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# ASYMPTOTIC ESTIMATES OF ENTIRE FUNCTIONS OF BOUNDED L-INDEX IN JOINT VARIABLES

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**Abstract.** In this paper growth estimates of entire in $\mathbb{C}^n$ function of bounded $\mathbf{L}$-index in joint variables are obtained. They describe the behavior of maximum modulus of an entire function on a skeleton in a polydisc by behavior of the function $\mathbf{L}(z) = (l_1(z), \ldots, l_n(z))$, where for every $j \in \{1, \ldots, n\}$ $l_j : \mathbb{C}^n \to \mathbb{R}_+$ is a continuous function. We generalized known results of W. K. Hayman, M. M. Sheremeta, A. D. Kuzyk and M. T. Borduyak to a wider class of functions $\mathbf{L}$. One of our estimates is sharper even for entire in $\mathbb{C}$ functions of bounded $l$-index than Sheremeta's estimate.

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## 1. Introduction

Let $l : \mathbb{C} \to \mathbb{R}_+$ be a fixed positive continuous function, where $\mathbb{R}_+ = (0, +\infty)$. An entire function $f$ is said to be of bounded $l-$index [13] if there exists an integer $m$, independent of $z$, such that for all $p$ and all $z \in \mathbb{C}$, $\frac{|f^{(p)}(z)|}{l^p(z)p!} \leq \max\{\frac{|f^{(s)}(z)|}{l^s(z)s!} : 0 \leq s \leq m\}$. The least such integer $m$ is called the $l-$index of $f(z)$ and is denoted by $N(f, l)$. If $l(z) \equiv 1$ then we obtain the definition of function of *bounded index* [14] and in this case we denote $N(f) := N(f, 1)$.

In 1970, W. J. Pugh and S. M. Shah [18] posed some questions about properties of entire functions of bounded index. One of those questions is following: *I. What are the growth properties of functions of bounded index: (c) is it possible to derive the boundedness (or the unboundedness) of the index from the asymptotic properties of the logarithm of the maximum modulus of $f(z)$, i.e., $\ln M(r, f)$?*

W. K. Hayman [12] proved that entire function of bounded index has exponential type which is not greater than $N(f) + 1$. Later A. D. Kuzyk and M. M. Sheremeta [13] obtained growth estimate of entire function of bounded $l-$index. It is known [9] that for every entire function $f$ with bounded multiplicities of

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zeros there exists a positive function $l(r)$ $(r = |z|)$, continuous on $[0; +\infty)$, such that $f$ is of bounded $l$-index. Thus, the concept of bounded $l$-index allows one to study growth properties of any entire functions with bounded multiplicities of zeros. The similar result was also obtained [7] for entire functions of several variables. M. M. Sheremeta [21] deduced analogous growth estimates for functions of bounded $l$-index which are analytic in a unit disc or in arbitrary complex domain.

Clearly, the question of Shah and Pugh can be formulated for entire functions in $\mathbb{C}^n$: *What are the growth properties of functions of bounded* $\mathbf{L}$*-index in joint variables? Is it possible to derive the boundedness (or the unboundedness) of the* $\mathbf{L}$*-index in joint variables from the asymptotic properties of the logarithm of the maximum modulus of* $F(z)$ *on a skeleton in a polydisc?*

M. T. Bordulyak and M. M. Sheremeta [10] gave an answer to the question if $\mathbf{L}(z) = (l_1(|z_1|), \ldots, l_n(|z_n|))$, and for every $j \in \{1, \ldots, n\}$ the function $l_j : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous. In this paper we extend their results for $\mathbf{L}(z) = (l_1(z), \ldots, l_n(z))$, where $l_j : \mathbb{C}^n \to \mathbb{R}_+$ is a continuous function for every $j \in \{1, \ldots, n\}$. We remark [5] that the entire function $f(z) = e^{z_1 \cdots z_n} : \mathbb{C}^n \to \mathbb{C}$ is of unbounded $\mathbf{L}$-index in joint variables for any function $\mathbf{L}(z) = (l_1(|z_1|), \ldots, l_n(|z_n|))$. But it has bounded $\mathbf{L}$-index in joint variables for the function $\mathbf{L}(z) = (l_1(z), \ldots, l_n(z))$, where $l_j(z) = |z_1 \cdots z_{j-1} z_{j+1} \cdots z_n| + 1$. In a some sense our results are new even in one-dimensional case (see below Corollaries 4.3 and 4.8). Note the obtained estimates are not direct analogs of the results of M. T. Bordulyak and M. M. Sheremeta. They are *sharper* (see Remark 4.9). Besides, we show that inequality (4.13) is exact (see the example at the end of the paper).

Note that there exists another approach to consider bounded index in $\mathbb{C}^n$ — so-called functions of bounded $L$-index in direction [4, 8, 2], where $L : \mathbb{C}^n \to \mathbb{R}_+$ is a continuous function.

The concepts of bounded $L$-index in a direction and bounded $\mathbf{L}$-index in joint variables have a few advantages in comparison with the traditional approaches to the study of the properties of entire solutions of partial differential equations. In particular, if an entire solution has bounded index [4, 2] then it immediately yields its growth estimates, an uniform distribution of its zeros in some sense, a certain regular behavior of the solution and etc.

## 2. Notations, definitions and main result

We need some standard notations. Let $\mathbb{R}_+ = (0, +\infty)$. Denote

$$\mathbf{0} = (0, \ldots, 0) \in \mathbb{R}^n, \quad \mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^n, \quad \mathbf{2} = (2, \ldots, 2) \in \mathbb{R}^n,$$

$$\mathbf{1}_j = (0, \ldots, 0, \underbrace{1}_{j\text{–th place}}, 0, \ldots, 0) \in \mathbb{R}^n, \quad [0, 2\pi]^n = \underbrace{[0, 2\pi] \times \cdots \times [0, 2\pi]}_{n\text{–th times}}.$$

For $R = (r_1, \ldots, r_n) \in \mathbb{R}^n_+$, $\Theta = (\theta_1, \ldots, \theta_n) \in [0, 2\pi]^n$ and $K = (k_1, \ldots, k_n) \in \mathbb{Z}^n_+$ let us to denote $\|K\| = k_1 + \cdots + k_n$, $Re^{i\Theta} = (r_1 e^{i\theta_1}, \ldots, r_n e^{i\theta_n})$, $K! =$

$k_1! \cdot \ldots \cdot k_n!$. For $A = (a_1, \ldots, a_n) \in \mathbb{C}^n$, $B = (b_1, \ldots, b_n) \in \mathbb{C}^n$, we will use formal notations without violation of the existence of these expressions

$$|A| = \Big(\sum_{j=1}^{n} |a_j|^2\Big)^{1/2}, A \pm B = (a_1 \pm b_1, \ldots, a_n \pm b_n), AB = (a_1b_1, \cdots, a_nb_n),$$

$$arg\, A = (arg\, a_1, \ldots, arg\, a_n), A/B = (a_1/b_1, \ldots, a_n/b_n), A^B = a_1^{b_1} a_2^{b_2} \cdot \ldots a_n^{b_n},$$

and the notation $A < B$ means that $a_j < b_j$ for all $j \in \{1, \ldots, n\}$; the relation $A \leq B$ is defined similarly. For $z \in \mathbb{C}^n$ and $w \in \mathbb{C}^n$ we define

$$\langle z, w\rangle = z_1\overline{w}_1 + \cdots + z_n\overline{w}_n,$$

where $\overline{w}_k$ is the complex conjugate of $w_k$.

The polydisc $\{z \in \mathbb{C}^n : \ |z_j - z_j^0| < r_j, \ j \in \{1, \ldots, n\}\}$ is denoted by $D^n(z^0, R)$, its skeleton $\{z \in \mathbb{C}^n : \ |z_j - z_j^0| = r_j, \ j \in \{1, \ldots, n\}\}$ is denoted by $T^n(z^0, R)$, and the closed polydisc $\{z \in \mathbb{C}^n : \ |z_j - z_j^0| \leq r_j, \ j \in \{1, \ldots, n\}\}$ is denoted by $D^n[z^0, R]$. For a partial derivative of entire function $F(z) = F(z_1, \ldots, z_n)$ we will use the notation

$$F^{(K)}(z) = \frac{\partial^{\|K\|}F}{\partial z^K} = \frac{\partial^{k_1+\cdots+k_n}F}{\partial z_1^{k_1}\ldots\partial z_n^{k_n}}, \text{ where } K = (k_1, \ldots, k_n) \in \mathbb{Z}_+^n.$$

Let $\mathbf{L}(z) = (l_1(z), \ldots, l_n(z))$, where $l_j(z)$ are positive continuous functions of variable $z \in \mathbb{C}^n$, $j \in \{1, 2, \ldots, n\}$.

An entire function $F(z)$ is called *a function of bounded* **L**-*index in joint variables,* [2, 3, 5] if there exists a number $m \in \mathbb{Z}_+$ such that for all $z \in \mathbb{C}^n$ and $J = (j_1, j_2, \ldots, j_n) \in \mathbb{Z}_+^n$

$$\frac{|F^{(J)}(z)|}{J!\mathbf{L}^J(z)} \leq \max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)} : \ K \in \mathbb{Z}_+^n, \ \|K\| \leq m\right\}. \tag{2.1}$$

If $l_j = l_j(|z_j|)$ then we obtain the concept of entire functions of bounded **L**-index in sense of definition in the paper [10]. If $l_j(z_j) \equiv 1, \ j \in \{1, 2, \ldots, n\}$, then the entire function is called a *function of bounded index in joint variables* [11, 15, 16, 17, 20]. Also there are papers about analytic functions of bounded **L**-index in joint variables [1, 6, 11].

The least integer $m$ for which inequality holds is called **L**-*index in joint variables of the function* $F$ and is denoted by $N(F, \mathbf{L})$.

For $R \in \mathbb{R}_+^n$, $j \in \{1, \ldots, n\}$ and $\mathbf{L}(z) = (l_1(z), \ldots, l_n(z))$ we define

$$\lambda_{1,j}(z_0, R) = \inf\left\{l_j(z)/l_j(z^0) \colon z \in D^n\left[z^0, R/\mathbf{L}(z^0)\right]\right\},$$

$$\lambda_{2,j}(z_0, R) = \sup\left\{l_j(z)/l_j(z^0) \colon z \in D^n\left[z^0, R/\mathbf{L}(z^0)\right]\right\},$$

$$\lambda_{1,j}(R) = \inf_{z^0\in\mathbb{C}^n} \lambda_{1,j}(z_0, R), \quad \lambda_{2,j}(R) = \sup_{z^0\in\mathbb{C}^n} \lambda_{2,j}(z_0, R),$$

$$\Lambda_k(R) = (\lambda_{k,j}(R), \ldots, \lambda_{k,n}(R)) \quad (k \in \{1, 2\}).$$

By $Q^n$ we denote the class of functions $\mathbf{L}(z)$ which for every $R \in \mathbb{R}^n_+$ satisfy the condition

$$0 < \Lambda_1(R) \le \Lambda_2(R) < +\infty. \tag{2.2}$$

Here we suppose that $\mathbf{L}(z) = \mathbf{L}(R)$, $R = (r_1, \dots, r_n)$, $z = (z_1, \dots, z_n)$, $r_k = |z_k|$ $(1 \le k \le n)$. Let us to denote $r^* = \max_{1 \le k \le n} r_k$.

Our main result is the following

**Theorem 2.1** (Main Theorem). *Let* $\mathbf{L}(R) = (l_1(R), \dots, l_n(R))$, $l_j(R)$ *be a positive continuously differentiable non-decreasing function in each variable* $r_k \in [0, +\infty)$, $k, j \in \{1, \dots, n\}$. *If an entire function* $F$ *has bounded* $\mathbf{L}$*-index* $N = N(F, \mathbf{L})$ *in joint variables then*

$$\varlimsup_{|R| \to +\infty} \frac{\ln \max\{|F(z)| : \ z \in T^n(\mathbf{0}, R)\}}{\int_0^1 \langle R, \mathbf{L}(\tau R) \rangle d\tau} \le N + 1.$$

This statement is a consequence of Theorem 4.6, which is obtained for a more general function $\mathbf{L}$.

## 3. Auxiliary propositions

**Lemma 3.1.** *Let* $\mathbf{L}(z) = (l_1(z), \dots, l_n(z))$, $l_j : \mathbb{C}^n \to \mathbb{C}$ *and* $\frac{\partial l_j}{\partial z_m}$ *be continuous functions in* $\mathbb{C}^n$ *for all* $j$, $m \in \{1, 2, \dots, n\}$. *If there exist numbers* $P > 0$ *and* $c > 0$ *such that for all* $z \in \mathbb{C}^n$ *and every* $j, m \in \{1, 2, \dots, n\}$

$$\frac{1}{c + |l_j(z)|} \left| \frac{\partial l_j(z)}{\partial z_m} \right| \le P \tag{3.1}$$

*then* $\mathbf{L}^* \in Q^n$, *where* $\mathbf{L}^*(z) = (c + |l_1(z)|, \dots, c + |l_n(z)|)$.

*Proof.* Clearly, the function $\mathbf{L}^*(z)$ is positive and continuous. For given $z \in \mathbb{C}^n$, $z^0 \in \mathbb{C}^n$ we define an analytic curve $\varphi : [0, 1] \to \mathbb{C}^n$

$$\varphi_j(\tau) = z_j^0 + \tau(z_j - z_j^0), \ j \in \{1, 2, \dots, n\},$$

where $\tau \in [0, 1]$. It is known that for every continuously differentiable function $g$ of real variable $\tau$ the inequality $\frac{d}{dt}|g(\tau)| \le |g'(\tau)|$ holds except the points where $g'(\tau) = 0$. Using restrictions of this lemma, we establish the upper estimate of $\lambda_{2,j}(z_0, R)$ :

$$\lambda_{2,j}(z_0, R) = \sup\left\{ \frac{c + |l_j(z)|}{c + |l_j(z^0)|} : z \in D^n\left[z^0, \frac{R}{\mathbf{L}_1(z^0)}\right] \right\} =$$
$$= \sup_{z \in D^n\left[z^0, \frac{R}{\mathbf{L}_1(z^0)}\right]} \left\{ \exp\left\{ \ln(c + |l_j(z)|) - \ln(c + |l_j(z^0)|) \right\} \right\} =$$
$$= \sup\left\{ \exp\left\{ \int_0^1 \frac{d(c + |l_j(\varphi(\tau))|)}{c + |l_j(\varphi(\tau))|} \right\} : z \in D^n\left[z^0, \frac{R}{\mathbf{L}_1(z^0)}\right] \right\} \le$$

$$\leq \sup_{z\in D^n\left[z^0,\frac{R}{\mathbf{L}_1(z^0)}\right]} \left\{\exp\left\{\int_0^1 \sum_{m=1}^n \frac{|\varphi_m'(\tau)|}{c+|l_j(\varphi(\tau))|}\left|\frac{\partial l_j(\varphi(\tau))}{\partial z_m}\right| d\tau\right\}\right\} \leq$$

$$\leq \sup_{z\in D^n\left[z^0,\frac{R}{\mathbf{L}_1(z^0)}\right]} \left\{\exp\left\{\int_0^1 \sum_{m=1}^n P|z_m - z_m^0| d\tau\right\}\right\} \leq$$

$$\leq \sup_{z\in D^n\left[z^0,\frac{R}{\mathbf{L}_1(z^0)}\right]} \left\{\exp\left\{\sum_{m=1}^n \frac{Pr_j}{c+|l_m(z^0)|}\right\}\right\} \leq \exp\left(\frac{P}{c}\sum_{m=1}^n r_j\right).$$

Hence, for all $R \geq \mathbf{0}$ $\lambda_{2,j}(R) = \sup_{z^0\in\mathbb{C}^n} \lambda_{2,j}(z^0,\eta) \leq \exp\left(\frac{P}{c}\sum\limits_{m=1}^n r_j\right) < \infty$. Using $\frac{d}{dt}|g(t)| \geq -|g'(t)|$ it can be proved that for every $\eta \geq 0$ $\lambda_{1,j}(R) \geq \exp\left(-\frac{P}{c}\sum\limits_{m=1}^n r_j\right) > 0$. Therefore, $\mathbf{L}^* \in Q^n$. □

To estimate the growth of entire functions of bounded **L**-index in joint variables we will use the following theorem which describes local behavior of these entire functions.

**Theorem 3.2** ([5, 2])**.** *Let* $\mathbf{L} \in Q^n$. *An entire function* $F$ *is of bounded* **L**-*index in joint variables if and only if there exist numbers* $R'$, $R''$, $\mathbf{0} < R' < \mathbf{1} < R''$, *and* $p_1 = p_1(R', R'') \geq 1$ *such that for every* $z^0 \in \mathbb{C}^n$

(3.2)
$$\max\left\{|F(z)|\colon z \in T^n\left(z^0, \frac{R''}{\mathbf{L}(z^0)}\right)\right\} \leq p_1 \max\left\{|F(z)|\colon z\in T^n\left(z^0, \frac{R'}{\mathbf{L}(z^0)}\right)\right\}.$$

Recently, Theorem 3.2 was slightly improved [3]. The condition $\mathbf{0} < R' < \mathbf{1} < R''$ was replaced by the condtion $\mathbf{0} < R' < R''$.

At first, we prove the following lemma.

**Lemma 3.3.** *If* $\mathbf{L} \in Q^n$, *then for every* $j \in \{1,\dots,n\}$ *and for every fixed* $z^* \in \mathbb{C}^n$, $|z_j| l_j(z^* + z_j\mathbf{1}_j) \to \infty$ *as* $|z_j| \to \infty$.

*Proof.* Assumen not. Then there exist a number $C > 0$ and a sequence $z_j^{(m)} \to \infty$ such that $|z_j^{(m)}| l_j(z^* + z_j^{(m)}\mathbf{1}_j) = k_m \leq C$, i. e. $|z_j^{(m)}| = \frac{k_m}{l_j(z^*+z_j^{(m)}\mathbf{1}_j)}$. Then

$$\frac{1}{l_j(z^*+z_j^{(m)}\mathbf{1}_j)} l_j\left(z^* + z_j^{(m)}\mathbf{1}_j - \frac{k_m e^{i\arg z_j^{(m)}}\mathbf{1}_j}{l_j(z^*+z_j^{(m)}\mathbf{1}_j)}\right) = \frac{|z_j^{(m)}|}{k_m} l_j(z^*) \to +\infty,$$

as $z_j^{(m)} \to \infty$, $m \to +\infty$, that is $\lambda_{2,j}(C\mathbf{1}_j) = +\infty$ and $\mathbf{L} \notin Q^n$. □

## 4. Estimates of growth of entire functions

By $K^n$ we denote a class of positive continuous functions $\mathbf{L}(z)$ for which there exists $c > 0$ such that for every $R \in \mathbb{R}_+^n$ and $j \in \{1,\dots,n\}$

$$\max_{\Theta_1,\Theta_2\in[0,2\pi]^n} \frac{l_j(Re^{i\Theta_2})}{l_j(Re^{i\Theta_1})} \leq c.$$

If $\mathbf{L}(z) = (l_1(|z_1|,\dots,|z_n|),\dots,l_n(|z_1|,\dots,|z_n|))$ then $\mathbf{L} \in K^n$. It is easy to prove that $|e^z|+1 \in Q^1 \setminus K^1$, but $e^{e^{|z|}} \in K^1 \setminus Q^1$, $z \in \mathbb{C}$. Besides, if $\mathbf{L}_1, \mathbf{L}_2 \in K^n$ then $\mathbf{L}_1 + \mathbf{L}_2 \in K^n$ and $\mathbf{L}_1\mathbf{L}_2 \in K^n$. For simplicity, let us to write $K \equiv K^1$ and $M(F,R) = \max\{|F(z)| \colon z \in T^n(\mathbf{0},R)\}$.

**Theorem 4.1.** *Let $\mathbf{L} \in Q^n \cap K^n$. If an entire function $F$ has bounded $\mathbf{L}$-index in joint variables, then*

(4.1)
$$\ln M(F,R) = O\left( \min_{\sigma_n \in \mathcal{S}_n} \min_{\Theta \in [0,2\pi]^n} \sum_{j=1}^{n} \int_0^{r_j} l_j(R(j,\sigma_n,t)e^{i\Theta})dt \right) \quad as\ |R| \to \infty,$$

*where $\sigma_n$ is a permutation of $\{1,\dots,n\}$, $\mathcal{S}_n$ is a set of all permutations of $\{1,\dots,n\}$, $R(j,\sigma_n,t) = (r_1',\dots,r_n')$,* $r_k' = \begin{cases} r_k^0, & \text{if } \sigma_n(k) < j, \\ t, & \text{if } k = j, \\ r_k, & \text{if } \sigma_n(k) > j, \end{cases}$ $k \in \{1,\dots,n\}$,
*$R^0 = (r_1^0,\dots,r_n^0)$ is sufficiently large radius.*

*Proof.* Let $R > 0$, $\Theta \in [0,2\pi]^n$ and a point $z^* \in T^n(\mathbf{0}, R + \frac{\mathbf{2}}{\mathbf{L}(Re^{i\Theta})})$ be a such that
$$|F(z^*)| = \max\left\{ |F(z)| : z \in T^n\left(\mathbf{0}, R + \frac{\mathbf{2}}{\mathbf{L}(Re^{i\Theta})}\right)\right\}.$$

Denote $z^0 = \frac{z^* R}{R + \mathbf{2}/\mathbf{L}(Re^{i\Theta})}$. Then for every $j \in \{1,\dots,n\}$

$$|z_j^0 - z_j^*| = \left| \frac{z_j^* r_j}{r_j + 2/l_j(Re^{i\Theta})} - z_j^* \right| = \left| \frac{2z_j^*/l_j(Re^{i\Theta})}{r_j + 2/l_j(Re^{i\Theta})} \right| = \frac{2}{l_j(Re^{i\Theta})} \text{ and}$$

$$\mathbf{L}(z^0) = \mathbf{L}\left( \frac{z^* R}{R + \mathbf{2}/\mathbf{L}(Re^{i\Theta})} \right) = \mathbf{L}\left( \frac{(R + \mathbf{2}/\mathbf{L}(Re^{i\Theta}))e^{i\arg z^*} R}{R + \mathbf{2}/\mathbf{L}(Re^{i\Theta})} \right) = \mathbf{L}(Re^{i\arg z^*}).$$

Since $\mathbf{L} \in K^n$ we have that $c\mathbf{L}(z^0) = c\mathbf{L}(Re^{i\arg z^*}) \geq \mathbf{L}(Re^{i\Theta}) \geq \frac{1}{c}\mathbf{L}(z^0)$. We consider two skeletons $T^n(z^0, \frac{\mathbf{1}}{\mathbf{L}(z^0)})$ and $T^n(z^0, \frac{\mathbf{2}}{\mathbf{L}(z^0)})$. By Theorem 3.2 there exists $p_1 = p_1(\frac{\mathbf{1}}{c}, c\mathbf{2}) \geq 1$ such that (3.2) holds with $R' = \frac{\mathbf{1}}{c}$, $R'' = c\mathbf{2}$, i.e.

$$\begin{aligned}
\max\left\{ |F(z)| \colon z \in T^n\left(\mathbf{0}, R + \frac{\mathbf{2}}{\mathbf{L}(Re^{i\Theta})}\right)\right\} &= |F(z^*)| \leq \\
\leq \max\left\{ |F(z)| \colon z \in T^n\left(z^0, \frac{\mathbf{2}}{\mathbf{L}(Re^{i\Theta})}\right)\right\} &\leq \\
\leq \max\left\{ |F(z)| \colon z \in T^n\left(z^0, \frac{c\mathbf{2}}{\mathbf{L}(z^0)}\right)\right\} &\leq \\
\leq p_1 \max\left\{ |F(z)| \colon z \in T^n\left(z^0, \frac{\mathbf{1}}{c\mathbf{L}(z^0)}\right)\right\} &\leq \\
\leq p_1 \max\left\{ |F(z)| : z \in T^n\left(\mathbf{0}, R + \frac{\mathbf{1}}{\mathbf{L}(Re^{i\Theta})}\right)\right\} & \qquad (4.2)
\end{aligned}$$

A function $\ln^+\max\{|F(z)|\colon z\in T^n(\mathbf{0},R)\}$ is a convex function of the variables $\ln r_1,\ \ldots,\ \ln r_n$ (see [19, p. 84]). Hence, the function admits a representation

$$\begin{aligned}&\ln^+\max\{|F(z)|\colon z\in T^n(\mathbf{0},R)\}-\\&-\ln^+\max\{|F(z)|\colon z\in T^n(\mathbf{0},R+(r_j^0-r_j)\mathbf{1}_j)\}=\\&=\int_{r_j^0}^{r_j}\frac{A_j(r_1,\ldots,r_{j-1},t,r_{j+1},\ldots,r_n)}{t}dt\end{aligned}\tag{4.3}$$

for arbitrary $0<r_j^0\le r_j<+\infty$, where the functions $A_j(r_1,\ldots,r_{j-1},t,r_{j+1},\ldots,r_n)$ are positive non-decreasing in variable $t\in(0;+\infty)$, $j\in\{1,\ldots,n\}$.

Using (4.2) we deduce

$$\begin{aligned}&\ln p_1\ge\ln\max\left\{|F(z)|\colon z\in T^n\left(\mathbf{0},R+\frac{\mathbf{2}}{\mathbf{L}(Re^{i\Theta})}\right)\right\}-\\&-\ln\max\left\{|F(z)|:z\in T^n\left(\mathbf{0},R+\frac{\mathbf{1}}{\mathbf{L}(Re^{i\Theta})}\right)\right\}=\\&=\sum_{j=1}^n\ln\max\left\{|F(z)|\colon z\in T^n\left(\mathbf{0},R+\frac{\mathbf{1}+\sum_{k=j}^n\mathbf{1}_k}{\mathbf{L}(Re^{i\Theta})}\right)\right\}-\\&-\ln\max\left\{|F(z)|\colon z\in T^n\left(\mathbf{0},R+\frac{\mathbf{1}+\sum_{k=j+1}^n\mathbf{1}_k}{\mathbf{L}(Re^{i\Theta})}\right)\right\}=\\&=\sum_{j=1}^n\int_{r_j+1/l_j(Re^{i\Theta})}^{r_j+2/l_j(Re^{i\Theta})}\frac{1}{t}A_j\left(r_1+\frac{1}{l_1(Re^{i\Theta})},\ldots,r_{j-1}+\frac{1}{l_{j-1}(Re^{i\Theta})},t,\right.\\&\left.r_{j+1}+\frac{2}{l_{j+1}(Re^{i\Theta})},\ldots,r_n+\frac{2}{l_n(Re^{i\Theta})}\right)dt\ge\sum_{j=1}^n\ln\left(1+\frac{1}{r_jl_j(Re^{i\Theta})+1}\right)\times\\&\times A_j\left(r_1+\frac{1}{l_1(Re^{i\Theta})},\ldots,r_{j-1}+\frac{1}{l_{j-1}(Re^{i\Theta})},r_j,r_{j+1}+\frac{2}{l_{j+1}(Re^{i\Theta})},\ldots,\right.\\&\left.r_n+\frac{2}{l_n(Re^{i\Theta})}\right)\end{aligned}\tag{4.4}$$

By Lemma 3.3 the function $r_jl_j(Re^{i\Theta})\to+\infty$ $(r_j\to+\infty)$. Hence, for $j\in\{1,\ldots,n\}$ and $r_j\ge r_j^0$

$$\ln\left(1+\frac{1}{r_jl_j(Re^{i\Theta})+1}\right)\sim\frac{1}{r_jl_j(Re^{i\Theta})+1}\ge\frac{1}{2r_jl_j(Re^{i\Theta})}.$$

Thus, for every $j\in\{1,\ldots,n\}$ inequality (4.4) implies that

$$\begin{aligned}&A_j\left(r_1+\frac{1}{l_1(Re^{i\Theta})},\ldots,r_{i-1}+\frac{1}{l_{j-1}(Re^{i\Theta})},r_j,r_{j+1}+\frac{2}{l_{j+1}(Re^{i\Theta})},\ldots,\right.\\&\left.r_n+\frac{2}{l_n(Re^{i\Theta})}\right)\le2\ln p_1\ r_jl_j(Re^{i\Theta}).\end{aligned}$$

Let $R^0=(r_1^0,\ldots,r_n^0)$, where every $r_j^0$ is above chosen. Applying (4.3) $n$ times consequently we obtain

$$\ln\max\{|F(z)|\colon z\in T^n(\mathbf{0},R)\}=$$
$$=\ln\max\{|F(z)|\colon z\in T^n(\mathbf{0},R+(r_1^0-r_1)\mathbf{1}_1)\}+\int_{r_1^0}^{r_1}\frac{A_1(t,r_2,\ldots,r_n)}{t}dt=$$
$$=\ln\max\{|F(z)|\colon z\in T^n(\mathbf{0},R+(r_1^0-r_1)\mathbf{1}_1+(r_2^0-r_2)\mathbf{1}_2)\}+$$
$$+\int_{r_1^0}^{r_1}\frac{A_1(t,r_2,\ldots,r_n)}{t}dt+\int_{r_2^0}^{r_2}\frac{A_2(r_1^0,t,r_3\ldots,r_n)}{t}dt=$$
$$=\ln\max\{|F(z)|\colon z\in T^n(\mathbf{0},R^0)\}+\sum_{j=1}^n\int_{r_j^0}^{r_j}\frac{A_j(r_1^0,\ldots,r_{j-1}^0,t,r_{j+1},\ldots,r_n)}{t}dt\leq$$
$$\leq\ln\max\{|F(z)|\colon z\in T^n(\mathbf{0},R^0)\}+$$
$$+2\ln p_1\sum_{j=1}^n\int_{r_j^0}^{r_j}l_j(r_1^0e^{i\theta_1},\ldots,r_{j-1}^0e^{i\theta_{j-1}},te^{i\theta_j},r_{j+1}e^{i\theta_{j+1}},\ldots,r_ne^{i\theta_n})dt\leq$$
$$\leq\ln\max\{|F(z)|\colon z\in T^n(\mathbf{0},R^0)\}+$$
$$+2\ln p_1\sum_{j=1}^n\int_0^{r_j}l_j(r_1^0e^{i\theta_1},\ldots,r_{j-1}^0e^{i\theta_{j-1}},te^{i\theta_j},r_{j+1}e^{i\theta_{j+1}},\ldots,r_ne^{i\theta_n})dt\leq$$
$$\leq C(R^0)\sum_{j=1}^n\int_0^{r_j}l_j(r_1^0e^{i\theta_1},\ldots,r_{j-1}^0e^{i\theta_{j-1}},te^{i\theta_j},r_{j+1}e^{i\theta_{j+1}},\ldots,r_ne^{i\theta_n})dt,$$

where $C(R^0)$ is some constant. The function $\ln\max\{|F(z)|\colon z\in T^n(\mathbf{0},R)\}$ is independent of $\Theta$. Thus, the following estimate holds

$$\ln\max\{|F(z)|\colon z\in T^n(\mathbf{0},R)\}=$$
$$=O\left(\min_{\Theta\in[0,2\pi]^n}\sum_{j=1}^n\int_0^{r_j}l_j(r_1^0e^{i\theta_1},\ldots,r_{j-1}^0e^{i\theta_{j-1}},te^{i\theta_j},r_{j+1}e^{i\theta_{j+1}},\ldots,r_ne^{i\theta_n})dt\right),$$

as $|R|\to+\infty$. It is obviously that similar equality can be proved for arbitrary permutation $\sigma_n$ of the set $\{1,2,\ldots,n\}$. Thus, estimate (4.1) holds. Theorem 4.1 is proved. □

**Corollary 4.2.** *If $\mathbf{L}\in Q^n\cap K^n$, $\min_{\Theta\in[0,2\pi]^n} l_j(Re^{i\Theta})$ is non-decreasing in each variable $r_k$, $k,\ j\in\{1,\ldots,n\}$, $k\neq j$ and an entire function $F$ has bounded $\mathbf{L}$-index in joint variables then*

$$\ln\max\{|F(z)|\colon z\in T^n(\mathbf{0},R)\}=O\left(\min_{\Theta\in[0,2\pi]^n}\sum_{j=1}^n\int_0^{r_j}l_j(R^{(j)}e^{i\Theta})dt\right)$$

*as $|R|\to\infty$, where $R^{(j)}=(r_1,\ldots,r_{j-1},t,r_{j+1},\ldots,r_n)$.*

Note that Theorem 4.1 is new too for $n = 1$ because we replace the condition $l = l(|z|)$ by the condition $l \in K$, i.e. there exists $c > 0$ such that for every $r > 0$ $\max\limits_{\theta_1,\theta_2\in[0,2\pi]} \frac{l(re^{i\theta_2})}{l(re^{i\theta_1})} \le c$. Particularly, the following proposition is valid.

**Corollary 4.3.** *If $l \in Q\cap K$ and an entire in $\mathbb{C}$ function $f$ has bounded $l$-index then*

$$\ln\max\{|f(z)|\colon |z| = r\} = O\left(\min_{\theta\in[0,2\pi]} \int_0^r l(te^{i\Theta})dt\right) \quad \textit{as } r \to \infty.$$

W. K. Hayman, A. D. Kuzyk, M M. Sheremeta [12, 13] improved estimate (4.1) by other conditions on the function $l$ for a case $n = 1$. M. T. Bordulyak and M. M. Sheremeta [10] deduced similar results for entire functions of bounded $\mathbf{L}$-index in joint variables, if $l_j = l_j(|z_j|)$, $j \in \{1, \dots, n\}$. Using their method we will generalize the estimate for $l_j : \mathbb{C}^n \to \mathbb{R}_+$.

Let us to denote $a^+ = \max\{a, 0\}$, $u_j(t) = u_j(t, R, \Theta) = l_j(\frac{tR}{r^*}e^{i\Theta})$, where $a \in \mathbb{R}$, $t \in \mathbb{R}_+$, $j \in \{1, \dots, n\}$, $r^* = \max_{1\le j\le n} r_j$.

**Theorem 4.4.** *Let $\mathbf{L}(Re^{i\Theta})$ be a positive continuously differentiable function in each variable $r_k \in [0, +\infty)$, $k \in \{1, \dots, n\}$, $\Theta \in [0, 2\pi]^n$. If an entire function $F$ has bounded $\mathbf{L}$-index $N = N(F, \mathbf{L})$ in joint variables then for every $\Theta \in [0, 2\pi]^n$ and for every $R \in \mathbb{R}^n_+$ and $S \in \mathbb{Z}^n_+$*

$$\ln\max\left\{\frac{|F^{(S)}(Re^{i\Theta})|}{S!\mathbf{L}^S(Re^{i\Theta})} : \ \|S\| \le N\right\} \le \ln\max\left\{\frac{|F^{(S)}(\mathbf{0})|}{S!\mathbf{L}^S(\mathbf{0})} : \ \|S\| \le N\right\} +$$

(4.5)

$$+\int_0^{r^*}\left(\max_{\|S\|\le N}\left\{\sum_{j=1}^n \frac{r_j}{r^*}(k_j+1)l_j\left(\frac{\tau}{r^*}Re^{i\Theta}\right)\right\} + \max_{\|S\|\le N}\left\{\sum_{j=1}^n \frac{k_j(-u_j'(\tau))^+}{l_j\left(\frac{\tau}{r^*}Re^{i\Theta}\right)}\right\}\right)d\tau.$$

*Proof.* Let $R \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, $\Theta \in [0, 2\pi]^n$. Denote $\alpha_j = \frac{r_j}{r^*}$, $j \in \{1, \dots, n\}$ and $A = (\alpha_1, \dots, \alpha_n)$. We consider the function

$$g(t) = \max\left\{\frac{|F^{(S)}(Ate^{i\Theta})|}{S!\mathbf{L}^S(Ate^{i\Theta})} : \ \|S\| \le N\right\}, \tag{4.6}$$

where $At = (\alpha_1 t, \dots, \alpha_n t)$, $Ate^{i\Theta} = (\alpha_1 te^{i\theta_1}, \dots, \alpha_n te^{i\theta_n})$.

Since the function $\frac{|F^{(S)}(Ate^{i\Theta})|}{K!\mathbf{L}^K(Ate^{i\Theta})}$ is continuously differentiable by real $t \in [0, +\infty)$, outside the zero set of function $|F^{(S)}(Ate^{i\Theta})|$, the function $g(t)$ is a continuously differentiable function on $[0, +\infty)$, except, perhaps, for a countable set of points.

Therefore, using the inequality $\frac{d}{dr}|g(r)| \le |g'(r)|$ which holds except for the points $r = t$ such that $g(t) = 0$, we deduce

$$\frac{d}{dt}\left(\frac{|F^{(S)}(Ate^{i\Theta})|}{S!\mathbf{L}^S(Ate^{i\Theta})}\right) = \frac{1}{S!\mathbf{L}^S(Ate^{i\Theta})}\frac{d}{dt}|F^{(S)}(Ate^{i\Theta})| +$$

$$+|F^{(S)}(Ate^{i\Theta})|\frac{d}{dt}\frac{1}{S!\mathbf{L}^S(Ate^{i\Theta})}\leq\frac{1}{S!\mathbf{L}^S(Ate^{i\Theta})}\left|\sum_{j=1}^n F^{(S+\mathbf{1}_j)}(Ate^{i\Theta})\alpha_j e^{i\theta_j}\right|-$$

$$-\frac{|F^{(S)}(Ate^{i\Theta})|}{S!\mathbf{L}^S(Ate^{i\Theta})}\sum_{j=1}^n\frac{s_j u_j'(t)}{l_j(Ate^{i\Theta})}\leq\sum_{j=1}^n\frac{|F^{(S+\mathbf{1}_j)}(Ate^{i\Theta})|}{(S+\mathbf{1}_j)!\mathbf{L}^{S+\mathbf{1}_j}(Ate^{i\Theta})}\alpha_j(s_j+1)l_j(Ate^{i\Theta})+$$

$$+\frac{|F^{(S)}(Ate^{i\Theta})|}{S!\mathbf{L}^S(Ate^{i\Theta})}\sum_{j=1}^n\frac{s_j(-u_j'(t))^+}{l_j(Ate^{i\Theta})} \tag{4.7}$$

For absolutely continuous functions $h_1$, $h_2$, …, $h_k$ and $h(x) := \max\{h_j(z) : 1\leq j\leq k\}$, $h'(x)\leq\max\{h_j'(x) : 1\leq j\leq k\}$, $x\in[a,b]$ (see [21, Lemma 4.1, p. 81]). The function $g$ is absolutely continuous, therefore, from (4.7) it follows that

$$g'(t)\leq\max\left\{\frac{d}{dt}\left(\frac{|F^{(S)}(Ate^{i\Theta})|}{S!\mathbf{L}^S(Ate^{i\Theta})}\right) : \|S\|\leq N\right\}\leq$$

$$\leq\max_{\|S\|\leq N}\left\{\sum_{j=1}^n\frac{\alpha_j(s_j+1)l_j(Ate^{i\Theta})|F^{(S+\mathbf{1}_j)}(Ate^{i\Theta})|}{(S+\mathbf{1}_j)!\mathbf{L}^{S+\mathbf{1}_j}(Ate^{i\Theta})}+\right.$$

$$\left.+\frac{|F^{(S)}(Ate^{i\Theta})|}{S!\mathbf{L}^S(Ate^{i\Theta})}\sum_{j=1}^n\frac{s_j(-u_j'(t))^+}{l_j(Ate^{i\Theta})}\right\}\leq$$

$$\leq g(t)\left(\max_{\|S\|\leq N}\left\{\sum_{j=1}^n\alpha_j(s_j+1)l_j(Ate^{i\Theta})\right\}+\max_{\|S\|\leq N}\left\{\sum_{j=1}^n\frac{s_j(-u_j'(t))^+}{l_j(Ate^{i\Theta})}\right\}\right)=$$

$$=g(t)(\beta(t)+\gamma(t)),$$

where

$$\beta(t)=\max_{\|S\|\leq N}\left\{\sum_{j=1}^n\alpha_j(s_j+1)l_j(Ate^{i\Theta})\right\},\gamma(t)=\max_{\|S\|\leq N}\left\{\sum_{j=1}^n\frac{s_j(-u_j'(t))^+}{l_j(Ate^{i\Theta})}\right\}.$$

Thus, $\frac{d}{dt}\ln g(t)\leq\beta(t)+\gamma(t)$ and

$$g(t)\leq g(0)\exp\int_0^t(\beta(\tau)+\gamma(\tau))d\tau, \tag{4.8}$$

because $g(0)\neq 0$. But $r^*A=R$. Substituting $t=r^*$ in (4.8) and taking into account (4.6), we deduce

$$\ln\max\left\{\frac{|F^{(S)}(Re^{i\Theta})|}{S!\mathbf{L}^S(Re^{i\Theta})} :\ \|S\|\leq N\right\}\leq\ln\max\left\{\frac{|F^{(S)}(\mathbf{0})|}{S!\mathbf{L}^S(\mathbf{0})} :\ \|S\|\leq N\right\}+$$

$$+\int_0^{r^*}\left(\max_{\|S\|\leq N}\left\{\sum_{j=1}^n\alpha_j(s_j+1)l_j(A\tau e^{i\Theta})\right\}+\max_{\|S\|\leq N}\left\{\sum_{j=1}^n\frac{s_j(-u_j'(\tau))^+}{l_j(A\tau e^{i\Theta})}\right\}\right)d\tau,$$

i.e. (4.5) is proved. □

**Theorem 4.5.** *Let* $\mathbf{L}(Re^{i\Theta})$ *be a positive continuously differentiable function in each variable* $r_k \in [0, +\infty)$, $k \in \{1, \ldots, n\}$, $\Theta \in [0, 2\pi]^n$. *If an entire function* $F$ *has bounded* $\mathbf{L}$*-index* $N = N(F, \mathbf{L})$ *in joint variables and there exists* $C > 0$ *such that the function* $\mathbf{L}$ *satisfies inequalities*

$$\sup_{R\in\mathbb{R}^n_+} \max_{t\in[0,r^*]} \max_{\Theta\in[0,2\pi]^n} \max_{1\le j\le n} \frac{(-(u_j(t,R,\Theta))'_t)^+}{\frac{r_j}{r^*} l_j^2(\frac{t}{r^*}Re^{i\Theta})} \le C, \tag{4.9}$$

$$\max_{\Theta\in[0,2\pi]^n} \int_0^1 \left\langle R, \mathbf{L}\left(\tau Re^{i\Theta}\right)\right\rangle d\tau \to +\infty \ \text{as} \ |R| \to +\infty \tag{4.10}$$

*then*

$$\varlimsup_{|R|\to+\infty} \frac{\ln\max\{|F(z)|:\ z \in T^n(\mathbf{0}, R)\}}{\max\limits_{\Theta\in[0,2\pi]^n} \int_0^1 \langle R, \mathbf{L}\left(\tau Re^{i\Theta}\right)\rangle d\tau} \le (C+1)N + 1. \tag{4.11}$$

*Proof.* Denote $\widetilde{\beta}(t) = \sum_{j=1}^n \alpha_j l_j(Ate^{i\Theta})$. If, in addition, (4.9)-(4.10) hold then for some $S^*$, $\|S^*\| \le N$ and $\widetilde{S}$, $\|\widetilde{S}\| \le N$,

$$\frac{\gamma(t)}{\widetilde{\beta}(t)} = \frac{\sum_{j=1}^n \frac{s_j^*(-u_j'(t))^+}{l_j(Ate^{i\Theta})}}{\sum_{j=1}^n \alpha_j l_j(Ate^{i\Theta})} \le \sum_{j=1}^n s_j^* \frac{(-u_j'(t))^+}{\alpha_j l_j^2(Ate^{i\Theta})} \le \sum_{j=1}^n s_j^* \cdot C \le NC \ \text{and}$$

$$\frac{\beta(t)}{\widetilde{\beta}(t)} = \frac{\sum_{j=1}^n \alpha_j(\tilde{s}_j+1) l_j(Ate^{i\Theta})}{\sum_{j=1}^n \alpha_j l_j(Ate^{i\Theta})} = 1 + \frac{\sum_{j=1}^n \alpha_j \tilde{s}_j l_j(Ate^{i\Theta})}{\sum_{j=1}^n \alpha_j l_j(Ate^{i\Theta})} \le$$

$$\le 1 + \sum_{j=1}^n \tilde{s}_j \le 1 + N.$$

In view of (4.8), we have $|F(Ate^{i\Theta})| \le g(t) \le g(0)\exp\int_0^t(\beta(\tau)+\gamma(\tau))d\tau$ and $r^*A = R$. Then we put $t = r^*$ and obtain

$$\ln\max\{|F(z)|:\ z\in T^n(\mathbf{0},R)\} = \ln \max_{\Theta\in[0,2\pi]^n} |F(Re^{i\Theta})| \le \ln \max_{\Theta\in[0,2\pi]^n} g(r^*) \le$$

$$\le \ln g(0) + \max_{\Theta\in[0,2\pi]^n} \int_0^{r^*} (\beta(\tau)+\gamma(\tau))d\tau \le$$

$$\le \ln g(0) + (NC+N+1) \max_{\Theta\in[0,2\pi]^n} \int_0^{r^*} \widetilde{\beta}(\tau)d\tau =$$

$$= \ln g(0) + (NC+N+1) \max_{\Theta\in[0,2\pi]^n} \int_0^{r^*} \sum_{j=1}^n \alpha_j l_j(A\tau e^{i\Theta})d\tau =$$

$$= \ln g(0) + (NC+N+1) \max_{\Theta\in[0,2\pi]^n} \int_0^{r^*} \sum_{j=1}^n \frac{r_j}{r^*} l_j\left(\frac{\tau}{r^*}Re^{i\Theta}\right)d\tau =$$

$$= \ln g(0) + (NC+N+1) \max_{\Theta\in[0,2\pi]^n} \int_0^1 \sum_{j=1}^n r_j l_j\left(\tau Re^{i\Theta}\right)d\tau.$$

Thus, we conclude that (4.11) holds. Theorem 4.5 is proved. □

**Theorem 4.6.** *Let $\mathbf{L}(Re^{i\Theta})$ be a positive continuously differentiable function in each variable $r_k \in [0,+\infty)$, $k \in \{1,\dots,n\}$, $\Theta \in [0,2\pi]^n$. If an entire function $F$ has bounded $\mathbf{L}$-index $N = N(F,\mathbf{L})$ in joint variables and*

$$r^*(-(u_j(t,R,\Theta))'_{t=r^*})^+/(r_j l_j^2(Re^{i\Theta})) \to 0 \tag{4.12}$$

*uniformly for all $\Theta \in [0,2\pi]^n$, $j \in \{1,\dots,n\}$, and (4.10) holds as $|R| \to +\infty$ then*

$$\varlimsup_{|R|\to+\infty} \frac{\ln\max\{|F(z)|\colon\ z \in T^n(\mathbf{0},R)\}}{\max\limits_{\Theta\in[0,2\pi]^n} \int_0^1 \langle R, \mathbf{L}(\tau Re^{i\Theta})\rangle d\tau} \le N+1. \tag{4.13}$$

Estimate (4.13) can be deduced by analogy to the proof of Theorem 4.5.

If $\mathbf{L}(z) = \mathbf{L}(R)$ then (4.10) and (4.12) can be written in a more simplified form.

**Corollary 4.7.** *Let $\mathbf{L}(R)$ be a positive continuously differentiable function in each variable $r_k \in [0,+\infty)$, $k \in \{1,\dots,n\}$. If an entire function $F$ has bounded $\mathbf{L}$-index $N = N(F,\mathbf{L})$ in joint variables and for every $j \in \{1,\dots,n\}$*

$$\frac{\langle R, \nabla l_j(R)\rangle}{r_j l_j^2(R)} \to 0, \quad \int_0^1 \langle R, \mathbf{L}(\tau R)\rangle\, d\tau \to +\infty \ \text{as } |R| \to +\infty$$

*then*

$$\varlimsup_{|R|\to+\infty} \frac{\ln\max\{|F(z)|\colon\ z \in T^n(\mathbf{0},R)\}}{\int_0^1 \langle R, \mathbf{L}(\tau R)\rangle d\tau} \le N+1,$$

*where $\nabla l_j(R) = (\frac{\partial l_j(R)}{\partial r_1}, \dots, \frac{\partial l_j(R)}{r_n})$.*

We will write $u(r,\theta) = l(re^{i\theta})$. Theorem 4.5 implies the following proposition for $n = 1$.

**Corollary 4.8.** *Let $l(re^{i\theta})$ be a positive continuously differentiable function in variable $r \in [0,+\infty)$ for every $\theta \in [0,2\pi]$. If an entire function $f$ has bounded $l$-index $N = N(f,l)$ and there exists $C > 0$ such that $\varlimsup\limits_{r\to+\infty} \max\limits_{\theta\in[0,2\pi]} \frac{(-u'_r(r,\theta))^+}{l^2(re^{i\Theta})} = C$ then*

$$\varlimsup_{r\to+\infty} \frac{\ln\max\{|f(z)\colon |z| = r\}}{\max\limits_{\theta\in[0,2\pi]} \int_0^r l(\tau e^{i\theta})\, d\tau} \le (C+1)N+1.$$

*Remark* 4.9. Our result is *sharper* than Sheremeta's result which is obtained in a case $n = 1$, $C \ne 0$ and $l(|z|)$. Indeed, corresponding theorem [21, p. 83] claims that

$$\varlimsup_{r\to+\infty} \frac{\ln\max\{|f(z)\colon |z| = r\}}{\int_0^r l(\tau) d\tau} \le (C+1)(N+1).$$

Obviously, that $NC + N + 1 < (C+1)(N+1)$ for $C \ne 0$ and $N \ne 0$.

Estimate (4.13) is sharp. It is easy to check for these functions $F(z_1,z_2) = \exp(z_1 z_2)$, $l_1(z_1,z_2) = |z_2| + 1$, $l_2(z_1,z_2) = |z_1| + 1$. Then $N(F,\mathbf{L}) = 0$ and $\ln\max\{|F(z)|\colon z \in T^2(\mathbf{0},R)\} = r_1 r_2$.

## References


[1] Bandura, A., and Skaskiv, O. Functions analytic in a unit ball of bounded **L**-index in joint variables. *J. Math. Sci.*.

[2] Bandura, A., and Skaskiv, O. *Entire functions of several variables of bounded index.* Lviv: Publisher I. E. Chyzhykov, Chyslo, 2016.

[3] Bandura, A., and Skaskiv, O. Entire functions of bounded L-index: its zeros and behavior of partial logarithmic derivatives. *J. Complex Anal.* (2017), Art. ID 3253095, 10.

[4] Bandura, A., Skaskiv, O., and Filevych, P. Properties of entire solutions of some linear PDE's. *J. Appl. Math. Comput. Mech. 16*, 2 (2017), 17–28.

[5] Bandura, A. I., Bordulyak, M. T., and Skaskiv, O. B. Sufficient conditions of boundedness of L-index in joint variables. *Mat. Stud. 45*, 1 (2016), 12–26.

[6] Bandura, A. I., and Petrechko, N. V. Properties of power series of analytic in a bidisc functions of bounded **L**-index in joint variables. *Carpathian Math. Publ. 9*, 1 (2017), 6–12.

[7] Bandura, A. I., and Skaskī v, O. B. Iyer's metric space, existence theorem and entire functions of bounded **L**-index in joint variables. *Bukovyn. Mat. Zh. 5*, 3-4 (2017), 8–14.

[8] Bandura, A. I., and Skaskī v, O. B. A logarithmic derivative in a direction and the distribution of zeros of an entire function of a bounded $L$-index in a direction. *Ukraïn. Mat. Zh. 69*, 3 (2017), 426–432.

[9] Bordulyak, M. T. A proof of a Sheremeta conjecture concerning entire function of bounded $l$-index. *Mat. Stud. 12*, 1 (1999), 108–110.

[10] Bordulyak, M. T., and Sheremeta, M. M. Boundedness of the $L$-index of an entire function of several variables. *Dopov./Dokl. Akad. Nauk Ukraïni*, 9 (1993), 10–13.

[11] Gopala Krishna, J., and Shah, S. M. Functions of bounded indices in one and several complex variables. In *Mathematical Essays Dedicated to A. J. Macintyre.* Ohio Univ. Press, Athens, Ohio, 1970, pp. 223–235.

[12] Hayman, W. K. Differential inequalities and local valency. *Pacific J. Math. 44* (1973), 117–137.

[13] Kuzyk, A. D., and Sheremeta, M. N. Entire functions of a bounded $l$-distribution of values. *Mat. Zametki 39*, 1 (1986), 3–13, 156.

[14] Lepson, B. Differential equations of infinite order, hyperdirichlet series and entire functions of bounded index. In *Entire Functions and Related Parts of Analysis (Proc. Sympos. Pure Math., LaJolla, Calif., 1966).* Amer. Math. Soc., Providence, R.I., 1968, pp. 298–307.

[15] Nuray, F., and Patterson, R. F. Entire bivariate functions of exponential type. *Bull. Math. Sci. 5*, 2 (2015), 171–177.

[16] Nuray, F., and Patterson, R. F. Multivalence of bivariate functions of bounded index. *Matematiche (Catania) 70*, 2 (2015), 225–233.

[17] Patterson, R. F., and Nuray, F. A characterization of holomorphic bivariate functions of bounded index. *Math. Slovaca 67*, 3 (2017), 731–736.

[18] Pugh, W. J., and Shah, S. M. On the growth of entire functions of bounded index. *Pacific J. Math. 33* (1970), 191–201.

[19] Ronkin, L. I. *Introduction to the theory of entire functions of several variables.* American Mathematical Society, Providence, R.I., 1974. Translated from the Russian by Israel Program for Scientific Translations, Translations of Mathematical Monographs, Vol. 44.

[20] Salmassi, M. Functions of bounded indices in several variables. *Indian J. Math. 31*, 3 (1989), 249–257.

[21] Sheremeta, M. *Analytic functions of bounded index*, vol. 6 of *Mathematical Studies Monograph Series.* VNTL Publishers, L'viv, 1999.

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