$, then we obtain a contradiction by the same arguments. Thus, $| a\alpha\alpha^{-1} | \neq 1$ and so there exist $x \in a\alpha^{-1}$ and $y \in b\alpha^{-1}$ such that $x \neq a$ and $y \neq a$. Let δ be the identity mapping on $\{x, y\}$. It is clear that $\delta \in C_a$ and so $\delta\alpha \in C$. Note that ran $\delta\alpha = \{a, b\}$ and $a \| b$. We obtain $\delta\alpha \notin C$ by Lemma 4.6, a contradiction. Hence, $a \notin \text{ran } \alpha$ or ran $\alpha \subseteq \{a-1, a, a+1\}$.

Up to now, we have shown that $\alpha \in C \subseteq Reg(OP_{\mathbb{N}})$ satisfies the conditions (i) and (ii) in Definition 4.1. Since $\alpha \notin C_a$, we conclude that $\alpha|_{\text{dom } \alpha \setminus \{a\}} \notin K_2$. Then there are $x_1, x_2 \in \text{ran } \alpha|_{\text{dom } \alpha \setminus \{a\}}$ with $x_1 \perp x_2$ as well as $a_1 \in x_1(\alpha|_{\text{dom } \alpha \setminus \{a\}})^{-1}$ and $a_2 \in x_2(\alpha|_{\text{dom } \alpha \setminus \{a\}})^{-1}$ such that $a_1 \| a_2$. Let γ be the identity mapping on $\{a_1, a_2\}$. It is clear that $\gamma \in C_a$ and so $\gamma\alpha \in C$. Now, we have that dom $\gamma\alpha = \{a_1, a_2\}$ and ran $\gamma\alpha = \{x_1, x_2\}$. Since $x_1 \perp x_2$ and $a_1 \| a_2$, we obtain $\gamma\alpha \notin Reg(OP_{\mathbb{N}})$ by Proposition 3.2. Therefore, $\gamma\alpha \notin C$, a contradiction.

Consequently, we obtain $C \setminus C_a = \emptyset$ and thus $C = C_a$, i.e. C_a is a maximal regular subsemigroup of $OP_{\mathbb{N}}$. □

After all together with Corollary 4.2, we obtain immediately:

Corollary 4.9. *There are countably infinitely many maximal regular subsemigroups of $OP_{\mathbb{N}}$.*

Proof. By Theorem 4.8 and Corollary 4.2, we have that $\{C_a : a \in \mathbb{N} \setminus \{1, 2, 3\}\}$ is a countably infinite set of maximal regular subsemigroups of $OP_{\mathbb{N}}$. □

We finish this section with a remark about the cardinalities of any semigroup C_a.

Proposition 4.10. *Let* $a \in \mathbb{N} \setminus \{1\}$. *Then* $\mid C_a \mid > \aleph_0$.

Proof. Let $A := \{a + 2n : n \in \mathbb{N}\}, B := \{a + 1, a + 3\}$, and let B^A be the set of all mappings from A to B. Further, let $\alpha \in B^A$. Clearly, α is a partial transformation on $\mathbb{N}$. If rank $\alpha = 1$, then $\alpha \in C_a$ by Remark 3.1. Suppose now that ran $\alpha = \{a+1, a+3\}$. Since $a+2n \| a+2m$ for all $n, m \in \mathbb{N}$ with $n \neq m$, we get immediately that $\alpha \in OP_{\mathbb{N}}$. Moreover, $a + 1 \| a + 3$ provides $\alpha \in Reg(OP_{\mathbb{N}})$ by Proposition 3.2. From ran $\alpha = \{a + 1, a + 3\}$ with $a + 1 \| a + 3$, we obtain $\alpha|_{\text{dom } \alpha \setminus \{a\}} \in K_2$. Together with $a \notin$ dom α and $a \notin$ ran α, we conclude that $\alpha \in C_a$. Thus, $B^A \subseteq C_a$, where $\mid C_a \mid \geq \mid B^A \mid = \mid B \mid^{|A|} = 2^{\aleph_0} > \aleph_0$. □

Proposition 4.10 shows us that C_a is an uncountably infinite set for all $a \in \mathbb{N} \setminus \{1\}$.

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