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ON THE TRIGONOMETRIC APPROXIMATION OF FUNCTIONS IN WEIGHTED LORENTZ SPACES USING CESÀRO SUBMETHOD

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Abstract. In this study, we investigate some trigonometric approximation problems in the weighted Lorentz spaces with Muckenhoupt weights using infinite lower triangular regular matrices obtained by Cesàro submethod.

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1. Introduction

We start with the definition of weighted Lorentz spaces.

Let $\mathbb{T} = [-\pi, \pi]$. If the set $w^{-1}(\{0, \infty\})$ has the Lebesgue measure zero then we say that a measurable, nonnegative function $w : \mathbb{T} \to [0, \infty]$ is a weight function. Let w be a weight function and μ be a measurable set. We put

$$w(\mu) = \int_{\mu} w(x)dx. \tag{1.1}$$

We define the decreasing rearrangement $f_w^\star(t)$ of $f : \mathbb{T} \to \mathbb{R}$ with respect to the Borel measure (1.1)

$$f_w^\star(t) = inf\{\lambda \geq 0 : w(\{x \in \mathbb{T} : |f(x)| > \lambda\} \leq t)\}.$$

Let $t > 0$. Then, the average function $f^{\star\star}(t)$ is defined by

$$f^{\star\star}(t) = \frac{1}{t}\int_0^t f_w^\star(u)du.$$

Let $f : \mathbb{T} \to \mathbb{R}$ be a measurable function and $1 < p, q < \infty$. The weighted Lorentz space $L_w^{pq}(\mathbb{T})$ is defined [10, p.20], [3, p.219] as the set of all measurable functions f such that $\|f\|_{pq,w} < \infty$, where

$$\|f\|_{pq,w} = \left(\int_{\mathbb{T}} (f^{\star\star}(t))^q t^{\frac{q}{p}} \frac{dt}{t}\right)^{\frac{1}{q}}.$$

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If $p=q$, $L_w^{pq}(\mathbb{T})$ turns into weighted Lebesgue space $L_w^p(\mathbb{T})$ [10, p.20].

Let $p^{'}=\frac{p}{p-1}$ and $1<p<\infty$. A weight function w belongs to the Muckenhoupt class $A_p(\mathbb{T})$ [29] if the condition

$$\sup \frac{1}{|J|}\int_J w(x)dx\left(\frac{1}{|J|}\int_J w^{1-p^{'}}(x)dx\right)^{p-1}<\infty$$

holds, where the supremum is taken with respect to all the intervals J with length $\leq 2\pi$ and $|J|$ denotes the length of J.

When $w\in A_p(\mathbb{T})$, $1<p,q<\infty$ the relation $L_w^{pq}(\mathbb{T})\subset L^1(\mathbb{T})$ holds [10, the proof of Prop. 3.3]. So the Fourier series and the conjugate Fourier series of $f\in L_w^{pq}(\mathbb{T})$ can be given as

$$f(x)\sim\frac{a_0(f)}{2}+\sum_{k=1}^{\infty}\left(a_k(f)\cos kx+b_k(f)\sin kx\right), \tag{1.2}$$

$$\tilde{f}(x)\sim\frac{a_0(f)}{2}+\sum_{k=1}^{\infty}\left(a_k(f)\sin kx-b_k(f)\cos kx\right)$$

respectively.

Here $a_0(f)$, $a_k(f)$, $b_k(f)$, $k=1,2,\ldots$, are Fourier coefficients of f. The nth partial sum of the series (1.2) will be denoted by $S_n(f,x),(n=0,1,2,\ldots)$ at the point x, that is,

$$S_n(f,x)=\sum_{k=0}^{n}U_k(f)(x),$$

where

$$U_0(f)(x):=\frac{a_0(f)}{2},$$

$$U_k(f)(x):=a_k(f)\cos kx+b_k(f)\sin kx,\quad k=1,2,\ldots.$$

By $E_n(f)_{L_\omega^{pq}}$ we will denote the best approximation of $f\in L_\omega^{pq}(\mathbb{T})$ by trigonometric polynomials of degree $\leq n$, i.e.,

$$E_n(f)_{L_\omega^{pq}}=\inf_{T_k\in\mathbf{T}_k}\left\|f-T_k\right\|_{pq,\omega},$$

where $\mathbf{T}_k$ is the set of all trigonometric polynomials of degree $k\leq n$.

The Hardy Littlewood Maximal operator is defined for $f\in L^1$ as

$$M(f)(x):=\sup_{x\in I}\frac{1}{|I|}\int_I|f(t)|dt,\quad x\in[0,2\pi]$$

where the supremum is taken over all subintervals I of $[0,2\pi]$ [7, p. 80].

When $w\in A_p(\mathbb{T})$ and $1<p,q<\infty$, the Hardy Littlewood maximal operator of $f\in L_w^{pq}(\mathbb{T})$ is bounded in $L_w^{pq}(\mathbb{T})$ [6].

The modulus of continuity of the function $f \in L^{pq}_w(\mathbb{T})$ is defined [19] as

$$\Omega(f,\delta)_{L^{pq}_w} = \sup_{|h|<\delta} \|A_h f\|_{pq,w}\,, \quad \delta > 0,$$

where

$$(A_h f)(x) := \frac{1}{h}\int_0^h |f(x+t) - f(x)|\,dt$$

is the Steklov operator.

The modulus of continuity $\Omega(f,\delta)_{L^{pq}_w}$ is defined in this way since the space $L^{pq}_w(\mathbb{T})$ is noninvariant under the usual shift $f(\cdot) \to f(\cdot + h)$. Due to boundedness of the Hardy Littlewood maximal operator the Steklov operator $A_h f$ is bounded in $L^{pq}_w(\mathbb{T})$. So $\Omega(f,\delta)_{L^{pq}_w}$ makes sense for $w \in A_p(\mathbb{T})$.

Furthermore, the modulus of continuity $\Omega(f,\delta)_{L^{pq}_w}$ is nondecreasing, nonnegative, continuous function satisfying the conditions

$$\lim_{\delta\to 0} \Omega(f,\delta)_{L^{pq}_w} = 0,\ \Omega(f_1+f_2,\delta)_{L^{pq}_w} \le \Omega(f_1,\delta)_{L^{pq}_w} + \Omega(f_2,\delta)_{L^{pq}_w}.$$

In Lebesgue spaces L^p $(1<p<\infty)$, the traditional modulus of continuity is defined as

$$\omega_p(f,\delta) = \sup_{0<h\le\delta} \|f(\cdot+h) - f(\cdot)\|_p\,,\ \delta > 0,$$

It is known that the modulus of continuity $\Omega(f,\delta)_{L^{pq}_w}$ and traditional modulus of continuity $\omega_p(f,\delta)$ are equivalent (see [19]).

Lipschitz class $Lip(\alpha, L^{pq}_w)$ is defined as

$$Lip(\alpha, L^{pq}_w) := \left\{ f \in L^{pq}_w(\mathbb{T}) : \Omega(f,\delta)_{L^{pq}_w} = O(\delta^\alpha),\ 0<\alpha\le 1 \right\}.$$

Let $(\lambda_n)_{n=1}^\infty$ be a strictly increasing sequence of positive integers. For a sequence (x_k) of the real or complex numbers, the Cesàro submethod C_λ is defined by

$$(C_\lambda x)_n := \frac{1}{\lambda_n}\sum_{k=1}^{\lambda_n} x_k,\ (n = 1,2,\ldots).$$

Particularly, when $\lambda_n = n$ we note that $(C_\lambda x)_n$ is the classical Cesàro method $(C,1)$ of (x_k). Therefore, the Cesàro submethod C_λ yields a subsequence of the Cesàro method $(C,1)$. The basic properties of the method C_λ were investigated firstly by Armitage and Maddox in [2] and Osikiewicz [30]. In these works, the relations between the classical Cesàro method and Cesàro submethod were obtained. Further information about the method C_λ can be found in [2, 30].

We denote by $T \equiv (a_{n,k})$ a lower triangular regular matrix with nonnegative entries and row sums 1. We define

$$\tau_n^\lambda(f,x) := \sum_{k=0}^{\lambda_n} a_{\lambda_n,k} S_k(f,x),\ n = 0,1,2,\ldots \tag{1.3}$$

and

$$T_n^{\lambda}(f;x)=\sum_{k=0}^{\lambda_n} a_{\lambda_n,\lambda_n-k}S_k(f;x),\ n=0,1,2,\ldots . \tag{1.4}$$

When $\lambda_n=n$, the method (1.3) turns into matrix method $\tau_n(f,x)$ defined by

$$\tau_n(f,x):=\sum_{k=0}^{n} a_{n,k}S_k(f,x),\ n=0,1,2,\ldots$$

and the method (1.4) turns into matrix method $T_n(f;x)$ defined by

$$T_n(f;x)=\sum_{k=0}^{n} a_{n,n-k}S_k(f;x), n=0,1,2,\ldots .$$

When $a_{\lambda_n,\lambda_n-k}=\frac{p_{\lambda_n-k}}{P_{\lambda_n}}$, the method T_n^{λ} turns into Nörlund submethod given as

$$N_n^{\lambda}(f;x):=\frac{1}{P_{\lambda_n}}\sum_{k=0}^{\lambda_n} p_{\lambda_n-k}S_k(f;x)$$

and when $a_{\lambda_n,k}=\frac{p_k}{P_{\lambda_n}}$, the method τ_n^{λ} turns into Riesz submethod given by

$$R_n^{\lambda}(f;x):=\frac{1}{P_{\lambda_n}}\sum_{k=0}^{\lambda_n} p_kS_k(f;x)$$

where

$$P_{\lambda_n}=p_0+p_1+p_2+\ldots+p_{\lambda_n}\neq 0 \quad (n\geqslant 0)$$

and by convention, $p_{-1}=P_{-1}=0$ [30].

Also, in the case $p_n=1$, $n\geq 0$, $\lambda_n=n$, both of $N_n^{\lambda}(f)(x)$ and $R_n^{\lambda}(f)(x)$ are equal to the Cesàro mean

$$\sigma_n(f)(x)=\frac{1}{n+1}\sum_{k=0}^{n} S_k(f;x).$$

A nonnegative sequence $u:=(u_n)$ is called almost monotone decreasing (increasing), if there exists a constant $K:=K(u)$, depending on the sequence u only, such that

$$u_n\leq Ku_m \ \ (Ku_n\geq u_m)$$

for all $n\geq m$. Such sequences will be denoted by $u\in AMDS$ $(u\in AMIS)$.

Let

$$A_{n,k}:=\frac{1}{k+1}\sum_{i=0}^{k} a_{n,i}.$$

If $\{A_{n,k}\} \in AMDS$ $(\{A_{n,k}\} \in AMIS)$, then we will say that $\{a_{n,k}\}$ is an almost monotone decreasing (increasing) mean sequence. Briefly, we will write $\{a_{n,k}\} \in AMDMS$ $(\{a_{n,k}\} \in AMIMS)$.

Let

$$\overset{*}{A}_{n,k} := \frac{1}{k+1} \sum_{i=n-k}^{n} a_{n,i}.$$

If $\left\{A^*_{n,k}\right\} \in AMDS$ $\left(\left\{A^*_{n,k}\right\} \in AMIS\right)$, then we will say that $\{a_{n,k}\}$ is an almost monotone decreasing (increasing) upper mean sequence. Briefly, $\{a_{n,k}\} \in AMDUMS$ $(\{a_{n,k}\} \in AMIUMS)$.

The following inclusions are valid [28, 32].

$$AMDS \subset AMIUMS \text{ and } AMIS \subset AMDUMS$$

We will use sums up to λ_n in S_n and σ_n and write these sums as S_n^λ and σ_n^λ, respectively.

The operator Δ_k is defined by

$$\Delta_k a_{n,k} = a_{n,k} - a_{n,k+1}.$$

The relation $\preceq$ is defined as "$A \preceq B$ $\Leftrightarrow$there exists a positive constant C, independent of essential parameters, such that $A \leq CB$".

We set

$$[x] := \max\{n \in \mathbb{Z} : n \leq x\}.$$

2. Historical Background

Approximation degree of trigonometric polynomials by means of different summability methods were investigated by many researchers in Lebesgue spaces. We will give some of these results in chronological order.

In [31], Quade proved an important theorem about trigonometric approximation without using the method given by Hardy and Littlewood [14] in 1928. The result of Quade is following:

Theorem 2.1. *Let* $f \in Lip(\alpha, p)$, $0 < \alpha \leq 1$. *Then*

$$\|f - \sigma_n(f)\|_p = O(n^{-\alpha})$$

for either

(i) $p > 1$ *and* $0 < \alpha \leq 1$ *or*

(ii) $p = 1$ *and* $0 < \alpha < 1$.

And if $p = \alpha = 1$ *then*

$$\|f - \sigma_n(f)\|_1 = O(n^{-1}\log(n+1)).$$

In [4, 5], Chandra extended the work of Quade [31] using $N_n(f;x)$ and $R_n(f;x)$ methods which are the generalizations of Cesàro method. These studies have given very satisfactory results about trigonometric approximation by means of $N_n(f;x)$ and $R_n(f;x)$ methods. Leindler [20] improved the theorems proved by Chandra [5] by weakened monotonicity conditions on sequences (P_n). The work of Chandra [5] was extended by Mittal et al. [22] using matrix method which is a generalization of $N_n(f;x)$ and $R_n(f;x)$ methods. The results given by Chandra [5] in classical Lebesgue spaces were extended by Guven [11] to weighted Lebesgue spaces. The approximation theorems proved by Chandra [5] and Leindler [20] were improved by Szal [32] under weakened assumptions, and they were also extended by Israfilov and Guven [13] to variable exponent weighted Lebesgue spaces. The trigonometric approximation problems investigated by Mittal et al. [22] their results were improved by Mittal et al. [23] by dropping the monotonicity on the elements of matrix rows. Also, they extended two theorems given by Leindler [20] to a more general class of triangular matrix methods. In [12], Guven extended the work of Mittal et al. [22] to variable exponent Lebesgue spaces using matrix method. In [28], Mohapatra and Szal investigated the results given in [5] with less stringent assumptions using matrix method. Deger and Kaya [9] and Deger et al. [8] generalized the results obtained by Chandra [5] and Leindler [20] taking $N_n^\lambda(f;x)$ and $R_n^\lambda(f;x)$ in the place of $N_n(f;x)$ and $R_n(f;x)$. Krasniqi [17] investigated the degree of approximation using Nörlund and Riesz submethods by modulus of continuity of first order in the weighted Lebesgue spaces. In [18], the results on the trigonometric approximation problems by means of $N_n^\lambda(f;x)$ and $R_n^\lambda(f;x)$ submethods were investigated for two new numerical sequences. In [24], Mittal and Singh improved the results given by Deger et al. [8] by dropping monotonicity conditions on the elements of matrix rows. In [9], Deger and Kaya extended the works of Leindler [20] and Mohapatra and Szal [28] under the perspective of $N_n^\lambda(f;x)$ and $R_n^\lambda(f;x)$ submethods.

In [25], Mittal and Singh examined the approximation rate of functions using $T_n^\lambda(f;x)$ and $\tau_n^\lambda(f;x)$ obtained by means of Cesàro Submethod in Lebesgue spaces.

Lebesgue space may be generalized in different ways. One of the important generalizations of this space is the Lorentz space. Lorentz space was firstly introduced by G. G. Lorentz in [21]. By means of the weight functions satisfying Muckenhoupt condition, the weighted Lorentz spaces were defined in [3, 10].

In weighted Lorentz spaces, some researchers obtained results about approximation theory using different methods [1, 16, 34, 33]. But in these papers degree of approximation using Cesàro submethod were not examined in the weighted Lorentz spaces. In this paper, we examine degree of approximation using this method in these spaces and generalize the results obtained by Mittal and Singh [25] to weighted Lorentz spaces.

3. Main results

In this study, we obtain the following results related to trigonometric approximation using matrix submethod of partial sums of Fourier series of functions f in weighted Lorentz spaces.

Theorem 3.1. *Let* $1 < p, q < \infty$, $w \in A_p(\mathbb{T})$, $f \in Lip(\alpha, L_w^{p,q})$ *and* $T \equiv (a_{n,k})$ *a lower triangular regular matrix with nonnegative entries and row sums* 1. *If one of the conditions*

(i) $0 < \alpha < 1$ *and* $\{a_{\lambda_n,k}\} \in AMIMS$,

(ii) $0 < \alpha < 1$, $\{a_{\lambda_n,k}\} \in AMDMS$ *and*

$$(\lambda_n + 1)a_{\lambda_n,0} = O(1),$$

(iii) $\alpha = 1$ *and*

$$\sum_{k=0}^{\lambda_n-2} |\Delta_k A_{\lambda_n,k}| = O\left(\lambda_n^{-1}\right)$$

is valid, then

$$\left\|f - \tau_n^{\lambda}(f)\right\|_{pq,w} = O(\lambda_n^{-\alpha}). \tag{3.1}$$

If we take $a_{\lambda_n,k} = \frac{p_k}{P_{\lambda_n}}$, we obtain the following corollary.

Corollary 3.2. *Let* $1 < p, q < \infty$, $w \in A_p(\mathbb{T})$, $f \in Lip(\alpha, L_w^{p,q})$ *and* (p_k) *be a positive sequence. If one of the conditions*

(i) $0 < \alpha < 1$ *and* $(p_k) \in AMIMS$,

(ii) $0 < \alpha < 1$, $(p_k) \in AMDMS$ *and*

$$(\lambda_n + 1)p_{\lambda_n} = O\left(P_{\lambda_n}\right),$$

(iii) $\alpha = 1$ *and*

$$\sum_{k=0}^{\lambda_n-1} \left|\Delta_k \frac{P_k}{k+1}\right| = O\left(\frac{P_{\lambda_n}}{\lambda_n}\right)$$

is valid, then

$$\left\|f - R_n^{\lambda}(f)\right\|_{pq,w} = O(\lambda_n^{-\alpha}).$$

Also, the result is similar to this corollary was obtained in Lebesgue space L_p, $1 < p < \infty$ in [9].

Theorem 3.3. *Let* $1 < p, q < \infty$, $w \in A_p(\mathbb{T})$, $f \in Lip(\alpha, L_w^{p,q})$ *and let* $T \equiv (a_{n,k})$ *be a lower triangular regular matrix with nonnegative entries and row sums 1. If one of the conditions*

(i) $0 < \alpha < 1$ *and* $\{a_{\lambda_n,k}\} \in AMDUMS$,

(ii) $0 < \alpha < 1, \{a_{\lambda_n,k}\} \in AMIUMS$ *and have*

$$(\lambda_n + 1)a_{\lambda_n,\lambda_n} = O(1),$$

(iii) $\alpha = 1$ *and*

$$\sum_{k=0}^{\lambda_n-2} \left|\Delta_k A^{*}_{\lambda_n,k}\right| = O\left(\lambda_n^{-1}\right)$$

is valid, then

$$\left\|f - T_n^{\lambda}(f)\right\|_{pq,w} = O(\lambda_n^{-\alpha}).$$

When we take $a_{\lambda_n,\lambda_n-k} = \frac{p_k}{P_{\lambda_n}}$ and $P_{\lambda_n,k} := \frac{1}{k+1}\sum\limits_{i=\lambda_n-k}^{\lambda_n} p_i$, we have the following corollary.

Corollary 3.4. *Let* $1 < p, q < \infty$, $\omega \in A_p(\mathbb{T})$, $f \in Lip(\alpha, L_w^{p;q})$ *and let* (p_k) *be a positive sequence. If one of the conditions*

(i) $0 < \alpha < 1$ *and* $(p_k) \in AMDUMS$,

(ii) $0 < \alpha < 1$, $(p_k) \in AMIUMS$ *and*

$$(\lambda_n + 1)p_{\lambda_n} = O\left(P_{\lambda_n}\right),$$

(iii) $\alpha = 1$ *and*

$$\sum_{k=0}^{\lambda_n-2} \left|\Delta_k P_{\lambda_n,k}\right| = O\left(\frac{P_{\lambda_n}}{\lambda_n}\right)$$

is valid, then

$$\left\|f - N_n^{\lambda}(f)\right\|_{pq,w} = O(\lambda_n^{-\alpha}).$$

Also, the result similar to this corollary was obtained in Lebesgue space L_p, $1 < p < \infty$ in [9].

4. Applications

In this section, we will give an application of digital filter, especially of modified Cesàro filter.

In (1.3), when we assume that (S_k) is the input sequence and $\left(\tau_n^{\lambda}\right)$ is the output sequence of information, we can say that $\left(\tau_n^{\lambda}\right)$ behaves as a digital filter. Then, $\left(\tau_n^{\lambda}\right)$ yields a modified version of Cesàro filter. When we use $a_{\lambda_n,k}$ instead of $a_{n,k}$, then we get sharper (good) estimates. So, we have a better performance of the digital filter. In [27], it has been mentioned that digital filters are widely used in many signal processing applications such as speech and image processing, radar, sonar and medical signal processing. Thus, the design of digital filters is transformed into approximation problems. Approximation method is one of the popular methods for the design of digital filters.

Example 4.1. To illustrate Gibb's phenomenon, consider the function defined by

$$f(t) = \begin{cases} -1, & -\pi \le t \le 0 \\ 1, & 0 \le t \le \pi. \end{cases}$$

The nth *partial sum of Fourier series of Gibb's phenomenon is*

$$S_n(t) = \sum_{k=-n,\ k\neq 0}^{n} \frac{i}{\pi}\frac{-1+(-1)^k}{k} e^{ikt}.$$

Then one can write that the nth Cesàro sum (filter) as follows

$$\sigma_n(t) = \sum_{k=1}^{n} \left(\frac{n-k}{n}\right)\left(\frac{-2}{\pi}\right)\left(\frac{-1+(-1)^k}{k}\right)\sin(kt).$$

For $n = 10$ using the above two definitions we plot the following figure.

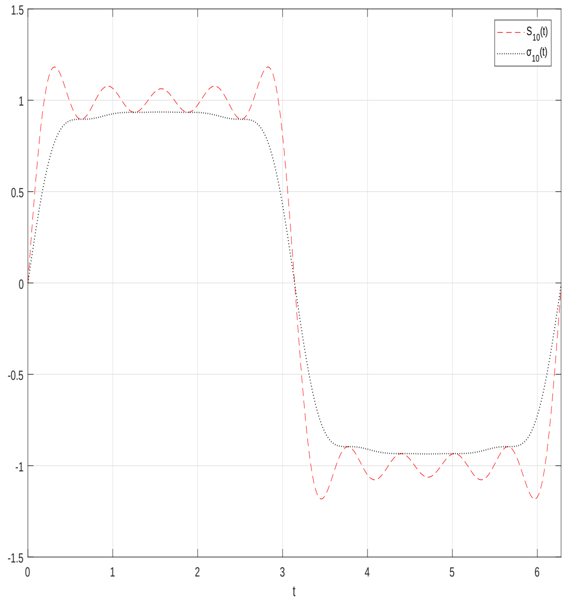

Figure 1:

For further information see *[26, p. 465-468]*. Note that the estimates in our main submethod results are sharper than the results obtained using Cesàro method because $(\lambda_n)^{-\alpha} \leq (n)^{-\alpha}$ for $0 < \alpha \leq 1$.

5. Auxiliary Results

We shall use the following lemmas for proving our main theorems.

Lemma 5.1. *Let $1 < p, q < \infty$, $w \in A_p(\mathbb{T})$. If $f \in Lip(1, L_w^{pq})$, then f is absolutely continuous and $f' \in L_w^{pq}(\mathbb{T})$.*

Proof. We follow the method in [19, Th. 3]. If $f \in L_w^{pq}(\mathbb{T})$, then there exists $p_0 > 1$ such that $f \in L^{p_0}$ and

(5.1) $$\|f\|_{L^{p_0}} \preceq \|f\|_{L_w^{pq}}$$

[16, Prop. 3.3]. Using (5.1) and the equivalence of traditional modulus of continuity $\omega(f,\delta)_{L^p}$ and $\Omega(f,\delta)_{L^{pq}_w}$, we get

$$\omega(f,\delta)_{L^{p_0}} \preceq \Omega(f,\delta)_{L^{pq}_w}.$$

Since $\Omega(f,\delta)_{L^{pq}_w} = O(\delta)$, the same estimate holds for $\omega(f,\delta)_{L^{p_0}}$, too. From here, we obtain that f is absolutely continuous in $[-\pi,\pi]$ and for almost every x

$$\frac{f(x+t)-f(x)}{t} \to f^{'}(x), \quad (t \to 0). \tag{5.2}$$

From (5.2) we write also for almost every x

$$\frac{2}{\delta}\int_{\frac{\delta}{2}}^{\delta} \frac{|f(x+t)-f(x)|}{t}dt \to \left|f^{'}(x)\right|, \quad (\delta \to 0_+).$$

From Fatou's Lemma

$$\begin{aligned}
\left\|f^{'}\right\|_{L^{pq}_w} &\le \liminf_{\delta\to 0_+}\left\|\frac{2}{\delta}\int_{\frac{\delta}{2}}^{\delta}\frac{|f(x+t)-f(x)|}{t}dt\right\|_{L^{pq}_w}\\
&\le \limsup_{\delta\to 0_+}\frac{4}{\delta}\left\|\frac{1}{\delta}\int_0^{\delta}|f(x+t)-f(x)|\,dt\right\|_{L^{pq}_w}\\
&\le \limsup_{\delta\to 0_+} 4\frac{\Omega(f,\delta)}{\delta} < \infty.
\end{aligned}$$

We have proved the lemma.

□

Lemma 5.2. *Let $1 < p,q < \infty$, $w \in A_p(\mathbb{T})$ and $f \in Lip(1, L^{pq}_w)$. Then, for $n = 1,2,\ldots,$ the estimate*

$$\|\sigma_n(f) - S_n(f)\|_{pq,w} = O(n^{-1})$$

holds.

Proof. If the Fourier series of f is

$$f(x) \sim \sum_{k=0}^{\infty} U_k(f)(x),$$

then the conjugate Fourier series of the function $f^{'}$

$$\tilde{f}^{'}(x) \sim \sum_{k=0}^{\infty} kU_k(f)(x).$$

On the other hand,

$$\begin{aligned} S_n(f)(x) - \sigma_n(f)(x) &= \sum_{k=1}^{n} \frac{k}{n+1} U_k(f)(x) \\ &= \frac{1}{n+1} S_n(\tilde{f}')(x). \end{aligned}$$

Since the partial sums and the conjugate operator are bounded in the space $L_w^{pq}(\mathbb{T})$ (see [16, Prop. 3.4],[15, Th. 6.6.2,],[35, Chap. VI]), we obtain that

$$\begin{aligned} \|S_n(f) - \sigma_n(f)\|_{pq,w} &= \frac{1}{n+1} \left\| S_n(\tilde{f}') \right\|_{pq,w} \preceq \frac{1}{n+1} \left\| \tilde{f}' \right\|_{pq,w} \\ &\preceq \frac{1}{n+1} \left\| f' \right\|_{pq,w} = O(n^{-1}) \end{aligned}$$

for $n = 1, 2, \ldots$. □

Lemma 5.3. *Let* $0 < \alpha \le 1$, $1 < p, q < \infty$, $w \in A_p(\mathbb{T})$ *and* $f \in Lip(1, L_w^{pq})$. *Then, the estimate*

$$\|f - S_n(f)\|_{pq,w} = O(n^{-\alpha})$$

is valid for $n = 1, 2, \ldots$.

Proof. Assume that t_n^* $(n = 0, 1, \ldots)$ is the best approximating trigonometric polynomial to f in L_w^{pq}. Then we have

$$\|f - t_n^*\|_{pq,w} = \inf_{t_n \in \mathbf{T}_n} \|f - t_n\|_{pq,w} .$$

From [34, Lemma 2.3], we have

$$\|f - t_n^*\|_{pq,w} = O(\Omega(f, 1/n)_{L_w^{pq}})$$

and hence

$$\|f - t_n^*\|_{pq,w} = O(n^{-\alpha}).$$

Using the uniform boundedness of the partial sums $S_n(f)$ in the space L_w^{pq}, we get

$$\begin{aligned} \|f - S_n(f)\|_{pq,w} &\le \|f - t_n^*\|_{pq,w} + \|t_n^* - S_n(f)\|_{pq,w} \\ &= \|f - t_n^*\|_{pq,w} + \|S_n(t_n^* - f)\|_{pq,w} \\ &= O(\|f - t_n^*\|_{pq,w}) \\ &= O(n^{-\alpha}). \end{aligned}$$

□

Lemma 5.4. *([25])*

Let $T \equiv (a_{n,k})$ *be an infinite lower triangular regular matrix with nonnegative entries and row sums* 1. *If either*

(i)

$$\{a_{\lambda_n,k}\} \in AMIMS$$

or

(ii)

$$\{a_{\lambda_n,k}\} \in AMDMS \text{ and } (\lambda_n + 1)a_{\lambda_n,0} = O(1),$$

then, for $0 < \alpha < 1$,

$$\sum_{k=0}^{\lambda_n} a_{\lambda_n,k} (k+1)^{-\alpha} = O\left(\lambda_n^{-\alpha}\right).$$

Lemma 5.5. *([25]) Let* $T \equiv (a_{n,k})$*be an infinite lower triangular regular matrix with nonnegative entries and row sums* 1*. If one of the conditions*

(i) $\{a_{\lambda_n,k}\} \in AMDUMS$

(ii) $\{a_{\lambda_n,k}\} \in AMIUMS$ *and*

$$(\lambda_n + 1)a_{\lambda_n,\lambda_n} = O(1)$$

holds, then for $0 < \alpha < 1$,

$$\sum_{k=0}^{\lambda_n} a_{\lambda_n,\lambda_n-k} (k+1)^{-\alpha} = O\left(\lambda_n^{-\alpha}\right).$$

6. Proof of Main Theorems

Proof of Theorem 3.1

Since

$$\tau_n^{\lambda}(f;x) - f(x) = \sum_{k=0}^{\lambda_n} a_{\lambda_n,k} S_k(f;x) - f(x),$$

using Lemma 5.3 and Lemma 5.4 we have

$$\begin{aligned} \left\|\tau_n^{\lambda}(f) - f\right\|_{pq,w} &\leq \sum_{k=0}^{\lambda_n} a_{\lambda_n,k} \left\|S_k(f) - f\right\|_{pq,w} \\ &= O\left(\sum_{k=0}^{\lambda_n} a_{\lambda_n,k}(k+1)^{-\alpha}\right) \\ &= O(\lambda_n^{-\alpha}). \end{aligned}$$

This completes the proof of (i) and (ii).

In the case of $\alpha = 1$, we will use S_{λ_n} as S_n^{λ}. Applying Abel's transformation two times,

$$\tau_n^{\lambda}(f;x) - f(x) = \sum_{k=0}^{\lambda_n - 1} \left[S_k(f;x) - S_{k+1}(f;x)\right] \sum_{i=0}^{k} a_{\lambda_n,i}$$

$$
\begin{aligned}
& + \left[S_n^\lambda f(x) - f(x) \right] \\
= \; & S_n^\lambda f(x) - f(x) - \sum_{k=0}^{\lambda_n - 1} (k+1) U_{k+1} f(x) A_{\lambda_n, k} \\
= \; & S_n^\lambda f(x) - f(x) - \sum_{k=0}^{\lambda_n - 2} (A_{\lambda_n, k} - A_{\lambda_n, k+1}) \times \\
& \sum_{i=0}^{k} (i+1) U_{i+1} f(x) - A_{\lambda_n, \lambda_{n-1}} \sum_{i=0}^{\lambda_n - 1} (i+1) U_{i+1} f(x) \\
= \; & S_n^\lambda f(x) - f(x) - \sum_{k=0}^{\lambda_n - 2} (A_{\lambda_n, k} - A_{\lambda_n, k+1}) \sum_{i=1}^{k+1} i U_i f(x) \\
& - \frac{1}{\lambda_n} \sum_{k=0}^{\lambda_n} a_{\lambda_n, k} \sum_{i=1}^{\lambda_n} i U_i f(x).
\end{aligned}
$$

Therefore by Minkowski inequality, we get

$$
\left\| \tau_n^\lambda(f) - f \right\|_{pq,w} \leq \left\| S_n^\lambda f - f \right\|_{pq,w} \tag{6.1}
$$

$$
+ \sum_{k=0}^{\lambda_n - 2} \left\| \sum_{i=1}^{k+1} i U_i f \right\|_{pq,w} \left| \Delta_k A^*_{\lambda_n, k} \right| + \frac{1}{\lambda_n} \left\| \sum_{i=1}^{\lambda_n} i U_i f \right\|_{pq,w}.
$$

We get

$$
S_n^\lambda f - \sigma_n^\lambda f = \frac{1}{\lambda_n + 1} \sum_{i=1}^{\lambda_n} i U_i f(x).
$$

From Lemma 5.2

$$
\left\| \sum_{i=1}^{\lambda_n} i U_i f \right\|_{pq,w} = (\lambda_n + 1) \left\| S_n^\lambda f - \sigma_n^\lambda f \right\|_{pq,w} = O(1). \tag{6.2}
$$

If

$$
\sum_{k=0}^{\lambda_n - 2} \left| \Delta_k A_{\lambda_n, k} \right| = O\left(\lambda_n^{-1} \right),
$$

then, from (6.1),(6.2) and Lemma 5.3, we write

$$
\begin{aligned}
\left\| \tau_n^\lambda(f) - f \right\|_{pq,w} &= O\left(\lambda_n^{-1} \right) + O(1) + \sum_{k=0}^{\lambda_n - 2} \left| \Delta_k A_{\lambda_n, k} \right| \\
&= O\left(\lambda_n^{-1} \right).
\end{aligned}
$$

Therefore, this yields (3.1).

Proof of Theorem 3.3

The proof of this theorem is similar to the proof of Theorem 3.1. Here, we use Lemma 5.5 instead of Lemma 5.4.

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