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ON GENERALIZED $C^{(n)}$-ALMOST PERIODIC SOLUTIONS OF ABSTRACT VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. The main purpose of this paper is to introduce the class of (equi-) Weyl $C^{(n)}$-almost periodic functions and the class of asymptotically (equi-) Weyl $C^{(n)}$-almost periodic functions as well as to examine the existence and uniqueness of solutions of abstract inhomogenous Volterra integro-differential equations belonging these classes of functions. The Besicovitch-Doss $C^{(n)}$-almost periodic functions and solutions of abstract inhomogenous Volterra integro-differential equations belonging this class of functions are also considered.

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1. Introduction and preliminaries

The class of scalar-valued $C^{(n)}$-almost periodic functions was introduced by M. Adamczak [1] in 1997, while the class of $C^{(n)}$-almost periodic functions with values in Banach spaces and the class of asymptotically $C^{(n)}$-almost periodic functions with values in Banach spaces were introduced by D. Bugajewski and G. M. N'Guérékata [8] in 2004. From then on, the analysis of various generalizations of (asymptotically) $C^{(n)}$-almost periodic functions and their applications to the qualitative analysis of abstract non-degenerate integro-differential equations in Banach spaces has attracted the attention of a great number of authors working in this subfield of functional analysis (see e.g. [5], [13]-[12], [16]-[17] and [25]). For example, J. B. Baillon, J. Blot, G. M. N'Guérékata and D. Pennequin [5] have analyzed the existence and uniqueness of $C^{(n)}$-almost periodic solutions to some nonautonomous differential equations in Banach spaces, while A. Elazzouzi [16] has analyzed $C^{(n)}$-almost periodic and $C^{(n)}$-almost automorphic solutions for a class of partial functional differential equations with finite delay.

In this paper, we continue our recent research studies [23]-[22] by investigating the Weyl $C^{(n)}$-almost periodic solutions, asymptotically Weyl $C^{(n)}$-almost periodic solutions and Besicovitch-Doss $C^{(n)}$-almost periodic solutions of abstract Volterra integro-differential equations and inclusions with multivalued linear operators (for the theories of abstract degenerate integro-differential

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equations and multivalued linear operators in Banach spaces, we refer the reader to the monographs [18] by A. Favini, A. Yagi, [10] by R. Cross and [19] by M. Kostić); in our forthcoming monograph [20], we will analyze similar concepts within the framework of theory of almost automorphic functions. Our results seem to be new even for abstract non-degenerate differential equations of the first order whose solutions are governed by strongly continuous semigroups of operators and for abstract non-degenerate differential equations with almost sectorial operators ([27]). The paper is very simply organized, with the main results clarified in Section 4, where we investigate the (asymptotically) Weyl $C^{(n)}$-almost periodic properties and Besicovitch-Doss $C^{(n)}$-almost periodic properties of convolution products as well as the so-called convolution invariance of introduced function spaces. In conclusion of this, introductory paragraph,, we only want to note that the classes of Weyl $C^{(n)}$-almost periodic functions, asymptotically Weyl $C^{(n)}$-almost periodic functions and Besicovitch-Doss $C^{(n)}$-almost periodic functions seem to be not considered elsewhere even in scalar-valued case. For the sake of brevity, we will not analyze abstract semilinear Cauchy inclusions and topologization of introduced function spaces here.

We use the standard notation throughout the paper. By X we denote a complex Banach space. If Y is likewise a complex Banach space, then by $L(X,Y)$ we denote the space consisting of all continuous linear mappings from X into Y; $L(X) \equiv L(X,X)$. The symbol $C_b([0,\infty) : X)$ stands for the space consisting of all bounded continuous functions from $[0,\infty)$ into X, while $C_0([0,\infty) : X)$ denotes the closed subspace of $C_b([0,\infty) : X)$ consisting of all such functions vanishing at infinity. By $BUC([0,\infty) : X)$ we designate the space consisting of all bounded uniformly continuous functions from $[0,\infty)$ to X. Equipped with the sup-norm, any of the above spaces is a Banach one.

Let $-\infty < a < b < \infty$. Recall, a function $f : [a,b] \to X$ is said to be absolutely continuous iff for every $\epsilon > 0$ there exists $\delta > 0$ such that for any finite collection of open subintervals (a_i, b_i), $1 \leq i \leq k$ of $[a,b]$ with $\sum_{i=1}^{k}(b_i - a_i) < \delta$, the following holds $\sum_{i=1}^{k} \|f(b_i) - f(a_i)\| < \epsilon$. If $I = \mathbb{R}$ or $I = [0,\infty)$, and $f : I \to X$, then we say that $f(\cdot)$ is locally absolutely continuous iff $f(\cdot)$ is absolutely continuous on any compact subinterval of I.

1.1. Asymptotically almost periodic functions and Stepanov generalizations

As it is well known, the notion of an almost periodic function was introduced by H. Bohr in 1925 and later generalized by many other authors (cf. [11], [26] and [24] for more details on the subject). Let $I = \mathbb{R}$ or $I = [0,\infty)$, and let $f : I \to X$ be continuous. Given $\epsilon > 0$, we call $\tau > 0$ an ϵ-period for $f(\cdot)$ iff $\|f(t+\tau) - f(t)\| \leq \epsilon$, $t \in I$. The set consisting of all ϵ-periods for $f(\cdot)$ is denoted by $\vartheta(f,\epsilon)$. We say that $f(\cdot)$ is almost periodic, a.p. for short, iff for each $\epsilon > 0$ the set $\vartheta(f,\epsilon)$ is relatively dense in I, which means that there exists $l > 0$ such that any subinterval of I of length l has a non-empty intersection with $\vartheta(f,\epsilon)$. The space consisting of all almost periodic functions from the interval I into X will be denoted by $AP(I : X)$.

The notion of an asymptotically almost periodic function was introduced by M. Fréchet in 1941 (for further information concerning the vector-valued asymptotically almost periodic functions, see e.g. [9], [11] and [26]). Let us recall that a function $f \in C_b([0,\infty) : X)$ is asymptotically almost periodic iff for every $\epsilon > 0$ we can find numbers $l > 0$ and $M > 0$ such that every subinterval of $[0,\infty)$ of length l contains, at least, one number τ such that $\|f(t+\tau)-f(t)\| \leq \epsilon$ for all $t \geq M$. The space consisting of all asymptotically almost periodic functions from $[0,\infty)$ into X will be denoted by $AAP([0,\infty) : X)$. It is well known that $f \in AAP([0,\infty) : X)$ iff there exist uniquely determined functions $g \in AP([0,\infty) : X)$ and $\phi \in C_0([0,\infty) : X)$ such that $f = g + \phi$.

Let $1 \leq p < \infty$, let $l > 0$, and let $f,\ g \in L^p_{loc}(I : X)$, where $I = \mathbb{R}$ or $I = [0,\infty)$. Define the Stepanov 'metric' by

$$D^p_{S_l}\big[f(\cdot), g(\cdot)\big] := \sup_{x\in I}\left[\frac{1}{l}\int_x^{x+l} \big\|f(t)-g(t)\big\|^p\,dt\right]^{1/p}.$$

Then, in scalar-valued case, there exists

$$D^p_W\big[f(\cdot), g(\cdot)\big] := \lim_{l\to\infty} D^p_{S_l}\big[f(\cdot), g(\cdot)\big] \tag{1.1}$$

in $[0,\infty]$. The distance appearing in (1.1) is called the Weyl 'distance' of $f(\cdot)$ and $g(\cdot)$. The Stepanov and Weyl 'norm' of $f(\cdot)$ are defined by

$$\big\|f\big\|_{S^p_l} := D^p_{S_l}\big[f(\cdot), 0\big] \quad \text{and} \quad \big\|f\big\|_{W^p} := D^p_W\big[f(\cdot), 0\big],$$

respectively.

It is said that a function $f \in L^p_{loc}(I : X)$ is Stepanov p-bounded, S^p-bounded shortly, iff

$$\|f\|_{S^p} := \sup_{t\in I}\left(\int_t^{t+1} \|f(s)\|^p\,ds\right)^{1/p} < \infty.$$

The space $L^p_S(I : X)$ consisting of all S^p-bounded functions is a Banach space when equipped with the above norm. A function $f \in L^p_S(I : X)$ is said to be Stepanov p-almost periodic, S^p-almost periodic for short, iff the function $\hat{f} : I \to L^p([0,1] : X)$, defined by

$$\hat{f}(t)(s) := f(t+s), \quad t \in I,\ s \in [0,1]$$

is almost periodic (cf. [2] for more details). We say that $f \in L^p_S([0,\infty) : X)$ is asymptotically Stepanov p-almost periodic, asymptotically S^p-almost periodic for short, iff $\hat{f} : [0,\infty) \to L^p([0,1] : X)$ is asymptotically almost periodic.

It is a well-known fact that if $f(\cdot)$ is an almost periodic (respectively, a.a.p.) function then $f(\cdot)$ is also S^p-almost periodic (resp., asymptotically S^p-a.a.p.) for $1 \leq p < \infty$. The converse statement is false, however.

Denote by $APS^p(I : X)$ and $AAPS^p([0,\infty) : X)$ the spaces consisting of all S^p-almost periodic functions $I \mapsto X$ and asymptotically S^p-almost periodic functions $[0,\infty) \mapsto X$, respectively. By $APS^p_b(I : X)$ we denote the space consisting of all essentially bounded functions from $APS^p(I : X)$.

Various classes of Stepanov-like $C^{(n)}$-pseudo almost automorphic functions have been considered by T. Diagana, V. Nelson and G. M. N'Guérékata in [13] (see e.g. [13, Definition 2.18]). For our work, the classes of Stepanov $C^{(n)}$-almost periodic functions and asymptotically Stepanov $C^{(n)}$-almost periodic functions will be general enough (the derivatives appearing below will be taken in distributional sense).

Definition 1.1. Let $1 \leq p < \infty$, let $n \in \mathbb{N}$, and let $f \in L^p_{loc}(I : X)$.

(i) It is said that the function $f(\cdot)$ is Stepanov-p-$C^{(n)}$-almost periodic, $f \in C^{(n)} - APS^p(I : X)$ for short, iff for each $k = 0, 1, \ldots, n$, we have that $f^{(k)} \in APS^p(I : X)$.

(ii) It is said that the function $f(\cdot)$ is asymptotically Stepanov-p-$C^{(n)}$-almost periodic, $f \in C^{(n)} - AAPS^p([0,\infty) : X)$ for short, iff for each $k = 0, 1, \ldots, n$, we have that $f^{(k)} \in AAPS^p([0,\infty) : X)$.

Before proceeding further, we would like to point out that the inclusion $f \in C^{(n)} - APS^p(I : X)$ does not imply that $f \in C^n(I : X)$, in general, where $C^n(I : X)$ denotes the space consisted of all n-times continuously differentiable functions $I \to X$. Speaking-matter-of-factly, the inclusion $f \in C^{(n)} - APS^p(I : X)$ or $f \in C^{(n)} - AAPS^p(I : X)$ implies that $f \in C^{n-1}(I : X)$ as well as that $f^{(n)} \in L^p_{loc}(I : X)$ and $f^{(n-1)}(\cdot)$ is locally absolutely continuous; see e.g. [4, Proposition 1.2.2] and [6, Chapter I, Section 2.2]. This observation also holds for any (asymptotically) generalized $C^{(n)}$-almost periodic function space considered below.

Now we would like to present the following example.

Example 1.2. Let $1 \leq p < \infty$, and let X be any non-trivial complex Banach space. Then, for any S^p-almost periodic function $f(\cdot)$ and for any real number $\delta \in (0,1)$, we define the function

$$f_\delta(t) := \frac{1}{\delta} \int_t^{t+\delta} f(s)\, ds, \quad t \in I.$$

Arguing as in scalar-valued case [7], we can prove that the function $f_\delta(\cdot)$ is almost periodic $(0 < \delta < 1)$ as well as that $\|f_\delta - f\|_{S^p} \to 0$ as $\delta \to 0^+$. Assume now that $f(\cdot)$ is S^p-almost periodic but not almost periodic. Then, for $0 < \delta < 1$, one has:

$$f'_\delta(t) := \frac{1}{\delta}\big[f(t+\delta) - f(t)\big] \text{ for a.e. } t \in I.$$

Hence, $f_\delta(\cdot) \in C^{(1)} - APS^p(I : X) \setminus C^{(1)} - AP(I : X)$ for $0 < \delta < 1$, so that $C^{(1)} - APS^p(I : X)$ is a strict extension of $C^{(1)} - AP(I : X)$; here, $C^{(1)} - AP(I : X)$ denotes the usually considered space of $C^{(1)}$-almost periodic functions $I \to X$.

We want also to introduce the following function spaces:

Definition 1.3. Let $1 \leq p < \infty$, let $n \in \mathbb{N}$, and let $f \in L^p_{loc}(I : X)$.

(i) It is said that the function $f(\cdot)$ is Stepanov-p-$C_b^{(n)}$-almost periodic, $f \in C_b^{(n)} - APS^p(I : X)$ for short, iff for each $k = 0, 1, \ldots, n$, we have that $f^{(k)} \in APS^p(I : X) \cap L^\infty(I : X)$.

(ii) Let $I = [0, \infty)$. It is said that the function $f(\cdot)$ is asymptotically Stepanov-p-$C_b^{(n)}$-almost periodic, $f \in C_b^{(n)} - AAPS^p([0, \infty) : X)$ for short, iff for each $k = 0, 1, \ldots, n$, we have that $f_b^{(k)} \in AAPS^p([0, \infty) : X)$.

Arguing similarly as in the proof of [11, Proposition 3.4, p. 81] and Theorem 4.10 below, we can prove the following result:

Proposition 1.4. *Let $n \in \mathbb{N}$, and let $f \in C_b^{(n)} - APS^1(\mathbb{R} : X)$, resp. $f \in APS^1_b(\mathbb{R} : X)$. Then, for every $g \in L^1(\mathbb{R})$, we have that $(g * f)(\cdot) := \int_{\mathbb{R}} f(\cdot - y)g(y)\,dy \in C_b^{(n)} - APS^1(\mathbb{R} : X)$, resp. $(g * f)(\cdot) \in APS^1_b(\mathbb{R} : X)$.*

2. On Weyl and Besicovitch-Doss generalizations of almost periodic functions

Let $I = \mathbb{R}$ or $I = [0, \infty)$, and let the underlying Banach space be denoted by X. The notion of an (equi-) Weyl almost periodic function is given as follows.

Definition 2.1. Let $1 \leq p < \infty$ and $f \in L^p_{loc}(I : X)$.

(i) It is said that the function $f(\cdot)$ is equi-Weyl-p-almost periodic, $f \in e - W^p_{ap}(I : X)$ for short, iff for each $\epsilon > 0$ we can find two real numbers $l > 0$ and $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that

$$\sup_{x \in I} \left[\frac{1}{l} \int_x^{x+l} \|f(t+\tau) - f(t)\|^p \, dt \right]^{1/p} \leq \epsilon,$$

$$\text{i.e., } D^p_{S_l}\big[f(\cdot + \tau), f(\cdot)\big] \leq \epsilon.$$

(ii) It is said that the function $f(\cdot)$ is Weyl-p-almost periodic, $f \in W^p_{ap}(I : X)$ for short, iff for each $\epsilon > 0$ we can find a real number $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that

$$\lim_{l \to \infty} \sup_{x \in I} \left[\frac{1}{l} \int_x^{x+l} \|f(t+\tau) - f(t)\|^p \, dt \right]^{1/p} \leq \epsilon,$$

$$\text{i.e., } \lim_{l \to \infty} D^p_{S_l}\big[f(\cdot + \tau), f(\cdot)\big] \leq \epsilon.$$

It is well-known that

$$APS^p(I : X) \subseteq e - W^p_{ap}(I : X) \subseteq W^p_{ap}(I : X)$$

in the set theoretical sense and that any of these two inclusions can be strict ([3]). Denote by $W^p_{ap,b}(I : X)$, resp. $e - W^p_{ap,b}(I : X)$, the space consisting of all essentially bounded functions from $W^p_{ap}(I : X)$, resp. $e - W^p_{ap}(I : X)$.

Now we will inscribe the basic definitions and results about asymptotically Weyl almost periodic functions ([23]). For any function $q \in L^p_{loc}([0,\infty) : X)$, we define $\mathbf{q}(\cdot,\cdot) : [0,\infty) \times [0,\infty) \to X$ by $\mathbf{q}(t,s) := q(t+s)$, t, $s \geq 0$.

Definition 2.2. We say that $q \in L^p_{loc}([0,\infty) : X)$ is Weyl-p-vanishing iff

(2.1)

$$\lim_{t\to\infty} \big\| \mathbf{q}(t,\cdot) \big\|_{W^p} = 0, \text{ i.e., } \lim_{t\to\infty} \lim_{l\to\infty} \sup_{x\geq 0} \Bigg[\frac{1}{l} \int_x^{x+l} \big\| q(t+s) \big\|^p \, ds \Bigg]^{1/p} = 0.$$

For any function $q \in L^p_{loc}([0,\infty) : X)$ we can reverse the order of the limits in (2.1). It is said that $q \in L^p_{loc}([0,\infty) : X)$ is equi-Weyl-p-vanishing iff

$$\lim_{l\to\infty} \lim_{t\to\infty} \sup_{x\geq 0} \Bigg[\frac{1}{l} \int_x^{x+l} \big\| q(t+s) \big\|^p \, ds \Bigg]^{1/p} = 0.$$

We know that, if $q \in L^p_{loc}([0,\infty) : X)$ and $q(\cdot)$ is equi-Weyl-p-vanishing, then $q(\cdot)$ is Weyl-p-vanishing. The converse statement does not hold, however. We also want to note that an equi-Weyl-p-vanishing function $q(\cdot)$ need not be bounded as $t \to +\infty$.

Denote by $W^p_0([0,\infty) : X)$ and $e - W^p_0([0,\infty) : X)$ the sets consisting of all Weyl-p-vanishing functions and equi-Weyl-p-vanishing functions, respectively. The symbol $S^p_0([0,\infty) : X)$ will be used to denote the set of all functions $q \in L^p_{loc}([0,\infty) : X)$ such that $\hat{q} \in C_0([0,\infty) : L^p([0,1] : X))$. Then the following inclusions hold

$$L^p([0,\infty) : X) \subseteq S^p_0([0,\infty) : X) \subseteq e - W^p_0([0,\infty) : X) \subseteq W^p_0([0,\infty) : X)$$

and any of them can be strict [23].

In [23], we have introduced the following function spaces:

$$AAPW^p([0,\infty) : X) := AP([0,\infty) : X) + W^p_0([0,\infty) : X),$$

$$e - AAPW^p([0,\infty) : X) := AP([0,\infty) : X) + e - W^p_0([0,\infty) : X),$$

$$AAPSW^p([0,\infty) : X) := APS^p([0,\infty) : X) + W^p_0([0,\infty) : X),$$

$$e - AAPSW^p([0,\infty) : X) := APS^p([0,\infty) : X) + e - W^p_0([0,\infty) : X),$$

$$e - W^p_{aap}([0,\infty) : X) := e - W^p_{ap}([0,\infty) : X) + W^p_0([0,\infty) : X),$$

$$ee - W^p_{aap}([0,\infty) : X) := e - W^p_{ap}([0,\infty) : X) + e - W^p_0([0,\infty) : X),$$

$$W^p_{aap}([0,\infty):X) := W^p_{ap}([0,\infty):X) + W^p_0([0,\infty):X),$$
$$W^p_{eaap}([0,\infty):X) := W^p_{ap}([0,\infty):X) + e-W^p_0([0,\infty):X).$$

Then

$$\begin{aligned} AAPW^p([0,\infty):X) &\subseteq AAPSW^p([0,\infty):X) \\ &\subseteq e-W^p_{aap}([0,\infty):X) \subseteq W^p_{aap}([0,\infty):X) \end{aligned}$$

and

$$\begin{aligned} e-AAPW^p([0,\infty):X) &\subseteq e-AAPSW^p([0,\infty):X) \\ &\subseteq ee-W^p_{aap}([0,\infty):X) \subseteq W^p_{eaap}([0,\infty):X), \end{aligned}$$

with any of these inclusions being strict, in general.

The sums defining $e-W^p_{aap}([0,\infty):X)$, $ee-W^p_{aap}([0,\infty):X)$, $W^p_{aap}([0,\infty):X)$ and $W^p_{eaap}([0,\infty):X)$ cannot be direct. For the spaces $AAPW^p([0,\infty):X)$, $e-AAPW^p([0,\infty):X)$, $AAPSW^p([0,\infty):X)$ and $e-AAPSW^p([0,\infty):X)$, the sums in their definitions are direct, which follows from the fact that, for $1 \leq p < \infty$, we have $W^p_0([0,\infty):X) \cap APS^p([0,\infty):X) = \{0\}$; see [23].

2.1. Besicovitch-Doss almost periodic functions

The class of Besicovitch almost periodic functions extends the classes of Stepanov and Weyl almost periodic functions. The standard procedure for the introducing the notion of Besicovitch almost periodic functions goes as follows. Assume that $1 \leq p < \infty$. Following A. S. Besicovitch [7], for every function $f \in L^p_{loc}(\mathbb{R}:X)$, we define

$$\|f\|_{\mathcal{M}^p} := \limsup_{t\to+\infty}\left[\frac{1}{2t}\int_{-t}^{t}\|f(s)\|^p\,ds\right]^{1/p};$$

if $f \in L^p_{loc}([0,\infty):X)$, then

$$\|f\|_{\mathcal{M}^p} := \limsup_{t\to+\infty}\left[\frac{1}{t}\int_{0}^{t}\|f(s)\|^p\,ds\right]^{1/p}.$$

In both cases, $\|\cdot\|_{\mathcal{M}^p}$ is a seminorm on the space $\mathcal{M}^p(I:X)$ consisting of those $L^p_{loc}(I:X)$-functions $f(\cdot)$ for which $\|f\|_{\mathcal{M}^p} < \infty$. Define $K_p(I:X) := \{f \in \mathcal{M}^p(I:X) : \|f\|_{\mathcal{M}^p} = 0\}$ and

$$M_p(I:X) := \mathcal{M}^p(I:X)/K_p(I:X).$$

The seminorm $\|\cdot\|_{\mathcal{M}^p}$ on $\mathcal{M}^p(I:X)$ induces the norm $\|\cdot\|_{M^p}$ on $M^p(I:X)$ for which $(M^p(I:X), \|\cdot\|_{M^p})$ is a Banach space.

Definition 2.3. Let $1 \leq p < \infty$. Then it is said that a function $f \in L^p_{loc}(I:X)$ is Besicovitch-p-almost periodic iff there exists a sequence of X-valued trigonometric polynomials converging to $f(\cdot)$ in $(M^p(I:X), \|\cdot\|_{M^p})$.

The vector space consisting of all Besicovitch-p-almost periodic functions $I \to X$ is denoted by $B^p(I : X)$. Then $B^p(I : X)$ is a closed subspace of $M^p(I : X)$.

The notion of Besicovitch-Doss-p-almost periodic function has been recently introduced in [22] following the fundamental characterization of scalar-valued Besicovitch almost periodic functions established by R. Doss in [14]-[15] (cf. also [3, pp. 160-161]):

Definition 2.4. Assume $1 \leq p < \infty$. We say that $f \in L^p_{loc}(I : X)$ is Besicovitch-Doss-p-almost periodic iff the following conditions hold:

(i) (B^p-boundedness) We have $\|f\|_{\mathcal{M}^p} < \infty$.

(ii) (B^p-continuity) We have

$$\lim_{\tau \to 0} \limsup_{t \to +\infty} \left[\frac{1}{2t} \int_{-t}^{t} \|f(s+\tau) - f(s)\|^p \, ds \right]^{1/p} = 0,$$

in the case that $I = \mathbb{R}$, resp.,

$$\lim_{\tau \to 0+} \limsup_{t \to +\infty} \left[\frac{1}{t} \int_{0}^{t} \|f(s+\tau) - f(s)\|^p \, ds \right]^{1/p} = 0,$$

in the case that $I = [0, \infty)$.

(iii) (Doss almost periodicity) For every $\epsilon > 0$, the set of numbers $\tau \in I$ for which

$$\limsup_{t \to +\infty} \left[\frac{1}{2t} \int_{-t}^{t} \|f(s+\tau) - f(s)\|^p \, ds \right]^{1/p} < \epsilon,$$

in the case that $I = \mathbb{R}$, resp.,

$$\limsup_{t \to +\infty} \left[\frac{1}{t} \int_{0}^{t} \|f(s+\tau) - f(s)\|^p \, ds \right]^{1/p} < \epsilon,$$

in the case that $I = [0, \infty)$, is relatively dense in I.

(iv) For every $\lambda \in \mathbb{R}$, we have that

$$\lim_{l \to +\infty} \limsup_{t \to +\infty} \frac{1}{l} \left[\frac{1}{2t} \int_{-t}^{t} \left\| \left(\int_{x}^{x+l} - \int_{0}^{l} \right) e^{i\lambda s} f(s) \, ds \right\|^p dx \right]^{1/p} = 0,$$

in the case that $I = \mathbb{R}$, resp.,

$$\lim_{l \to +\infty} \limsup_{t \to +\infty} \frac{1}{l} \left[\frac{1}{t} \int_{0}^{t} \left\| \left(\int_{x}^{x+l} - \int_{0}^{l} \right) e^{i\lambda s} f(s) \, ds \right\|^p dx \right]^{1/p} = 0,$$

in the case that $I = [0, \infty)$.

The vector space consisting of all Besicovitch-Doss-p-almost periodic functions $I \to X$ is denoted by $\mathrm{B}^p(I : X)$. If $X = \mathbb{C}$, then the fundamental result of R. Doss says that $\mathrm{B}^p(I : X) = B^p(I : X)$. In [22], we have raised the following problem: Assume $1 \leq p < \infty$, and X is a Banach space. Is it true that $\mathrm{B}^p(I : X) = B^p(I : X)$ in the set theoretical sense?

In the sequel of this paper, we will consider both spaces $B^p(I : X)$ and $\mathrm{B}^p(I : X)$, although in the present situation we do not know whether they are equal and although our main results will be clarified only for the class of Besicovitch-Doss $C^{(n)}$-almost periodic functions.

Definition 2.5. [22] It is said that $q \in L^p_{loc}([0,\infty) : X)$ is Besicovitch-p-vanishing iff

$$\lim_{t\to\infty} \big\|\mathbf{q}(t,\cdot)\big\|_{\mathcal{M}^p} = 0, \text{ i.e., } \lim_{t\to+\infty} \limsup_{s\to+\infty} \left[\frac{1}{s}\int_0^s \big\|q(t+r)\big\|^p\, dr\right]^{1/p} = 0.$$

The class consisting of all Besicovitch-p-vanishing functions extends the corresponding class consisting of all Weyl-p-vanishing functions. For the sequel, it would be very important to note that the class of Besicovitch-p-vanishing functions is equal to the class of p-locally integrable X-valued functions whose Besicovitch seminorm is equal to zero. This basically follows from the analysis of R. Doss [14, p. 478], showing that, for every $q \in L^p_{loc}([0,\infty) : X)$, we have $\big\|\mathbf{q}(t,\cdot)\big\|_{\mathcal{M}^p} = \|q\|_{\mathcal{M}^p}$, $t \geq 0$. As a simple corollary of this equality, we have the following result [22]: Let $1 \leq p < \infty$, and let $q \in L^p_{loc}([0,\infty) : X)$. Then $q(\cdot)$ is Besicovitch-p-vanishing iff $\|q\|_{\mathcal{M}^p} = 0$ iff $q \in K_p([0,\infty) : X)$ iff

$$\lim_{s\to+\infty} \frac{1}{s}\int_0^s \big\|q(r)\big\|^p\, dr = 0.$$

Denote by $B^p_0([0,\infty) : X)$ the set consisting of all Besicovitch-p-vanishing functions. Then $B^p_0([0,\infty) : X)$ has a linear vector structure. As pointed out in [22], it is ridiculous to introduce the space of asymptotically Besicovitch almost periodic functions and asymptotically Besicovitch-Doss almost periodic functions since the sum of space $B^p(I : X)$ ($\mathrm{B}^p([0,\infty) : X)$) and $B^p_0([0,\infty) : X)$ is again $B^p(I : X)$ ($\mathrm{B}^p([0,\infty) : X)$).

3. Generalized $C^{(n)}$-almost periodic functions and generalized asymptotically $C^{(n)}$-almost periodic functions

We start this section by introducing the following notions (as before, the derivatives appearing below will be taken in distributional sense).

Definition 3.1. Let $1 \leq p < \infty$, let $n \in \mathbb{N}$, and let $f \in L^p_{loc}(I : X)$.

(i) It is said that the function $f(\cdot)$ is equi-Weyl-p-$C^{(n)}$-almost periodic, $f \in e - C^{(n)} - W^p_{ap}(I : X)$ for short, iff for each $k = 0, 1, \ldots, n$, we have that $f^{(k)} \in e - W^p_{ap}(I : X)$.

(ii) It is said that the function $f(\cdot)$ is Weyl-p-$C^{(n)}$-almost periodic, $f \in C^{(n)} - W^p_{ap}(I : X)$ for short, iff for each $k = 0, 1, \ldots, n$, we have that $f^{(k)} \in W^p_{ap}(I : X)$.

The spaces $C^{(n)} - AAPW^p([0,\infty) : X)$, $e - C^{(n)} - AAPW^p([0,\infty) : X)$, $C^{(n)} - AAPSW^p([0,\infty) : X)$, $e - C^{(n)} - AAPSW^p([0,\infty) : X)$, $e - C^{(n)} - W^p_{aap}([0,\infty) : X)$, $ee - C^{(n)} - W^p_{aap}([0,\infty) : X)$, $C^{(n)} - W^p_{aap}([0,\infty) : X)$ and $C^{(n)} - W^p_{eaap}([0,\infty) : X)$ are introduced similarly.

We continue by providing three illustrative examples.

Example 3.2. Define the function $f : \mathbb{R} \to \mathbb{C}$ by

$$f(x) := \chi_{(-\infty,0]}(x) + x\chi_{[0,1/2]}(x) + \chi_{(1/2,\infty)}(x), \quad x \in \mathbb{R}.$$

Then it is easily verified that its first distributional derivative is the locally integrable function $g : \mathbb{R} \to \mathbb{C}$ given by $g(x) := \chi_{(0,1/2)}(x)$, $x \in \mathbb{R}$, which is not Stepanov almost periodic but equi-Weyl-1-almost periodic (see e.g. [3, Example 4.27]) and therefore Weyl-1-almost periodic. Denote by $H(x) := \chi_{(0,\infty)}(x)$, $x \in \mathbb{R}$ the well known Heaviside function, for which we know that it is Weyl-1-almost periodic but not equi-Weyl-1-almost periodic [3]. Hence, in order to see that $f \in C^{(1)} - W^1_{ap}(\mathbb{R} : X) \setminus e - C^{(1)} - W^1_{ap}(\mathbb{R} : X)$, we only need to prove that the function $h(x) := x\chi_{[0,1/2]}(x)$, $x \in \mathbb{R}$ is equi-Weyl-1-almost periodic. But, this simply follows from definition of an equi-Weyl-1-almost periodic function and the following obvious estimates:

$$\begin{aligned} &\frac{1}{l}\int_x^{x+l} \bigl\|h(t+\tau) - h(t)\bigr\|\,dt \\ &\quad \leq \frac{1}{l}\int_x^{x+l} \bigl\|h(t+\tau)\bigr\|\,dt + \frac{1}{l}\int_x^{x+l} \bigl\|h(t)\bigr\|\,dt \\ &\quad \leq \frac{2}{l}\int_0^{1/2} \bigl\|h(t)\bigr\|\,dt \leq \frac{1}{2l}, \quad l > 0,\ \tau \in \mathbb{R}. \end{aligned}$$

It is clear that $f \notin C^{(2)} - W^1_{ap}(\mathbb{R} : X)$ since the weak derivative of $g(\cdot)$ is not a regular distribution. At the end of this example, we would like to observe that we can construct a great number of other function spaces by using a 'Sobolev type idea' from Definition 3.1, allowing possibly that the function and its derivatives belong to different function spaces (here, concretely, $f \in W^1_{ap}(\mathbb{R} : X)$ and $f' \in e - W^1_{ap}(\mathbb{R} : X)$ see also Example 1.2). It would take too long to consider such spaces here.

Example 3.3. Define

$$\mathcal{A} := \bigl\{f \in L^\infty(I : X) : \operatorname{supp}(f) \text{ is compact}\bigr\}.$$

Then the computation used in the former example shows that $\mathcal{A} \subseteq e - W^1_{ap}(I : X)$. On the other hand, any non-trivial function from $\mathcal{A}$ cannot be Stepanov almost periodic; if we suppose the contrary, then the identity [4, (4.24)] would imply

$$\sup_{t\in I}\int_t^{t+1}\|f(s)\|\,ds=\sup_{t\geq t_0}\int_t^{t+1}\|f(s)\|\,ds,\quad t_0\in I;$$

by choosing t_0 arbitrarily large, the above would imply $\sup_{t\in I}\int_t^{t+1}\|f(s)\|\,ds = 0$ for all $t\in I$ and therefore $f(s)=0$ a.e. $s\in I$. This can be easily employed for the construction of a function $f\in e-C^{(1)}-W^1_{ap}(I:X)\setminus APS^1(I:X)$; a typical example is given by

$$\begin{aligned} f(x):=&\chi_{(-\infty,-1)}(x)+2(x+1)\chi_{[-1,-1/2)}(x)+\chi_{[-1/2,1/2]}(x)\\ &-2(x-1)\chi_{(1/2,1)}(x)+\chi_{(1,\infty)}(x),\quad x\in\mathbb{R}. \end{aligned}$$

Example 3.4. Let $f(\cdot)$ be defined by the last formula, and let $f^{[1]}(\cdot)$ be its first anti-derivative, defined as usually. Then it can be simply verified that $f^{[1]}\in C^{(2)}-W^1_{ap}(\mathbb{R}:X)$ but $f^{[1]}\notin e-W^1_{ap}(\mathbb{R}:X)$.

We introduce the notions of a Besicovitch $C^{(n)}$-almost periodic function and a Besicovitch-Doss $C^{(n)}$-almost periodic function similarly as in Definition 3.1.

Definition 3.5. Let $1\leq p<\infty$, let $n\in\mathbb{N}$, and let $f\in L^p_{loc}(I:X)$.

(i) It is said that the function $f(\cdot)$ is Besicovitch-p-$C^{(n)}$-almost periodic, $f\in C^{(n)}-B^p(I:X)$ for short, iff for each $k=0,1,\ldots,n$, we have that $f^{(k)}\in B^p(I:X)$.

(ii) It is said that the function $f(\cdot)$ is Besicovitch-Doss-p-$C^{(n)}$-almost periodic, $f\in C^{(n)}-\mathrm{B}^p(I:X)$ for short, iff for each $k=0,1,\ldots,n$, we have that $f^{(k)}\in \mathrm{B}^p(I:X)$.

A fairly complete analysis of introduced function spaces is outside the scope of this paper and we deeply believe that this theme could be interesting and motivating to our readers. Now we would like to illustrate the importance of such function spaces in the qualitative analysis of solutions of abstract inhomogenous Volterra integro-differential equations:

4. Weyl, asymptotically Weyl and Besicovitch-Doss $C^{(n)}$-almost periodic properties of convolution products

In this section, we examine the Weyl, asymptotically Weyl and Besicovitch-Doss $C^{(n)}$-almost periodic properties of convolution products (Stepanov case can be considered similarly [21]). It seems that choosing the value $p=1$ is almost inevitable for saying anything relevant concerning this subject. In addition to the above, we analyze the convolution invariance of introduced function spaces.

For our investigation of Weyl-$C^{(n)}$-almost periodicity of infinite convolution products, it will be very important to introduce the following subclasses of (equi-) Weyl-$C^{(n)}$-almost periodic functions.

Definition 4.1. Let $1 \leq p < \infty$, let $n \in \mathbb{N}$, and let $f \in L^p_{loc}(I : X)$.

(i) It is said that the function $f(\cdot)$ is equi-Weyl-p-$C_b^{(n)}$-almost periodic, $f \in e - C_b^{(n)} - W^p_{ap}(I : X)$ for short, iff for each $k = 0, 1, \ldots, n$, we have that $f^{(k)} \in e - W^p_{ap}(I : X) \cap L^\infty(I : X)$.

(ii) It is said that the function $f(\cdot)$ is Weyl-p-$C_b^{(n)}$-almost periodic, $f \in C_b^{(n)} - W^p_{ap}(I : X)$ for short, iff for each $k = 0, 1, \ldots, n$, we have that $f^{(k)} \in W^p_{ap}(I : X) \cap L^\infty(I : X)$.

Now we are ready to state the following proposition.

Proposition 4.2. *(i) Suppose that $n \in \mathbb{N}$ and $(R(t))_{t>0} \subseteq L(X, Y)$ is a strongly continuous operator family satisfying $\int_0^\infty \|R(s)\|\, ds < \infty$. If $g : \mathbb{R} \to X$ is in the class $e - C_b^{(n)} - W^1_{ap}(\mathbb{R} : X)$, resp. $C_b^{(n)} - W^1_{ap}(\mathbb{R} : X)$, then the function $G(\cdot)$, given by*

$$G(t) := \int_{-\infty}^{t} R(t-s)g(s)\, ds, \quad t \in \mathbb{R}, \tag{4.1}$$

is in the same class, as well.

(ii) Suppose that $(R(t))_{t>0} \subseteq L(X, Y)$ is a strongly continuous operator family satisfying $M = \sum_{k=0}^{\infty} \|R(\cdot)\|_{L^\infty[k,k+1]} < \infty$. If $g : \mathbb{R} \to X$ is in the class $C^{(n)} - W^1_{ap}(I : X)$, then the function $G(\cdot)$, given by (4.1), is in the class $C_b^{(n)} - W^1_{ap}(I : X)$.

Proof. By [23, Proposition 5.1(i)], we have that $G(\cdot)$ is bounded and belongs to the class $e - W^1_{ap}(\mathbb{R} : X)$, resp. $W^1_{ap}(\mathbb{R} : X)$. The final conclusion follows from the fact, for $0 \leq l \leq n$, we have

$$G^{(l)}(t) = \int_0^\infty R(s)g^{(l)}(t-s)\, ds, \quad t \in \mathbb{R}, \tag{4.2}$$

which can be simply proved by applying the dominated convergence theorem. To prove the second part, we first observe that [23, Proposition 5.1(ii)] implies that $G(\cdot)$ is bounded and equi-Weyl-1-almost periodic. Therefore, it suffices to show that the equation (4.2) holds for $0 \leq l \leq n$. This follows by induction and its validity for $l = 1$, which is a consequence of the following computation involving the Fubini theorem:

$$\begin{aligned} G(t) &= G(0) + \int_0^\infty R(v)[g(t-v) - g(v)]\, dv \\ &= G(0) + \int_0^\infty \int_0^t R(v)g'(s-v)\, ds\, dv \\ &= G(0) + \int_0^t \int_0^\infty R(v)g'(s-v)\, ds\, dv, \quad t \in \mathbb{R}. \end{aligned}$$

□

The following notion is very similar to the corresponding one already introduced in Definition 4.1.

Definition 4.3. Let $1 \leq p < \infty$, let $n \in \mathbb{N}$, and let $f \in L^p_{loc}(I : X)$.

(i) It is said that the function $f(\cdot)$ is Besicovitch-p-$C_b^{(n)}$-almost periodic, $f \in C_b^{(n)} - B^p(I : X)$ for short, iff for each $k = 0, 1, \ldots, n$, we have that $f^{(k)} \in B^p(I : X) \cap L^\infty(I : X)$.

(ii) It is said that the function $f(\cdot)$ is Besicovitch-Doss-p-$C_b^{(n)}$-almost periodic, $f \in C_b^{(n)} - \mathrm{B}^p(I : X)$ for short, iff for each $k = 0, 1, \ldots, n$, we have that $f^{(k)} \in \mathrm{B}^p(I : X) \cap L^\infty(I : X)$.

Keeping in mind [22, Theorem 3.1] and the proof of the first part of Proposition 4.2, we have the following result:

Proposition 4.4. *Suppose that $n \in \mathbb{N}$ and $(R(t))_{t>0} \subseteq L(X, Y)$ is a strongly continuous operator family satisfying $\int_0^\infty (1 + s)\|R(s)\|\, ds < \infty$. If $g : \mathbb{R} \to X$ is in the class $C_b^{(n)} - \mathrm{B}^1(I : X)$, then the function $G(\cdot)$, given by (4.1), is in the same class, as well.*

We continue by stating some results about the asymptotical Weyl-$C^{(n)}$-almost periodicity of finite convolution product. For this purpose, we need to recall the following notion from [23]: For any function $q \in L^1_{loc}(\mathbb{R} : X)$ with $q^{(k)} \in L^1_{loc}(\mathbb{R} : X)$ for $0 \leq k \leq n$, and for any strongly continuous operator family $(R(t))_{t>0} \subseteq L(X, Y)$ satisfying $\int_0^\infty \|R(s)\|\, ds < \infty$, we formally set

$$J_k(t, l) := \sup_{x \geq 0} \left\{ \int_0^{x+t} \left[\frac{1}{l} \int_{x+t-r}^{x+t-r+l} \big\|R(v)\big\|\, dv \right] \big\|q^{(k)}(r)\big\|\, dr \right\},$$

for $t > 0,\ l > 0,\ 0 \leq k \leq n$. The subsequent conditions will play an important role for us ($0 \leq k \leq n$):

$$\lim_{t \to \infty} \lim_{l \to \infty} J_k(t, l) = 0, \tag{4.3}$$

and

$$\lim_{l \to \infty} \lim_{t \to \infty} J_k(t, l) = 0. \tag{4.4}$$

The main purpose of next proposition is to look into the asymptotically Weyl-$C^{(n)}$-almost periodic properties of finite convolution product.

Proposition 4.5. *Let $n \in \mathbb{N}$.*

(i) Suppose that $(R(t))_{t>0} \subseteq L(X, Y)$ is a strongly continuous operator family satisfying $\int_0^\infty \|R(s)\|\, ds < \infty$. If $g : \mathbb{R} \to X$ is in the class $e - C_b^{(n)} - W^1_{ap}(\mathbb{R} : X)$, as well as $q^{(k)} \in W^1_0([0, \infty) : X)$, resp., $q^{(k)} \in$

$e-W_0^1([0,\infty):X)$ for $0\leq k\leq n$, (4.3), resp., (4.4), holds for $0\leq k\leq n$, then the function $F(\cdot)$, given by

$$F(t):=\int_0^t R(t-s)[g(s)+q(s)]\,ds,\quad t\geq 0, \tag{4.5}$$

is in the class $C^{(n)}-W^1_{aap}([0,\infty):Y)$, resp., $C^{(n)}-W^1_{eaap}([0,\infty):Y)$, provided that

$$(g+q)^{(k)}(0)=0,\quad 0\leq k\leq n-1. \tag{4.6}$$

(ii) *Let the requirements of part (i) hold with $g\in e-C_b^{(n)}-W^1_{ap}(\mathbb{R}:X)$ as well as with the function $q(\cdot)$ satisfying the same conditions as in (i). Let the condition (4.6) hold. Then the function $F(\cdot)$, given by (4.5), is in the class $e-C^{(n)}-W^1_{aap}([0,\infty):X)$, resp., $ee-C^{(n)}-W^1_{aap}([0,\infty):X)$.*

(iii) *Suppose that $(R(t))_{t>0}\subseteq L(X,Y)$ is a strongly continuous operator family satisfying $M=\sum_{k=0}^{\infty}\|R(\cdot)\|_{L^\infty[k,k+1]}<\infty$. If $g:\mathbb{R}\to X$ is in the class $e-C^{(n)}-W^1_{ap}(\mathbb{R}:X)$ and the function $q(\cdot)$ satisfying the same conditions as in (i), then the function $F(\cdot)$, given by (4.5), is in the class $e-C^{(n)}-W^1_{aap}([0,\infty):X)$, resp., $ee-C^{(n)}-W^1_{aap}([0,\infty):X)$, provided that the condition (4.6) holds.*

Proof. The proof of proposition follows almost immediately by applying [23, Proposition 5.3] and the identity

$$F^{(k)}(t):=\int_0^t R(t-s)\big[g^{(k)}(s)+q^{(k)}(s)\big]\,ds,\quad t\geq 0,\ 0\leq k\leq n,$$

which can be proved by using the arguments contained in the proof of [4, Proposition 1.3.6] and the condition (4.6). □

The conditions (4.3) and (4.4) are satisfied for a large class of functions $q(\cdot)$, provided that $(R(t))_{t>0}$ is locally integrable at zero and has a certain polynomial or exponential decaying growth order at infinity; see [23] for further information.

Appealing to [22, Theorem 3.3] in place of [23, Proposition 5.3] leads us to the following result:

Proposition 4.6. *Suppose that $n\in\mathbb{N}$, and $(R(t))_{t>0}\subseteq L(X,Y)$ is a strongly continuous operator family satisfying $\int_0^\infty(1+s)\|R(s)\|\,ds<\infty$. If $g:\mathbb{R}\to X$ is in the class $C_b^{(n)}-\mathrm{B}^1(I:X)$, as well as $q^{(k)}\in L^1_{loc}([0,\infty):X)$ is Besicovitch-1-vanishing for $0\leq k\leq n$, then the function $F(\cdot)$, given by (4.5), is in the class $f\in C^{(n)}-\mathrm{B}^1(I:X)$, as long as the condition (4.6) holds.*

Now we will investigate the situation in which the condition (4.6) is no longer satisfied, with the integer $n\in\mathbb{N}$ given in advance. It will be always assumed that the resolvent $(R(t))_{t>0}\subseteq L(X,Y)$ is locally integrable at zero and $(n-1)$-times continuously differentiable for $t>0$. Assuming $f,\ f' \in L^1_{loc}([0,\infty):X)$, the proofs of [4, Proposition 1.3.4, Proposition 1.3.6] show that the function $u(t):=\int_0^t R(t-s)f'(s)\,ds+R(t)f(0)$, $t\geq 0$ is locally integrable on $[0,\infty)$ and continuous on $(0,\infty)$, as well as that $(d/dt)\int_0^t R(t-s)f(s)\,ds=u(t)$ for all $t>0$. Repeating this argument, we obtain inductively that, for every $t>0$,

$$\frac{d^k}{dt^k}\int_0^t R(t-s)f(s)\,ds=\int_0^t R(t-s)f^{(k)}(s)\,ds+\sum_{j=0}^{k-1}R^{(k-1-j)}(t)g^{(j)}(0),$$

provided that $f,\ f',\ldots,f^{(n)}\in L^1_{loc}([0,\infty):X)$ and $0\leq k\leq n$. Keeping in mind this equality and our previous investigation of case in which the condition (4.6) is satisfied, the following results can be deduced:

Proposition 4.7. *Let $n\in\mathbb{N}$.*

(i) Suppose that $(R(t))_{t>0}\subseteq L(X,Y)$ is a strongly continuous operator family satisfying $\int_0^\infty \|R(s)\|\,ds<\infty$. If $g:\mathbb{R}\rightarrow X$ is in the class $e-C_b^{(n)}-W^1_{ap}(\mathbb{R}:X)$, as well as $q^{(k)}\in W^1_0([0,\infty):X)$, resp., $q^{(k)}\in e-W^1_0([0,\infty):X)$ for $0\leq k\leq n$, (4.3), resp., (4.4), holds for $0\leq k\leq n$, then the function $F(\cdot)$, given by (4.5), is in the class $C^{(n)}-W^1_{aap}([0,\infty):Y)$, resp., $C^{(n)}-W^1_{eaap}([0,\infty):Y)$, provided that $(R(t))_{t>0}$ is $(n-1)$-times continuously differentiable for $t>0$ and $R^{(k-1-j)}(\cdot)$ is pointwise in the class $W^1_{aap}([0,\infty):Y)$, resp., $W^1_{eaap}([0,\infty):Y)$, for any $0\leq k\leq n$ and $0\leq j\leq k-1$ such that $(g+q)^{(j)}(0)\neq 0$.

(ii) Let the requirements of part (i) hold with $g\in e-C_b^{(n)}-W^1_{ap}(\mathbb{R}:X)$ as well as with the function $q(\cdot)$ satisfying the same conditions as in (i). Let $(R(t))_{t>0}$ be $(n-1)$-times continuously differentiable for $t>0$, and let $R^{(k-1-j)}(\cdot)$ be pointwise in the class $e-W^1_{aap}([0,\infty):X)$, resp., $ee-W^1_{aap}([0,\infty):X)$, for any $0\leq k\leq n$ and $0\leq j\leq k-1$ such that $(g+q)^{(j)}(0)\neq 0$. Then the function $F(\cdot)$, given by (4.5), is in the class $e-C^{(n)}-W^1_{aap}([0,\infty):X)$, resp., $ee-C^{(n)}-W^1_{aap}([0,\infty):X)$.

(iii) Suppose that $(R(t))_{t>0}\subseteq L(X,Y)$ is a strongly continuous operator family satisfying $M=\sum_{k=0}^\infty \|R(\cdot)\|_{L^\infty[k,k+1]}<\infty$. If $g:\mathbb{R}\rightarrow X$ is in the class $e-C^{(n)}-W^1_{ap}(\mathbb{R}:X)$ and the function $q(\cdot)$ satisfying the same conditions as in (i), then the function $F(\cdot)$, given by (4.5), is in the class $e-C^{(n)}-W^1_{aap}([0,\infty):X)$, resp., $ee-C^{(n)}-W^1_{aap}([0,\infty):X)$, provided that $(R(t))_{t>0}$ is $(n-1)$-times continuously differentiable for $t>0$ and $R^{(k-1-j)}(\cdot)$ is pointwise in the class $e-W^1_{aap}([0,\infty):Y)$, resp., $ee-W^1_{aap}([0,\infty):Y)$, for any $0\leq k\leq n$ and $0\leq j\leq k-1$ such that $(g+q)^{(j)}(0)\neq 0$.

Proposition 4.8. *Suppose that $n\in\mathbb{N}$, and $(R(t))_{t>0}\subseteq L(X,Y)$ is a strongly continuous operator family satisfying $\int_0^\infty (1+s)\|R(s)\|\,ds<\infty$. If $g:\mathbb{R}\rightarrow X$ is* *in the class* $C_b^{(n)} - \mathrm{B}^1(I : X)$*, as well as* $q^{(k)} \in L^1_{loc}([0,\infty) : X)$ *is Besicovitch-1-vanishing for* $0 \leq k \leq n$*, then the function* $F(\cdot)$*, given by* (4.5)*, is in the class* $f \in C^{(n)} - \mathrm{B}^1(I : X)$*, as long as* $(R(t))_{t>0}$ *is* $(n-1)$*-times continuously differentiable for* $t > 0$ *and* $R^{(k-1-j)}(\cdot)$ *is pointwise in the class* $\mathrm{B}^1(I : Y)$ *for any* $0 \leq k \leq n$ *and* $0 \leq j \leq k-1$ *such that* $(g+q)^{(j)}(0) \neq 0$.

Remark 4.9. The condition that $(R(t))_{t>0} \subseteq L(X,Y)$ is $(n-1)$-times continuously differentiable for $t > 0$ holds, in particular, if $(R(t))_{t>0}$ is analytic in a sector around the positive real axis (or that $n = 1$). Then we can apply the Cauchy integral formula to see that all derivatives of $R(\cdot)$ decay polynomially or exponentially at infinity, so that the conditions stated in the formulations of the above two propositions hold for a large class of solution operator families $(R(t))_{t>0}$ examined in our previous research studies; see e.g. [20] for more details.

Concerning applications to abstract Volterra integro-differential equations, it should be noted that we can apply our results from this section, e.g., in the analysis of existence and uniqueness of Weyl $C^{(n)}-$almost periodic solutions and Besicovitch-Doss $C^{(n)}-$almost periodic solutions of abstract fractional inclusion

$$D^{\gamma}_{t,+}u(t) \in \mathcal{A}u(t) + f(t),\ t \in \mathbb{R},$$

with the Riemann-Liouville derivative of $D^{\gamma}_{t,+}$ of order $\gamma \in (0,1]$, as well as asymptotically Weyl $C^{(n)}-$almost periodic solutions and Besicovitch-Doss $C^{(n)}-$almost periodic solutions of the fractional relaxation inclusion

$$(\mathrm{DFP})_{f,\gamma} : \begin{cases} \mathbf{D}^{\gamma}_t u(t) \in \mathcal{A}u(t) + f(t),\ t > 0, \\ \quad u(0) = x_0, \end{cases}$$

where $\mathbf{D}^{\gamma}_t$ denotes the Caputo fractional derivative of order $\gamma \in (0,1]$, $x_0 \in X$ and the multivalued linear operator $\mathcal{A}$ satisfies the condition [18, (P), p. 47] introduced by A. Favini and A. Yagi:

(P) There exist finite constants $c,\ M > 0$ and $\beta \in (0,1]$ such that

$$\Psi := \Psi_c := \Big\{ \lambda \in \mathbb{C} : \Re\lambda \geq -c\big(|\Im\lambda| + 1\big) \Big\} \subseteq \rho(\mathcal{A})$$

and

$$\|R(\lambda : \mathcal{A})\| \leq M\big(1 + |\lambda|\big)^{-\beta}, \quad \lambda \in \Psi.$$

In such a way, we can consider the qualitative properties of solutions of the famous Poisson heat equations [18]

$$\begin{cases} \frac{\partial}{\partial x}[m(x)v(t,x)] = (\Delta - b)v(t,x) + f(t,x), \quad t \in \mathbb{R},\ x \in \Omega; \\ v(t,x) = 0, \quad (t,x) \in [0,\infty) \times \partial\Omega, \end{cases}$$

and

$$\begin{cases} \frac{\partial}{\partial x}[m(x)v(t,x)] = (\Delta - b)v(t,x) + f(t,x), \quad t \geq 0,\ x \in \Omega; \\ v(t,x) = 0, \quad (t,x) \in [0,\infty) \times \partial\Omega, \\ m(x)v(0,x) = u_0(x), \quad x \in \Omega, \end{cases}$$

in the space $X := L^p(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^n$, $b > 0$, $m(x) \geq 0$ a.e. $x \in \Omega$, $m \in L^\infty(\Omega)$ and $1 < p < \infty$, as well as their fractional analogons associated with the use of Weyl-Liouville or Caputo derivatives. See [20] for more details.

The following result on the convolution invariance of introduced function spaces belonging to the Weyl class holds good:

Theorem 4.10. *Let $n \in \mathbb{N}$, and let $f \in C_b^{(n)} - W_{ap}^1(\mathbb{R} : X)$ ($f \in W_{ap,b}^1(\mathbb{R} : X)$), resp. $f \in e - C_b^{(n)} - W_{ap}^1(\mathbb{R} : X)$ ($f \in e - W_{ap,b}^1(\mathbb{R} : X)$). Then, for every $g \in L^1(\mathbb{R})$, we have that $g * f \in C_b^{(n)} - W_{ap}^1(\mathbb{R} : X)$ ($g * f \in W_{ap,b}^1(\mathbb{R} : X)$), resp. $f \in e - C_b^{(n)} - W_{ap}^1(\mathbb{R} : X)$ ($f \in e - W_{ap,b}^1(\mathbb{R} : X)$).*

Proof. The proof is a consequence of the following simple calculation

$$\begin{aligned}
&\frac{1}{l}\int_x^{x+l}\left\| \int_{-\infty}^{\infty} [f(t+\tau-y) - f(t-y)]g(y)\,dy \right\| dt \\
&\leq \frac{1}{l}\int_x^{x+l}\int_{-\infty}^{\infty} \|f(t+\tau-y) - f(t-y)\| \|g(y)\|\,dy\,dt \\
&= \int_{-\infty}^{\infty}\left[\frac{1}{l}\int_x^{x+l} \|f(t+\tau-y) - f(t-y)\|\,dt\right]\|g(y)\|\,dy \\
&= \int_{-\infty}^{\infty}\left[\frac{1}{l}\int_{x-y}^{x+l-y} \|f(r+\tau) - f(r)\|\,dr\right]\|g(y)\|\,dy \\
&\leq \left[\int_{-\infty}^{\infty}\|g(y)\|\,dy\right]\cdot \sup_{x\in\mathbb{R}}\left[\frac{1}{l}\int_x^{x+l}\|f(r+\tau) - f(r)\|\,dr\right],
\end{aligned}$$

the well-known distributional equality $(g * f)^{(n)} = g * f^{(n)}$ and elementary definitions. □

The interested reader may try to examine whether we can transfer the above assertion to Besicovitch-Doss $C_b^{(n)}$-almost periodic classes of functions.

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References

[1] Adamczak, M. $C^{(n)}$-almost periodic functions. *Comment. Math. (Prace Mat.) 37* (1997), 1–12.

[2] Amerio, L., and Prouse, G. *Almost-periodic functions and functional equations.* Van Nostrand Reinhold Co., New York-Toronto, Ont.-Melbourne, 1971.

[3] Andres, J., Bersani, A. M., and Grande, R. F. Hierarchy of almost-periodic function spaces. *Rend. Mat. Appl. (7) 26*, 2 (2006), 121–188.

[4] Arendt, W., Batty, C. J. K., Hieber, M., and Neubrander, F. *Vector-valued Laplace transforms and Cauchy problems*, vol. 96 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 2001.

[5] Baillon, J.-B., Blot, J., N'Guérékata, G. M., and Pennequin, D. On $C^{(n)}$-almost periodic solutions to some nonautonomous differential equations in Banach spaces. *Comment. Math. (Prace Mat.) 46*, 2 (2006), 263–273.

[6] Barbu, V. *Nonlinear semigroups and differential equations in Banach spaces*. Editura Academiei Republicii Socialiste România, Bucharest; Noordhoff International Publishing, Leiden, 1976. Translated from the Romanian.

[7] Besicovitch, A. S. *Almost periodic functions*. Dover Publications, Inc., New York, 1955.

[8] Bugajewski, D., and N'Guérékata, G. M. On some classes of almost periodic functions in abstract spaces. *Int. J. Math. Math. Sci.*, 61-64 (2004), 3237–3247.

[9] Cheban, D. N. *Asymptotically almost periodic solutions of differential equations*. Hindawi Publishing Corporation, New York, 2009.

[10] Cross, R. *Multivalued linear operators*, vol. 213 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker, Inc., New York, 1998.

[11] Diagana, T. *Almost automorphic type and almost periodic type functions in abstract spaces*. Springer, Cham, 2013.

[12] Diagana, T., and Nelson, V. $C^{(n)}$-pseudo almost automorphy and its applications to some higher-order differential equations. *Nonlinear Stud. 19*, 3 (2012), 443–455.

[13] Diagana, T., Nelson, V., and N'Guérékata, G. M. Stepanov-like $C^{(n)}$-pseudo almost automorphy and applications to some nonautonomous higher-order differential equations. *Opuscula Math. 32*, 3 (2012), 455–471.

[14] Doss, R. On generalized almost periodic functions. *Ann. of Math. (2) 59* (1954), 477–489.

[15] Doss, R. On generalized almost periodic functions. II. *J. London Math. Soc. 37* (1962), 133–140.

[16] Elazzouzi, A. $C^{(n)}$-almost periodic and $C^{(n)}$-almost automorphic solutions for a class of partial functional differential equations with finite delay. *Nonlinear Anal. Hybrid Syst. 4*, 4 (2010), 672–688.

[17] Ezzinbi, K., Nelson, V., and N'Guérékata, G. $C^{(n)}$-almost automorphic solutions of some nonautonomous differential equations. *Cubo 10*, 2 (2008), 61–74.

[18] Favini, A., and Yagi, A. *Degenerate differential equations in Banach spaces*, vol. 215 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker, Inc., New York, 1999.

[19] Kostić, M. *Abstract Degenerate Volterra Integro-Differential Equations: Linear Theory and Applications*. Book Manuscript, 2016.

[20] Kostić, M. *Abstract Volterra Integro-Differential Equations: Almost Periodicity and Asymptotically Almost Periodic Properties of Solutions*. Book Manuscript, 2017.

[21] Kostić, M. Abstract volterra integro-differential equations: generalized almost periodicity and asymptotical almost periodicity of solutions. *Electronic J. Diff. Equ.* (submitted).

[22] Kostić, M. On besicovitch-doss almost periodic solutions of abstract volterra integro-differential equations. *Novi Sad J. Math.* (submitted).

[23] Kostić, M. Weyl-almost periodic solutions and asymptotically weyl-almost periodic solutions of abstract volterra integro-differential equations. *J. Math. Anal. Appl.* (submitted).

[24] Levitan, B. M., and Zhikov, V. V. *Almost periodic functions and differential equations.* Cambridge University Press, Cambridge-New York, 1982. Translated from the Russian by L. W. Longdon.

[25] Liang, J., Maniar, L., N'Guérékata, G. M., and Xiao, T.-J. Existence and uniqueness of $C^{(n)}$-almost periodic solutions to some ordinary differential equations. *Nonlinear Anal. 66*, 9 (2007), 1899–1910.

[26] N'Guerekata, G. M. *Almost automorphic and almost periodic functions in abstract spaces.* Kluwer Academic/Plenum Publishers, New York, 2001.

[27] Periago, F., and Straub, B. A functional calculus for almost sectorial operators and applications to abstract evolution equations. *J. Evol. Equ. 2*, 1 (2002), 41–68.

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