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ON SOME CHARACTERIZATIONS OF REGULAR AND POTENT RINGS RELATIVE TO RIGHT IDEALS

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Abstract. In this paper we study the notion of regular rings relative to right ideals, and we give another characterization of these rings. Also, we introduce the concept of an annihilator relative to a right ideal. Basic properties of this concept are proved. New results obtained include necessary and sufficient conditions for a ring to be regular (potent) relative to right ideal.

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1. Introduction

In 1987, V. A. Andrunakievich and Yu. M. Ryabukhin [2], were the first who studied the concept of quasi-regularity and primitivity relative to a right ideal. Later, V. A. Andrunakievich and A. V. Andrunakievich [1] in 1991, studied the concept of regularity relative to right ideals. They show that every ring is regular relative to the intersection of a finite number of maximal right ideals. Also, they proved that a ring R is regular relative to a right ideal $P \neq R$ if and only if for every $a \in R$, $aR + P = eR + P$ where $e \in R$ is idempotent relative to P. In 2011, P. Dheena and S. Manivasan [3] studied quasi-ideals of a $P-$regular near-rings. In section 2 in the present paper, it is proved that a ring R is regular relative to a right ideal $P \neq R$ if and only if for every $a \in R$, $aR + P = r_P(1 - ab)$ for some $b \in R$. Interesting corollaries are obtained for a right annihilator relative to right ideals. In section 3, we characterize the potent rings relative to right ideal. The main result in this section states that a ring R is $P-$potent if and only if for every $a \in R$ there exists $b \in R$ such that $r_P(1 - ba) = eR + P$ where $e \in R$ is $P-$idempotent.

Throughout in this paper rings R, are associative with identity unless otherwise indicated. We denote the Jacobson radical of a ring R by $J(R)$. A ring R is called regular in the sense of Von Neumann if for every $a \in R$, $a = aba$ for some $b \in R$. Recall that a ring R is semi-potent, also called I_0-ring by Nicholson [5], if every principal left (resp. right) ideal not contained in $J(R)$ contains a nonzero idempotent.

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2. P-Regular Rings

We start this section with the following basic properties:

Lemma 2.1. *Let R be a ring and $P \neq R$ be a right ideal of R. The following statements hold:*
(1) *For every $a, b \in R$, $R = (aR + P) + ((1 - ab)R + P)$.*
(2) *For every $a, b \in R$ such that $abP \subseteq P$,*

$$(a - aba)R + P = (aR + P) \cap ((1 - ab)R + P).$$

Proof. (1). Let $a, b \in R$. We have $1 = ab + (1 - ab)$, so for every $p_0 \in P$, $1 = (ab + p_0) + ((1 - ab) - p_0)$, which implies:

$$R \subseteq (ab + p_0)R + ((1 - ab) - p_0)R \subseteq$$

$$\subseteq abR + p_0R + (1 - ab)R + (-p_0)R \subseteq$$

$$\subseteq (aR + P) + ((1 - ab)R + P) \subseteq R.$$

(2). Let $a, b \in R$ with $abP \subseteq P$. We have

$$(a - aba)R + P = a(1 - ba)R + P \subseteq aR + P$$

$$(a - aba)R + P = (1 - ab)aR + P \subseteq (1 - ab)R + P,$$

so $(a - aba)R + P \subseteq (aR + P) \cap ((1 - ab)R + P)$.
Let $x \in (aR + P) \cap ((1 - ab)R + P)$, then $x = ax_1 + p_1 = (1 - ab)x_2 + p_2$, where $x_1, x_2 \in R$, $p_1, p_2 \in P$ and $x_2 = x + abx_2 - p_2 = ax_1 + p_1 + abx_2 - p_2 = a(x_1 + bx_2) + (p_1 - p_2) = ax_0 + p_0$, where $x_0 = x_1 + bx_2 \in R$, $p_0 = p_1 - p_2 \in P$. Thus

$$x = (1 - ab)(ax_0 + p_0) + p_2 =$$

$$= (a - aba)x_0 + p_0 - abp_0 + p_2 =$$

$$= (a - aba)x_0 + ab(-p_0) + p_0 + p_2 \in$$

$$\in (a - aba)R + abP + P \subseteq (a - aba)R + P.$$

$\square$

Note that in Lemma 2.1 and for $P = 0$ we derive the following:

Corollary 2.2. *Let R be a ring. Then for any $a, b \in R$ the following statements hold:*
(1) *$R = aR + (1 - ab)R$.*
(2) *$(a - aba)R = aR \cap (1 - ab)R$.*

Lemma 2.3. *Let R be a ring and $P \neq R$ be a right ideal of R. The following statements hold:*
(1) *For every $a \in R$ the set $r_P(a) = \{x : x \in R; \;\; ax \in P\}$ is a right ideal in R.*
(2) *For every $a \in P$, $r_P(a) = R$ and $r_P(1) = P$.*
(3) *For every $a \in R$, $aP \subseteq P$ if and only if $P \subseteq r_P(a)$.*

Proof. It is clear. □

In Lemma 2.3 we call the set $r_P(a)$ the right annihilator of a in R relative to P or $P-$annihilator of a in R. Note that in Lemma 2.3 and for $P = 0$ we found that $r_0(a) = r(a)$ is the right annihilator for all $a \in R$. We prove some basic properties of the right annihilator relative to a right ideal.

Lemma 2.4. *Let R be a ring and $P \neq R$ be a right ideal of R. The following statements hold:*
(1) *If $bP \subseteq P$, then $r_P(a) \cap r_P(1 - ba) \subseteq P$.*
(2) *If $aP \subseteq P$ and $bP \subseteq P$, then $r_P(a) \cap r_P(1 - ba) = P$.*
(3) *If $aP \subseteq P$ and $bP \subseteq P$, then $r_P(a - aba) = r_P(a) + r_P(1 - ba)$.*

Proof. (1). Assume that $bP \subseteq P$. Let $x \in r_P(a) \cap r_P(1 - ba)$, then $ax \in P$ and $(1 - ba)x \in P$. Suppose that $ax = p_1$ and $(1 - ba)x = p_2$ where $p_1, p_2 \in P$, then $x = bax + p_2 = bp_1 + p_2 \in bP + P \subseteq P$.
(2). Assume that $aP \subseteq P$ and $bP \subseteq P$. By (1) we have $r_P(a) \cap r_P(1-ba) \subseteq P$. Let $p_0 \in P$, then $ap_0 \in aP \subseteq P$, so $p_0 \in r_P(a)$. Also, $(1 - ba)p_0 = p_0 - bap_0 = p_0 + ba(-p_0) \in P + baP \subseteq P + bP \subseteq P$ so $p_0 \in r_P(1 - ba)$. Thus $P \subseteq r_P(a) \cap r_P(1 - ba)$.
(3). Assume that $aP \subseteq P$ and $bP \subseteq P$. Let $x \in r_P(a)$, then $ax \in P$ and $(a - aba)x = ax - abax \in P + abP \subseteq P + aP \subseteq P$ so $x \in r_P(a - aba)$ i.e. $r_P(a) \subseteq r_P(a - aba)$.
Let $y \in r_P(1-ba)$. Then $(1-ba)y \in P$ and $(a-aba)y = a(1-ba)y \in aP \subseteq P$. So $y \in r_P(a - aba)$, i.e. $r_P(1 - ba) \subseteq r_P(a - aba)$, thus $r_P(a) + r_P(1 - ba) \subseteq r_P(a - aba)$.
Let $z \in r_P(a - aba)$, then $(a - aba)z \in P$. Since $z = baz - (1 - ba)z$ and $baz \in r_P(1 - ba)$, $(1 - ba)z \in r_P(a)$. Hence $(1 - ba)baz = baz - babaz = b(a - aba)z \in bP \subseteq P$ and $a(1 - ba)z = (a - aba)z \in P$ implies $r_P(a - aba) \subseteq r_P(a) + r_P(1 - ba)$. □

Note that in Lemma 2.4 and for $P = 0$ we derive the following:

Corollary 2.5. *Let R be a ring. Then for any $a, b \in R$ the following statements hold:*
(1) $r(a) \cap r(1 - ba) = 0$.
(2) $(a - aba)R = r(a) + r(1 - ba)$.

Lemma 2.6. *Let R be a ring and $P \neq R$ be a right ideal of R. If $a, b \in R$ such that $aP \subseteq P$, $bP \subseteq P$, then the following statements are equivalent:*
(1) $r_P(1 - ab) = P$.
(2) $r_P(1 - ba) = P$.
(3) $r_P(a - aba) = r_P(a)$.

Proof. (1) $\Rightarrow$ (2). Suppose that $r_P(1 - ab) = P$. Let $x \in r_P(1 - ba)$, then $(1 - ba)x \in P$. Assume that $(1 - ba)x = p_0$, where $p_0 \in P$, then $(1 - ab)ax = ap_0 \in aP \subseteq P$, which implies that $ax \in r_P(1 - ab) = P$. Since $x = bax + (1 - ba)x \in bP + P \subseteq P$, thus $r_P(1 - ba) \subseteq P$.
On the other hand, for every $t \in P$ we have $(1 - ba)t = t - bat = t + ba(-t) \in$

$P+baP \subseteq P+bP \subseteq P$, which implies that $t \in r_P(1-ba)$, thus $P \subseteq r_P(1-ba)$.
(2) $\Rightarrow$ (3). Suppose that $r_P(1-ba) = P$. Then by assumption and Lemma 2.4(3), we infer that $r_P(a-aba) = r_P(a)+P$. Since $aP \subseteq P$ and by Lemma 2.3(3) $P \subseteq r_P(a)$, thus $r_P(a-aba) = r_P(a)+P = r_P(a)$.
(3) $\Rightarrow$ (1). Let $y \in r_P(1-ab)$, then $(1-ab)y \in P$. Suppose that $(1-ab)y = p_0$, where $p_0 \in P$, so $(a-aba)by = ab(1-ab)y \in abP \subseteq aP \subseteq P$. This implies that $by \in r_P(a-aba) = r_P(a)$, so $aby \in P$ and $y = aby+(1-ab)y \in P$, thus $r_P(1-ab) \subseteq P$. On the other hand, for every $t \in P$, $(1-ab)t = t+ab(-t) \in P+abP \subseteq P+aP \subseteq P$, which implies $t \in r_P(1-ab)$, thus $P \subseteq r_P(1-ab)$. $\square$

Note that in Lemma 2.6 and for $P = 0$, we drive the following:

Corollary 2.7. *Let R be a ring. Then for any $a, b \in R$ the following are equivalent:*
(1) $r(1-ab) = 0$.
(2) $r(1-ba) = 0$.
(3) $r(a-aba) = r(a)$.

Let R be a ring and $P \neq R$ be a right ideal of R, recall that an element $e \in R$ is $P-$ idempotent [1], if $e^2 - e \in P$, and $eP \subseteq P$.

Lemma 2.8. *Let R be a ring and $P \neq R$ be a right ideal of R. If $e \in R$ is $P-$idempotent, then the following statements hold:*
(1) *An element $1-e \in R$ is $P-$idempotent.*
(2) $r_P(e) = (1-e)R+P$ *and* $r_P(1-e) = eR+P$.
(3) *If $e \in J(R)$, then $e \in P$.*

Proof. It is obvious. $\square$

Let R be a ring and $P \neq R$ be a right ideal of R. Recall that a ring R is $P-$regular [1], if for every $a \in R$ there exists $b \in R$ such that $aba - a \in P$ and $abP \subseteq P$. If $P = 0$, then R is a Von Neumann regular ring.

Theorem 2.9. *Let R be a ring and $P \neq R$ be a right ideal of R. The following are equivalent:*
(1) *R is a $P-$regular ring.*
(2) *For every $a \in R$, $aR+P = eR+P$ where $e \in R$ is a $P-$idempotent element.*
(3) *For every $a \in R$ there exists $b \in R$ such that $(aR+P) \cap ((1-ab)R+P) = P$.*

Proof. (1) $\Leftrightarrow$ (2). By [[1], Theorem 1].
(1) $\Rightarrow$ (3). Let $a \in R$, then by assumption there exists $b \in R$ such that $a - aba \in P$ and $abP \subseteq P$. Let $aba - a = p_0$ where $p_0 \in P$. It is clear that $P \subseteq (aR+P) \cap ((1-ab)R+P)$. Let $x \in (aR+P) \cap ((1-ab)R+P)$. Then $x = ax_1 + p_1 = (1-ab)x_2 + p_2$, where $x_1, x_2 \in R$, $p_1, p_2 \in P$. Hence $x_2 = x+abx_2-p_2 = (ax_1+abx_2)+(p_1-p_2) = ax_3+p_3$, where $x_3 = x_1+bx_2 \in R$, $p_3 = p_1 - p_2 \in P$. Thus $x = (1-ab)x_2 + p_2 = (1-ab)(ax_3+p_3)+p_2 = (a-aba)x_3 + (1-ab)p_3 + p_2$. Since $(1-ab)p_3 = p_3 + ab(-p_3) \in P + abP \subseteq P$, we obtain that $x = p_0p_3 + (1-ab)p_3 + p_2 \in P$.
(3) $\Rightarrow$ (1). Let $a \in R$, then by assumption $(aR+P) \cap ((1-ab)R+P) = P$ for

some $b \in R$, Since $(a-aba)R+P \subseteq aR+P$ and $(a-aba)R+P \subseteq (1-ab)R+P$, hence $a-aba \in (a-aba)R+P \subseteq P$.
We will prove that $abP \subseteq P$. Suppose that $a = aba + p'$, where $p' \in P$, then $ab = abab + p'b$, so for every $t \in P$, $abt = ababt + p'bt \in aR + p'R \subseteq aR + PR \subseteq aR + P$. Also, since $t = abt + (1-ab)t$, $abt = (1-ab)(-t) + t \in (1-ab)R + P$, thus $abt \in (aR+P) \cap ((1-ab)R+P) = P$, which implies $abP \subseteq P$. □

Note that in Theorem 2.9 and for $P = 0$ we derive the following:

Corollary 2.10. *For any ring R the following are equivalent:*
(1) *R is regular.*
(2) *For every $a \in R$, $aR = eR$, where $e \in R$ is an idempotent element.*
(3) *For every $a \in R$ there exists $b \in R$ such that $aR \cap (1-ab)R = 0$.*

R. Ming in [[4], Proposition 1] proved that a ring R without non-zero nilpotent elements is regular if and only if every principal left ideal of R is the left annihilator of an element of R. This fact is generalized as following:

Proposition 2.11. *For any ring R the following are equivalent:*
(1) *R is regular.*
(2) *For every $a \in R$ there exists $b \in R$ such that $Ra = \ell(1-ba)$.*
(3) *For every $a \in R$ there exists $b \in R$ such that $aR = r(1-ab)$.*

Proof. (1) $\Rightarrow$ (2). Let $a \in R$, then $a = aba$ for some $b \in R$, so $e = ba$ is an idempotent of R and $\ell(1-ba) = \ell(1-e) = Re = Ra$.
(2) $\Rightarrow$ (1). If $\ell(1-ba) = Ra$ for some $b \in R$, then $a \in \ell(1-ba)$, which implies that $a = aba$. Similarly we can prove the equivalence (1) $\Leftrightarrow$ (3). □

Proposition 2.11 is generalized as following:

Theorem 2.12. *Let R be a ring and $P \neq R$ be a right ideal of R. The following are equivalent:*
(1) *R is $P-$regular.*
(2) *For every $a \in R$ there exists $b \in R$ such that $aR + P = r_P(1-ab)$.*

Proof. (1) $\Rightarrow$ (2). Let $a \in R$, then $aba - a \in P$ and $abP \subseteq P$ for some $b \in R$. So, for every $x \in R$ and every $p_0 \in P$,

$$(1-ab)(ax+p_0) = (1-ab)ax + (1-ab)p_0 =$$

$$= (a-aba)x + p_0 + ab(-p_0) \in PR + P + abP \subseteq P$$

which implies that $aR + P \subseteq r_P(1-ab)$.
Let $y \in r_P(1-ab)$, then $(1-ab)y = p_1 \in P$ so $y = aby + p_1 \in aR + P$.
(2) $\Rightarrow$ (1). Let $a \in R$, then by assumption $aR+P = r_P(1-ab)$ for some $b \in R$. Since $P \subseteq r_P(1-ab)$, for every $t \in P$, $abt = t + (1-ab)(-t) \in P$, which implies that $abP \subseteq P$. Also, since $a \in r_P(1-ab)$, $aba - a \in P$, thus R is $P-$regular. □

Corollary 2.13. *Let R be a ring and $P \neq R$ be a right ideal of R. If R is a $P-$regular ring, then $J(R) \subseteq P$.*

Proof. Let $a \in J(R)$, then $aba - a \in P$ and $abP \subseteq P$ for some $b \in R$. Since $ba \in J(R)$, $1 - ba$ has a right inverse so $(1 - ba)t = 1$ for some $t \in R$. Suppose that $aba - a = p_0$, where $p_0 \in P$, then $a = a(1 - ba)t = (aba - a)t = -p_0 t \in PR \subseteq P$. $\square$

3. P-Potent Rings

Let R be a ring and $P \neq R$ be a right ideal of R. We say that R is a $P-$potent ring if for every $a \in R$ there exists $b \in R$ such that $bab - b \in P$ and $baP \subseteq P$.

Theorem 3.1. *Let R be a ring and $P \neq R$ be a right ideal of R. The following are equivalent:*
(1) *R is $P-$potent.*
(2) *For every $a \in R$ there exists $b \in R$ such that $r_P(1 - ba) = eR + P$ where $e \in R$ is $P-$idempotent.*
(3) *For every $a \in R$ there exists $b \in R$ such that $(bR+P)\cap((1-ba)R+P) = P$.*

Proof. (1) $\Rightarrow$ (2). Let $a \in R$, then $bab - b \in P$ and $baP \subseteq P$ for some $b \in R$. Let $e = ba$, then $e \in R$ is $P-$idempotent so, $e^2 = e + p_0$ for some $p_0 \in P$. Let $x \in r_P(1 - ba)$, then $(1 - ba)x \in P$ and $x = bax + p_1 = ex + p_1 \in eR + P$. Let $z \in eR + P$, then $z = ey + p_2$ where $y \in R$, $p_2 \in P$ and

$$(1 - ba)z = (1 - ba)(ey + p_2) = ey - baey + (1 - ba)p_2 =$$

$$= ey - e^2 y + (1 - ba)p_2 = (e - e^2)y + (1 - ba)p_2 \in P,$$

so $eR + P \subseteq r_P(1 - ba)$.
(2) $\Rightarrow$ (1). Let $a \in R$, then by assumption there exists $b \in R$ such that $r_P(1 - ba) = eR + P$, where $e \in R$ is $P-$idempotent. Since $e \in r_P(1 - ba)$, $(1 - ba)e \in P$, so $e = bae + p_0$, where $p_0 \in P$ and $eb = baeb + p_0 b$, $e^2 b = (eb)a(eb) + ep_0 b$. Since $e^2 = e + p_1$ for some $p_1 \in P$, $(e + p_1)b = (eb)a(eb) + ep_0 b$ and

$$eb = (eb)a(eb - p_1) + ep_0 b,$$

so $eb = (eb)a(eb) - p_2$, where $p_2 = -p_1 b + ep_0 b \in P$. Let $eb = y$, then $yay - y \in P$ and for every $t \in P \subseteq eR + P$, $t \in r_P(1 - ba)$, so $t - bat \in P$. Thus $bat \in P$ and $yaP = ebaP \subseteq eP \subseteq P$, which shows that R is $P-$potent.
(1) $\Rightarrow$ (3). Let $a \in R$, then by assumption there exists $b \in R$ such that $bab - b \in P$, $baP \subseteq P$. Since $baP \subseteq P$, Lemma 2.1 implies that

$$(bR + P) \cap ((1 - ba)R + P) = (b - bab)R + P \subseteq PR + P \subseteq P$$

(3) $\Rightarrow$ (1). Let $a \in R$, then $(bR + P) \cap ((1 - ba)R + P) = P$ for some $b \in R$. Since

$$(b - bab)R + P = (1 - ba)bR + P \subseteq (1 - ba)R + P$$

$$(b - bab)R + P = b(1 - ab)R + P \subseteq bR + P$$

implies that $(b-bab)R+P=P$, hence $bab-b\in P$. Suppose that $b=bab+p'$, where $p'\in P$. Then $ba=baba+p'a$, so for every $t\in P$, $bat=babat+p'at\in bR+p'R\subseteq bR+P$. Also, since $t=bat+(1-ba)t$,

$$bat=(1-ba)(-t)+t\in(1-ba)R+P$$

implies that $bat\in P$, so $baP\subseteq P$. This shows that R is $P-$potent. $\square$

Note that using Theorem 3.1 with $P=0$, we derive the following:

Corollary 3.2. *For any ring R the following are equivalent:*
(1) *R is $0-$potent.*
(2) *For every $a\in R$ there exists $b\in R$ such that $r(1-ba)=eR$, where $e^2=e\in R$.*
(3) *For every $a\in R$ there exists $b\in R$ such that $bR\cap(1-ba)R=0$.*

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