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OPERATORS INDUCED BY WEIGHTED TOEPLITZ AND WEIGHTED HANKEL OPERATORS

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Abstract. In this paper, the notion of weighted Toep-Hank operator G_{ϕ}^{β}, induced by the symbol $\phi \in L^{\infty}(\beta)$, on the space $H^2(\beta)$, $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ being a semi-dual sequence of positive numbers with $\beta_0 = 1$, is introduced. Symbols are identified for the induced weighted Toep-Hank operator to be co-isometry, normal and hyponormal.

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1. Preliminaries and Introduction

Let $\mathbb{C}$ and $\mathbb{Z}$ denote the set of all complex numbers and integers, respectively. We consider the spaces $L^2(\beta)$, $H^2(\beta)$, $L^{\infty}(\beta)$ and $H^{\infty}(\beta)$ under the assumption that $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ is a semi-dual sequence of positive numbers (that is $\beta_n = \beta_{-n}$ for each n) with $\beta_0 = 1$, $r \leq \frac{\beta_n}{\beta_{n+1}} \leq 1$ for $n \geq 0$, for some $r > 0$. Any additional condition if needed, is stated explicitly. If there is no confusion about the sequence, we denote it by $\beta = \{\beta_n\}_{n\geq 0}$. We begin with the following notational familiarity needed in the paper, for the details of which we refer to [3],[9] and the references therein.

The space $L^2(\beta)$ consists of all formal Laurent series of the form $f(z) = \sum\limits_{n=-\infty}^{\infty} a_n z^n$, $a_n \in \mathbb{C}$ (whether or not the series converges for any values of z) for which $\|f\|_{\beta} < \infty$, where $\|f\|_{\beta}$ is defined as

$$\|f\|_{\beta}^2 = \sum_{n=-\infty}^{\infty} |a_n|^2 {\beta_n}^2.$$

The space $L^2(\beta)$ is a Hilbert space with the norm $\|\cdot\|_{\beta}$ induced by the inner product

$$\langle f, g\rangle = \sum_{n=-\infty}^{\infty} a_n\, \bar{b}_n {\beta_n}^2,$$

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for $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n,\ g(z) = \sum_{n=-\infty}^{\infty} b_n z^n$. The collection $\{e_n(z) = z^n/\beta_n\}_{n\in\mathbb{Z}}$ form an orthonormal basis for $L^2(\beta)$.

The collection of all $f(z) = \sum_{n=0}^{\infty} a_n z^n$ (formal power series) for which $\|f\|_\beta^2 = \sum_{n=0}^{\infty} |a_n|^2 {\beta_n}^2 < \infty$, is denoted by $H^2(\beta)$. $H^2(\beta)$ is a subspace of $L^2(\beta)$.

Let $L^\infty(\beta)$ denote the set of formal Laurent series $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ such that $\phi L^2(\beta) \subseteq L^2(\beta)$ and there exists some $c > 0$ satisfying $\|\phi f\|_\beta \leq c\|f\|_\beta$ for each $f \in L^2(\beta)$. For $\phi \in L^\infty(\beta)$, define the norm $\|\phi\|_\infty$ as

$$\|\phi\|_\infty = \inf\{c > 0 : \|\phi f\|_\beta \leq c\|f\|_\beta \text{ for each } f \in L^2(\beta)\}.$$

$L^\infty(\beta)$ is a Banach space with respect to $\|\cdot\|_\infty$. Also, $L^\infty(\beta) \subseteq L^2(\beta)$. $H^\infty(\beta)$ denotes the set of formal power series ϕ such that $\phi H^2(\beta) \subseteq H^2(\beta)$. These weighted sequence spaces cover Bergman, Hardy, Dirichlet and Fischer spaces for specifically designed sequences $\beta = \{\beta_n\}$ and thus become more demanding.

A huge literature is available on the study of Toeplitz and Hankel operators on the Hardy spaces, for which we refer [[8],[3],[6]] and the references therein. A class of operators induced from these operators was discussed in [2] and named as the class of *Toep-Hank operators, whose matrix representation provides a Hankel matrix if only even columns are considered and a Toeplitz matrix if only odd columns are considered.*

The study of multiplication or Laurent operators was extended to the space $L^2(\beta)$ by Shields [9] in the year 1974. The notions of Toeplitz and Hankel operators were lifted to weighted Toeplitz and weighted Hankel operators on weighted sequence spaces $H^2(\beta)$ and $L^2(\beta)$ in [7] and [3], respectively. In this paper, we are now interested to extend the notion of Toep-Hank operators to the weighted Hardy space $H^2(\beta)$ and call these operators as weighted Toep-Hank operators. In the second section of this paper, some algebraic properties of these operators are discussed and a necessary condition is obtained for the adjoint of the weighted Toep-Hank operator to be an isometry. However, it is seen that there is no isometric weighted Toep-Hank operator on $H^2(\beta)$. In the third section, an attempt is made to study the compactness, hyponormality and normality of the weighted Toep-Hank operators.

2. Weighted Toep-Hank operators

We begin with the definition of weighted Toeplitz, weighted Hankel and Toep-Hank operators, which are frequently used in the paper.

Definition 2.1 ([7])**.** For $\phi \in L^\infty(\beta)$, a weighted Toeplitz operator T_ϕ^β on the space $H^2(\beta)$ is an operator given by $T_\phi^\beta = P^\beta M_\phi^\beta\big|_{H^2(\beta)}$, where $P^\beta : L^2(\beta) \to H^2(\beta)$ is the orthogonal projection of $L^2(\beta)$ onto $H^2(\beta)$ and M_ϕ^β is the weighted Laurent operator on $L^2(\beta)$.

Definition 2.2 ([3])**.** For $\phi \in L^\infty(\beta)$, a weighted Hankel operator H_ϕ^β is an operator on $H^2(\beta)$ given by $H_\phi^\beta = P^\beta J^\beta M_\phi^\beta\big|_{H^2(\beta)}$, where J^β is the reflection operator on $L^2(\beta)$ given by $J^\beta(e_n) = e_{-n}$ for $n \in \mathbb{Z}$.

We recall that for ϕ given by $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$, $a_n \in \mathbb{C}$, the symbol $\widetilde{\phi}$ means the expression $\widetilde{\phi}(z) = \sum_{n=-\infty}^{\infty} a_{-n} z^n$. It is easy to see that if $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ is a semi-dual sequence and $\phi \in L^\infty(\beta)$ then $\widetilde{\phi} \in L^\infty(\beta)$.

If $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$, then T_ϕ^β and H_ϕ^β satisfy that for each $j \geq 0$,

$$T_\phi^\beta e_j = \frac{1}{\beta_j}\sum_{n=0}^{\infty} a_{n-j}\beta_n e_n \ ; \ T_\phi^{\beta^*} e_j = \beta_j \sum_{n=0}^{\infty} \overline{a}_{j-n}\frac{e_n}{\beta_n}$$

$$H_\phi^\beta e_j = \frac{1}{\beta_j}\sum_{n=0}^{\infty} a_{-n-j}\beta_n e_n \ ; \ H_\phi^{\beta^*} e_j = \beta_j \sum_{n=0}^{\infty} \overline{a}_{-n-j}\frac{e_n}{\beta_n}.$$

Definition 2.3 ([2])**.** Let $\phi \in L^\infty(\mathbb{T})$. A Toep-Hank operator G_ϕ on $H^2(\mathbb{T})$ induced by ϕ is given by $G_\phi = H_\phi\Lambda + T_{z\widetilde{\phi}}V$, where V and Λ are operators on $H^2(\mathbb{T})$ defined as

$$V(e_n) = \begin{cases} e_{\frac{n-1}{2}} & \text{if n is odd,} \\ 0 & \text{if n is even} \end{cases} \quad \text{and} \quad \Lambda(e_n) = \begin{cases} e_{\frac{n}{2}} & \text{if n is even,} \\ 0 & \text{if n is odd.} \end{cases}$$

Each Toep-Hank operator can be expressed as $G_\phi = PJM_\phi K$, K being an operator from $H^2(\mathbb{T})$ to $L^2(\mathbb{T})$ defined as $K(e_{2n}) = e_n$, $K(e_{2n+1}) = e_{-n-1}$ for all $n \geq 0$.

We now extend the notion of Toep-Hank operator to $H^2(\beta)$ as follows.

Definition 2.4. Let $\phi \in L^\infty(\beta)$. A weighted Toep-Hank operator G_ϕ^β on $H^2(\beta)$ is given by $G_\phi^\beta = H_\phi^\beta\Lambda^\beta + T_{z\widetilde{\phi}}^\beta V^\beta$, where we define operators V^β and Λ^β on $H^2(\beta)$ as

$$V^\beta(e_n) = \begin{cases} e_{\frac{n-1}{2}} & \text{if n is odd,} \\ 0 & \text{if n is even} \end{cases} \quad \text{and} \quad \Lambda^\beta(e_n) = \begin{cases} e_{\frac{n}{2}} & \text{if n is even,} \\ 0 & \text{if n is odd} \end{cases}$$

for $n \geq 0$.

Clearly, $\|G_\phi^\beta\| \leq 2\|\phi\|_\infty$. It is trivial to conclude that $G_\phi^\beta = 0$ for $\phi = 0$. The matrix of G_ϕ^β with respect to the orthonormal basis $\{e_n : n \geq 0\}$ of $H^2(\beta)$

is of the form

$$\begin{bmatrix} a_0\frac{\beta_0}{\beta_0} & a_1\frac{\beta_0}{\beta_0} & a_{-1}\frac{\beta_0}{\beta_1} & a_2\frac{\beta_0}{\beta_1} & a_{-2}\frac{\beta_0}{\beta_2} & a_3\frac{\beta_0}{\beta_2} & a_{-3}\frac{\beta_0}{\beta_3} & \cdots \\ a_{-1}\frac{\beta_1}{\beta_0} & a_0\frac{\beta_1}{\beta_0} & a_{-2}\frac{\beta_1}{\beta_1} & a_1\frac{\beta_1}{\beta_1} & a_{-3}\frac{\beta_1}{\beta_2} & a_2\frac{\beta_1}{\beta_2} & a_{-4}\frac{\beta_1}{\beta_3} & \cdots \\ a_{-2}\frac{\beta_2}{\beta_0} & a_{-1}\frac{\beta_2}{\beta_0} & a_{-3}\frac{\beta_2}{\beta_1} & a_0\frac{\beta_2}{\beta_1} & a_{-4}\frac{\beta_2}{\beta_2} & a_1\frac{\beta_2}{\beta_2} & a_{-5}\frac{\beta_2}{\beta_3} & \cdots \\ a_{-3}\frac{\beta_3}{\beta_0} & a_{-2}\frac{\beta_3}{\beta_0} & a_{-4}\frac{\beta_3}{\beta_1} & a_{-1}\frac{\beta_3}{\beta_1} & a_{-5}\frac{\beta_3}{\beta_2} & a_0\frac{\beta_3}{\beta_2} & a_{-6}\frac{\beta_3}{\beta_3} & \cdots \\ a_{-4}\frac{\beta_4}{\beta_0} & a_{-3}\frac{\beta_4}{\beta_0} & a_{-5}\frac{\beta_4}{\beta_1} & a_{-2}\frac{\beta_4}{\beta_1} & a_{-6}\frac{\beta_4}{\beta_2} & a_{-1}\frac{\beta_4}{\beta_2} & a_{-7}\frac{\beta_4}{\beta_3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

As is observed in the case of Toep-Hank operators, the matrix of weighted Toep-Hank operator G_ϕ^β *provides the matrix of weighted Hankel operator* H_ϕ^β *if only even columns are considered and the matrix of weighted Toeplitz operator* $T_{z\widetilde{\phi}}^\beta$ *if only odd columns are considered. Further, if* $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ *is the Fourier expansion of* ϕ *and* $\{\alpha_{i,j}\}_{i,j\geq 0}$ *denotes the matrix of the operator* G_ϕ^β*, then the* $(i,j)^{th}$ *entry is given by* $\langle \alpha_{i,j}\rangle = \langle a_{-i-n}\frac{\beta_i}{\beta_n}\rangle$*, if* $j = 2n$ *and* $\langle \alpha_{i,j}\rangle = \langle a_{-i+n+1}\frac{\beta_i}{\beta_n}\rangle$*, if* $j = 2n+1$*,* $n \geq 0$*. Clearly,* $\{\alpha_{i,j}\}_{i,j\geq 0}$ *satisfies the following relations:*

$$\tag{2.1} \begin{cases} \frac{\beta_{j-1}}{\beta_{k+j}}\alpha_{k+j,2j-1} = \frac{\beta_0}{\beta_k}\alpha_{k,0} & \text{for } k \geq 0,\ j \geq 1, \\ \frac{\beta_j+1}{\beta_{i-1}}\alpha_{i-1,2j+2} = \frac{\beta_j}{\beta_i}\alpha_{i,2j} = \frac{\beta_0}{\beta_i}\alpha_{i,0} & \text{for } i \geq 1,\ j \geq 0, \\ \frac{\beta_{k+j}}{\beta_k}\alpha_{k,2k+2j+1} = \frac{\beta_j}{\beta_0}\alpha_{0,2j+1} & \text{for } k \geq 0,\ j \geq 0. \end{cases}$$

In [10], Zorboska discussed the notion of composition operator C_ϕ^β*, with the symbol* ϕ *(non constant analytic), defined on* $H^2(\beta)$ *to* $H^2(\beta)$ *as* $(C_\phi^\beta f)(z) = f(\phi(z))$*, for all* f *in* $H^2(\beta)$*. It is evident from here that for a bounded sequence* $\beta = \{\beta_n\}_{n\geq 0}$*, the composition operator* $C_{z^2}^\beta$ *is a bounded operator on* $H^2(\beta)$*. Clearly, if* $\beta_n = 1$ *for each* n*, then the operator* $C_{z^2}^\beta$ *coincides with the composition operator* C_{z^2} *on* $H^2(\mathbb{T})$*. Further, it is proved in [2] that* AC_{z^2} *is a Hankel operator for every Toep-Hank operator* A *on* $H^2(\mathbb{T})$*. However, we will see that this is not the situation in case of a weighted Toep-Hank operator.*

An infinite matrix $\{\gamma_{i,j}\}_{i,j\geq 0}$ *is called a weighted Hankel matrix [4] with respect to a semi-dual sequence* $\beta = \{\beta_n\}_{n\geq 0}$ *if* $\frac{\beta_j}{\beta_i}\gamma_{i,j} = \frac{\beta_{j+1}}{\beta_{i-1}}\gamma_{i-1,j+1}$ *for each* $i > 0, j \geq 0$*. Under the additional assumption of* $\{\beta_n\}_{n\in\mathbb{Z}}$ *being bounded, it is shown in [4, Theorem 2.10] that an operator on* $H^2(\beta)$ *is weighted Hankel operator if and only if its matrix is a weighted Hankel matrix. We show through next example that for a weighted Toep-Hank operator* A *on* $H^2(\beta)$*,* $AC_{z^2}^\beta$ *need not be a weighted Hankel operator.*

Example 2.5. Let $\phi(z) = 2z^{-2}$ and $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ be defined as

$$\beta_n = \begin{cases} 1 & \text{if } n = 0, 1, -1 \\ 2 & \text{otherwise} \end{cases}.$$

Then $\{\beta_n\}$ is a bounded semi-dual sequence satisfying $\frac{1}{2} \leq \frac{\beta_n}{\beta_{n+1}} \leq 1$ for $n \geq 0$ and $\phi \in L^\infty(\beta)$. Consider the weighted Toep-Hank operator A $(= G_\phi^\beta)$ on $H^2(\beta)$. Let $\{\alpha_{i,j}\}_{i,j\geq 0}$ and $\{\gamma_{i,j}\}_{i,j\geq 0}$ be the matrices of A and $AC_{z^2}^\beta$, respectively with respect to the usual basis of $H^2(\beta)$. Then $\{\alpha_{i,j}\}_{i,j\geq 0}$ satisfies the relation (2.1). But for $i \geq 1, j \geq 0$,

$$\begin{aligned}\frac{\beta_{j+1}}{\beta_{i-1}}\gamma_{i-1,j+1} &= \frac{\beta_{j+1}}{\beta_{i-1}}\langle AC_{z^2}^\beta e_{j+1}, e_{i-1}\rangle = \frac{\beta_{j+1}}{\beta_{i-1}}\langle \frac{\beta_{2j+2}}{\beta_{j+1}} Ae_{2j+2}, e_{i-1}\rangle \\ &= \frac{\beta_{2j+2}}{\beta_{i-1}}\alpha_{i-1,2j+2} = \frac{\beta_{2j+2}}{\beta_{j+1}}\frac{\beta_j}{\beta_i}\alpha_{i,2j}\end{aligned}$$

and $\frac{\beta_j}{\beta_i}\gamma_{i,j} = \frac{\beta_{2j}}{\beta_i}\alpha_{i,2j}$. In particular, for $i = 1, j = 1$, we find that $\frac{\beta_{j+1}}{\beta_{i-1}}\gamma_{i-1,j+1} = 2 \neq 4 = \frac{\beta_j}{\beta_i}\gamma_{i,j}$. Thus, $AC_{z^2}^\beta$ can not be a weighted Hankel operator.

In order to derive a weighted Hankel operator from a given weighted Toep-Hank operator, we proceed to define the following operator.

Definition 2.6. For $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n \in H^2(\beta)$, an operator $\hat{C}_{z^2}^\beta$ from $H^2(\beta)$ to $H^2(\beta)$ is defined as $\hat{C}_{z^2}^\beta(f(z)) = \sum\limits_{n=0}^{\infty} a_n \frac{\beta_n}{\beta_{2n}} z^{2n}$.

$\hat{C}_{z^2}^\beta$ is a bounded linear operator on $H^2(\beta)$ with norm 1. Further, $\hat{C}_{z^2}^\beta(e_n) = \hat{C}_{z^2}^\beta(\frac{z^n}{\beta_n}) = \frac{z^{2n}}{\beta_{2n}} = e_{2n}$ for each $n \geq 0$. The following result is now immediate.

Proposition 2.7. *Let $\beta = \{\beta_n\}_{n\geq 0}$ be bounded. If matrix of any bounded linear operator A defined on $H^2(\beta)$ is a weighted Toep-Hank matrix, then $A\hat{C}_{z^2}^\beta$ is a weighted Hankel operator on $H^2(\beta)$.*

It is worth noticing that weighted Hankel and weighted Toeplitz operators are linear with respect to their symbols. Thus the class of all weighted Toep-Hank operators on $H^2(\beta)$ is a linear subspace of $\mathfrak{B}(H^2(\beta))$, the space of all bounded operators on $H^2(\beta)$. Furthermore, the correspondence $\phi \to G_\phi^\beta$ is an injective linear mapping from $L^\infty(\beta)$ into $\mathfrak{B}(H^2(\beta))$.

Now one can easily see that the adjoint $G_\phi^{\beta^*}$ of a weighted Toep-Hank operator G_ϕ^β is nothing but an operator on $H^2(\beta)$ satisfying $G_\phi^{\beta^*} = \Lambda^{\beta^*} H_\phi^{\beta^*} + V^{\beta^*} T_{z\tilde{\phi}}^{\beta^*}$, where Λ^{β^*} and V^{β^*} on $H^2(\beta)$ are defined as $\Lambda^{\beta^*}(e_n) = e_{2n}$ and $V^{\beta^*}(e_n) = e_{2n+1}$ for $n \geq 0$. The operators Λ^β and V^β also satisfies $\Lambda^\beta\Lambda^{\beta^*} = I$, $V^\beta V^{\beta^*} = I$, $V^\beta\Lambda^{\beta^*} = 0$ and $\Lambda^\beta V^{\beta^*} = 0$.

In [5], it is shown that if the sequence $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ is such that $\{\frac{\beta_{2n}}{\beta_n}\}_{n\in\mathbb{Z}}$ is bounded then $\phi(z^2) \in L^\infty(\beta)$ for each $\phi \in L^\infty(\beta)$. We now proceed ahead to discuss the product of weighted Toep-Hank operators with the weighted Toeplitz operators on the space $H^2(\beta)$ for some specific symbols. We begin with the following result.

Proposition 2.8. *Let $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ be a sequence such that $\{\frac{\beta_{2n}}{\beta_n}\}_{n\in\mathbb{Z}}$ is bounded. Then for each $\phi \in L^\infty(\beta)$ of the form $\phi(z) = \sum\limits_{n\in\mathbb{Z}} a_n z^n$ with $a_p \neq 0$ for at least one positive integer p, we have the following:*

1. *$\Lambda^\beta T^\beta_{\phi(z^2)} = T^\beta_\phi \Lambda^\beta$ if and only if there exists a positive real number $a \geq 1$ such that $\{\beta_n\}_{n\in\mathbb{Z}}$ is given by*
$$\beta_n = \begin{cases} 1 & if\ n = 0 \\ a & otherwise \end{cases}.$$
2. *A necessary condition for the operator equation $V^\beta T^\beta_{\phi(z^2)} = T^\beta_\phi V^\beta$ to hold is that the sequence $\{\frac{\beta_{2np+1}}{\beta_{np}}\}_{n\geq 1}$ is constant with each term equal to β_1.*
3. *For $0 \neq \psi \in L^\infty(\beta)$ and $\phi \in H^\infty(\beta)$, $G^\beta_\psi \Lambda^{\beta^*} \Lambda^\beta T^\beta_{\phi(z^2)} = H^\beta_{\phi\psi}\Lambda^\beta$ if $\beta_n = a$ for each $n \neq 0$ and for some positive real number $a \geq 1$.*

Proof. Let $\Lambda^\beta T^\beta_{\phi(z^2)} = T^\beta_\phi \Lambda^\beta$ for the above ϕ. Hence, for $k \geq 0$,
$$\Lambda^\beta T^\beta_{\phi(z^2)} e_{2k}(z) = T^\beta_\phi \Lambda^\beta e_{2k}(z),$$
which yields that
$$\frac{1}{\beta_{2k}}\Big(\sum_{n=0}^{\infty} a_n \beta_{2k+2n} e_{k+n}\Big) = \frac{1}{\beta_k}\Big(\sum_{n=0}^{\infty} a_n \beta_{k+n} e_{k+n}\Big) \tag{2.2}$$
for each $k \geq 0$. On comparing the coefficients of e_{n+k} both sides of equation (2.2) for $n = p$ and $k = mp$ for $m \geq 0$, we get that $\beta_{(m+1)p} = \beta_{2(m+1)p}$ for $m \geq 0$. As a consequence, we have $\beta_n = \beta_p$ for each $n \geq p$. Similarly, on comparing the coefficients by setting $n = p$ and taking the values $k = 1, 2, 3, \cdots, p-1$ successively, we obtain that
$$\beta_1 = \beta_2 = \beta_4 = \cdots = \beta_{p-1} = \beta_{2p-2}.$$
Therefore, $\beta_n = \beta_1$ for $n \geq 1$ and $\beta_1 \geq 1$.

Converse follows immediately as for each $n \geq 0$,
$$\begin{aligned}
\Lambda^\beta T^\beta_{\phi(z^2)} e_n &= \begin{cases} \frac{1}{\beta_{2k}}\big(\sum\limits_{m=0}^{\infty} a_m \beta_{2k+2m} e_{k+m}\big) & if\ n = 2k, k \geq 0 \\ 0 & if\ n = 2k+1, k \geq 0 \end{cases} \\
&= \begin{cases} \frac{1}{\beta_k}\big(\sum\limits_{m=0}^{\infty} a_m \beta_{k+m} e_{k+m}\big) & if\ n = 2k, k \geq 0 \\ 0 & if\ n = 2k+1, k \geq 0 \end{cases} \\
&= T^\beta_\phi \Lambda^\beta e_n.
\end{aligned}$$
This completes the proof of (1).

For (2), suppose $V^{\beta}T^{\beta}_{\phi(z^2)} = T^{\beta}_{\phi}V^{\beta}$. This provides that for each $k \geq 0$, $V^{\beta}T^{\beta}_{\phi(z^2)}e_{2k+1}(z) = T^{\beta}_{\phi}V^{\beta}e_{2k+1}(z)$. As a consequence, for each $k \geq 0$

$$\frac{1}{\beta_{2k+1}}\Big(\sum_{n=0}^{\infty} a_n\beta_{2k+2n+1}e_{k+n}\Big) = \frac{1}{\beta_k}\Big(\sum_{n=0}^{\infty} a_n\beta_{k+n}e_{k+n}\Big). \tag{2.3}$$

On setting $n = p$ and applying equation (2.3) successively for $k = mp$ for $m \geq 0$, we get the result.

Proof of (3) follows using (1) and the facts that $\Lambda^{\beta}\Lambda^{\beta^*} = I$ and $V^{\beta}\Lambda^{\beta^*} = 0$. This completes the proof. $\square$

However, we find that the condition $\frac{\beta_{2np+1}}{\beta_{np}} = \beta_1$ *for each* $n \geq 1$*, in the Proposition 2.8 is only necessary but not sufficient. For,* $\phi(z) = z^2$ *and the semi-dual sequence* $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ *defined as*

$$\beta_n = \begin{cases} 1 & if\ n = 0, 1, -1 \\ 2 & otherwise \end{cases},$$

we have $\frac{\beta_{4n+1}}{\beta_{2n}} = 1 = \beta_1$ *for each* $n \geq 1$*. Although,* $V^{\beta}T^{\beta}_{\phi(z^2)}e_3 = e_3 \neq 2e_3 = T^{\beta}_{\phi}V^{\beta}e_3$*.*

It is known [3, Theorem 4.2] that for the symbols $\phi, \psi \in L^{\infty}(\beta)$*,* $T^{\beta}_{\widetilde{\psi}}H^{\beta}_{\phi} = H^{\beta}_{\phi}T^{\beta}_{\psi}$ *if and only if* $\psi \in H^{\infty}(\beta)$*. Further it is proved [1] that* $T^{\beta}_{\phi}T^{\beta}_{\psi} = T^{\beta}_{\phi\psi}$ *if* ψ *is analytic. In accordance with these observations, our next result calculates the product of a weighted Toep-Hank operator* G^{β}_{ϕ} *with* $T^{\beta}_{\widetilde{\psi}}$ *on* $H^2(\beta)$*, which we state without proof.*

Proposition 2.9. *Let* $\phi(z) = \sum_{n=-\infty}^{1} a_n z^n \in L^{\infty}(\beta)$*. Then* $T^{\beta}_{\widetilde{\psi}}G^{\beta}_{\phi} = G^{\beta}_{\phi\psi}$ *for each* $\psi \in H^{\infty}(\beta)$*. In particular,* $T^{\beta}_{z^{-n}}G^{\beta}_{\phi} = G^{\beta}_{z^n\phi}$ *for each* $n \geq 0$*.*

Recall that for a semi-dual sequence $\{\beta_n\}_{n\in\mathbb{Z}}$*, all the functions* $\widetilde{\phi}$*,* $\phi + \widetilde{\phi}$ *and* $\phi\widetilde{\phi}$ *belong to* $L^{\infty}(\beta)$ *provided* $\phi \in L^{\infty}(\beta)$*. In [3], commutativity of the weighted Hankel operator* $H^{\beta}_{z\phi}$ *with the weighted Toeplitz operator* T^{β}_{ϕ} *has been established for those symbols* $\phi \in L^{\infty}(\beta)$ *such that* $\phi + \widetilde{\phi}$ *and* $\phi\widetilde{\phi}$ *are constants. Using this fact, the following can be easily attained.*

Proposition 2.10. *Let* $\phi \in L^{\infty}(\beta)$ *be such that* $\phi + \widetilde{\phi}$ *and* $\phi\widetilde{\phi}$ *are constants. Then*

1. $T^{\beta}_{\phi}G^{\beta}_{z\phi} = H^{\beta}_{z\phi^2}\Lambda^{\beta} + T^{\beta}_{\phi}T^{\beta}_{\widetilde{\phi}}V^{\beta}$ *if* $\phi \in H^{\infty}(\beta)$*.*

2. $T^{\beta}_{\phi}G^{\beta}_{z\phi} = H^{\beta}_{z\phi}T^{\beta}_{\phi}\Lambda^{\beta} + T^{\beta}_{\phi\widetilde{\phi}}V^{\beta}$ *if* $\widetilde{\phi} \in H^{\infty}(\beta)$*.*

It is known from [3] that there does not exist any $\phi \in L^{\infty}(\beta)$ *inducing isometric weighted Hankel operators* H^{β}_{ϕ}*. In the next result, we see that the weighted Toep-Hank operator* G^{β}_{ϕ}*,* $\phi \in L^{\infty}(\beta)$*, fails to become an isometry.*

Theorem 2.11. *A weighted Toep-Hank operator on $H^2(\beta)$ cannot be an isometry.*

Proof. Suppose that a weighted Toep-Hank operator G^β_ϕ on $H^2(\beta)$, where $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n \in L^\infty(\beta)$, is an isometry. Then for each $j \geq 0$, $\|G^\beta_\phi e_{2j}\|^2 = \|H^\beta_\phi e_j\|^2 = \frac{1}{\beta_j^2} \sum_{n=0}^{\infty} |a_{-n-j}|^2 {\beta_n}^2 = 1$, which implies that

$$\sum_{n=0}^{\infty} |a_{-n-j}|^2 {\beta_n}^2 = \beta_j^2. \tag{2.4}$$

For $j = 0$, equation (2.4) means that $\sum_{n=0}^{\infty} |a_{-n}|^2 {\beta_n}^2 = 1$ and hence we have

$$1 \leq {\beta_j}^2 = \sum_{n=0}^{\infty} |a_{-n-j}|^2 {\beta_n}^2 \leq \sum_{n=0}^{\infty} |a_{-n}|^2 {\beta_n}^2 = 1$$

for each $j \geq 1$. This yields that $\beta_n = 1$ for all n.

Now on replacing j by $j+1$ in equation (2.4) and then on subtracting it from equation (2.4), we obtain that $a_{-n-j} = 0$ for each $n, j \geq 0$. This implies that $\|G^\beta_\phi e_{2j}\|^2 = 0$. This contradicts our assumption. Hence, G^β_ϕ can not be an isometry. □

Now, we discuss the isometric behavior of the adjoint of weighted Toep-Hank operators and obtain a necessary condition for such operators when induced by a specific symbol. Almost along the same arguments as in the above theorem, we can prove the following.

Theorem 2.12. *A necessary condition for the adjoint of a weighted Toep-Hank operator, induced by the symbol $\phi(z) = \sum_{i=-\infty}^{-1} a_i z^i \in L^\infty(\beta)$, to be an isometry is that $\beta_n = 1$ for each $n \in \mathbb{Z}$ and $\sum_{i=-\infty}^{-1} |a_i|^2 = 1$.*

Proof. Proof can be obtained using the inequalities $\frac{1}{\beta_j^2} \leq 1$ and $\frac{1}{\beta_{j+1}^2} \leq \frac{1}{\beta_j^2}$ for each $j \geq 1$. □

The conditions obtained in above theorem are not sufficient for the adjoint of the given weighted Toep-Hank operator to be an isometry and this can be justified through the following example.

Example 2.13. Consider the sequence $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ such that $\beta_n = 1$ for each n. Let $\phi(z) = \frac{1}{\sqrt{2}} z^{-1} + \frac{1}{\sqrt{2}} z^{-2}$, where $(\frac{1}{\sqrt{2}})^2 + (\frac{1}{\sqrt{2}})^2 = 1$. Then $\phi \in L^\infty(\beta)$ as $\|\phi f\|_\beta \leq \sqrt{2}\|f\|_\beta$. But for $f(z) = 2e_0 + 3e_1 \in H^2(\beta)$, we have $\|G^{\beta^*}_\phi f\|^2 = 19 \neq 13 = \|f\|^2$.

In [2, Theorem 2.5], it is proved that ${G_\phi}^*$ is an isometry on $H^2(\mathbb{T})$ if and only if $\widetilde{\phi}\phi^* = 1$. If we take the case of $\beta_n = 1$ for each $n \in \mathbb{Z}$, then the weighted Toep-Hank operator G_ϕ^β on $H^2(\beta)$ becomes the Toep-Hank operator G_ϕ on $H^2(\mathbb{T})$. The above mentioned ϕ satisfies $\widetilde{\phi}\phi^* = 1 + \frac{1}{2}z + \frac{1}{2}z^{-1} \neq 1$. As a consequence, $G_\phi^{\beta^*}$ can't be an isometry.

3. Compact, Hyponormal and Hilbert-Schmidt Operators

This section is devoted to study some basic structural properties of the weighted Toep-Hank operators on $H^2(\beta)$. It is also proved that for $G_\phi^{\beta^*}$, $\phi(z) = \sum\limits_{n=-p}^{-1} a_n z^n \in L^\infty(\beta)$ with $a_{-p} \neq 0$ for $p \geq 1$, to be hyponormal, we have $\beta_n = 1$ for $0 \leq n \leq 2p$. We, however, see in the next result that the only Hilbert-Schmidt weighted Toep-Hank operator is the zero operator.

Theorem 3.1. *G_ϕ^β is a Hilbert-Schmidt operator if and only if $\phi = 0$.*

Proof. Let G_ϕ^β be a Hilbert-Schmidt operator, where $\phi = \sum\limits_{i=-\infty}^{\infty} a_i z^i \in L^\infty(\beta)$. Then

$$\begin{aligned}
\sum_{n=0}^{\infty} \|G_\phi^\beta e_n\|^2 &= \sum_{n=0}^{\infty} \langle G_\phi^\beta e_n, G_\phi^\beta e_n \rangle \\
&= \sum_{n=0}^{\infty} \langle G_\phi^\beta e_{2n}, G_\phi^\beta e_{2n} \rangle + \sum_{n=0}^{\infty} \langle G_\phi^\beta e_{2n+1}, G_\phi^\beta e_{2n+1} \rangle \\
&= \sum_{n=0}^{\infty} \langle H_\phi^\beta e_n, H_\phi^\beta e_n \rangle + \sum_{n=0}^{\infty} \langle T_{z\widetilde{\phi}}^\beta e_n, T_{z\widetilde{\phi}}^\beta e_n \rangle \\
&= \sum_{n=0}^{\infty} \frac{1}{\beta_n^2} \big(\sum_{i=0}^{\infty} |a_{-i-n}|^2 \beta_i^2 \big) + \sum_{n=0}^{\infty} \frac{1}{\beta_n^2} \big(\sum_{i=0}^{\infty} |a_{-i+n+1}|^2 \beta_i^2 \big) \\
&= \sum_{n=0}^{\infty} \frac{1}{\beta_n^2} \big(\sum_{i=0}^{\infty} |a_{-i-n}|^2 \beta_i^2 \big) + \Big(\sum_{i=1}^{\infty} |a_i|^2 \big(\sum_{n=0}^{\infty} \frac{\beta_n^2}{\beta_{n+i-1}^2} \big) + \\
&\quad \sum_{i=-\infty}^{0} |a_i|^2 \big(\sum_{n=0}^{\infty} \frac{\beta_{n-i+1}^2}{\beta_n^2} \big) \Big).
\end{aligned}$$

As $\sum\limits_{n=0}^{\infty} \|G_\phi^\beta e_n\|^2$ is finite, we have $|a_i|^2 \big(\sum\limits_{n=0}^{\infty} \frac{\beta_n^2}{\beta_{n+i-1}^2} \big)$ is finite for each $i \geq 1$ and $|a_i|^2 \big(\sum\limits_{n=0}^{\infty} \frac{\beta_{n-i+1}^2}{\beta_n^2} \big)$ is finite for each $i \leq 0$. Now the former implies $a_i = 0$ for $n \geq 1$ because the series $\sum\limits_{n=0}^{\infty} \frac{\beta_n^2}{\beta_{n+i-1}^2}$ is divergent for each $i \geq 1$. The latter holds only if $a_i = 0$ for each $i \leq 0$ as each term of the series $\sum\limits_{n=0}^{\infty} \frac{\beta_{n-i+1}^2}{\beta_n^2}$ satisfies $\frac{\beta_{n-i+1}^2}{\beta_n^2} \geq 1$. Hence $\phi = 0$.

Converse follows evidently. $\square$

In [2], it has been proved that a Toep-Hank operator on the space $H^2(\mathbb{T})$ is compact if and only if $\phi = 0$. Along similar lines, we show that for the bounded sequences $\beta = \{\beta_n\}_{n\geq 0}$, there is a dearth of compact weighted Toep-Hank operators on $H^2(\beta)$. In fact, the only compact weighted Toep-Hank operator is the zero operator.

Theorem 3.2. *For bounded sequences $\{\beta_n\}_{n\geq 0}$, the operator G_ϕ^β on $H^2(\beta)$ is compact if and only if $\phi = 0$.*

Proof. Let G_ϕ^β be compact, where $\phi = \sum\limits_{i=-\infty}^{\infty} a_i z^i \in L^\infty(\beta)$. Since $e_n \to 0$ weakly, we have $\|G_\phi^\beta e_{2n+1}\|^2 = \sum\limits_{i=0}^{\infty} |a_{-i+n+1}|^2 \beta_i^2 \to 0$ as $n \to \infty$. It is easy to conclude from here that $|a_i| = 0$ for each $i \in \mathbb{Z}$. Hence, $\phi = 0$.

Nothing needs to be proved for the converse. $\square$

In the next result, we investigate the self-adjoint nature of weighted Toep-Hank operators induced by the symbols of the form $\phi(z) = \sum\limits_{n=-m}^{\infty} a_n z^n$ or $\phi(z) = \sum\limits_{n=-\infty}^{-m} a_n z^n$, where $a_n \in \mathbb{C}$ and $a_{-m} \neq 0$ for $m > 0$. For $\phi(z) = \sum\limits_{n=-m}^{\infty} a_n z^n \in L^\infty(\beta)$, we have

$$T_{z\tilde{\phi}}^\beta e_j = \frac{1}{\beta_j} \sum_{n=0}^{\infty} a_{-n+j+1} \beta_n e_n \text{ and } T_{z\tilde{\phi}}^{\beta\,*} e_j = \beta_j \sum_{n=0}^{\infty} \frac{\overline{a}_{n-j+1}}{\beta_n} e_n.$$

Theorem 3.3. *The weighted Toep-Hank operator G_ϕ^β on $H^2(\beta)$, induced by $\phi(z) = \sum\limits_{n=-m}^{\infty} a_n z^n$ or $\phi(z) = \sum\limits_{n=-\infty}^{-m} a_n z^n$, where $a_n \in \mathbb{C}$ and $a_{-m} \neq 0$ for $m > 0$, can not be self-adjoint.*

Proof. For $\phi(z) = \sum\limits_{n=-m}^{\infty} a_n z^n$, if we assume that G_ϕ^β is self-adjoint then we must have $G_\phi^\beta e_{2j} = G_\phi^{\beta^*} e_{2j}$ for each $j \geq 0$. This provides that

$$\frac{1}{\beta_j}\Big(\sum_{n=0}^{-j+m} a_{-n-j} \beta_n e_n \Big) = \beta_{2j} \Big(\sum_{n=0}^{-2j+m} \overline{a}_{-n-2j} \frac{e_{2n}}{\beta_n} + \sum_{n=2j-m-1}^{\infty} \overline{a}_{n-2j+1} \frac{e_{2n+1}}{\beta_n} \Big)$$

for $j \geq 0$. If $j = m$ it implies that $\frac{a_{-m}}{\beta_m} e_0 = \beta_{2m}\big(\sum\limits_{n=m-1}^{\infty} \overline{a}_{n-2m+1} \frac{e_{2n+1}}{\beta_n} \big)$, where the series on right side does not include an appearance of e_0. Hence $a_{-m} = 0$, which is absurd. Hence G_ϕ^β can not be self-adjoint.

Similarly, for $\phi(z) = \sum_{n=-\infty}^{-m} a_n z^n$, we can check that $G_\phi^\beta e_0 \neq G_\phi^{\beta^*} e_0$. Hence the result. □

Towards the end, we discuss the hyponormality and normality of weighted Toep-Hank operators for the symbol $\phi \in L^\infty(\beta)$ *of the form* $\phi(z) = \sum_{n=-m}^{\infty} a_n z^n$, $a_{-m} \neq 0$ *for* $m > 0$. *It is known that an operator* T *on* $H^2(\beta)$ *is hyponormal if* $\|Tf\|^2 \geq \|T^* f\|^2$ *for each* $f \in H^2(\beta)$. *The following can be obtained without any extra efforts.*

Theorem 3.4. *The weighted Toep-Hank operator* G_ϕ^β *on* $H^2(\beta)$, *induced by* $\phi(z) = \sum_{n=-m}^{\infty} a_n z^n \in L^\infty(\beta)$, $a_{-m} \neq 0$ *for* $m > 0$, *can not be hyponormal.*

Proof. If we assume that the symbol $\phi(z) = \sum_{n=-m}^{\infty} a_n z^n$, $m > 0$, is such that G_ϕ^β is hyponormal, then $\|G_\phi^\beta e_{2m+2}\|^2 \geq \|G_\phi^{\beta^*} e_{2m+2}\|^2$. This yields that

$$0 \geq |a_{-m}|^2 \frac{\beta_{2m+2}^2}{\beta_{m+1}^2} + |a_{-m+1}|^2 \frac{\beta_{2m+2}^2}{\beta_{m+2}^2} + |a_{-m+2}|^2 \frac{\beta_{2m+2}^2}{\beta_{m+3}^2} + \cdots ,$$

which is possible only if $a_i = 0$ for each $i \geq -m$. This implies $a_{-m} = 0$ which is absurd. This completes the proof. □

Every normal operator is hyponormal so the above theorem leads to the following.

Corollary 3.5. *No weighted Toep-Hank operator* G_ϕ^β *on the weighted Hardy space* $H^2(\beta)$, *for* $\phi(z) = \sum_{n=-m}^{\infty} a_n z^n \in L^\infty(\beta)$, $a_{-m} \neq 0$ *for* $m > 0$, *is normal.*

From Corollary 3.5, we can also conclude that the trigonometric polynomials of the form $\phi(z) = \sum_{n=-m}^{l} a_n z^n$ *with* $a_{-m}, a_l \neq 0$, *can never induce a normal weighted Toep-Hank operator on* $H^2(\beta)$.

Through our next result, we discuss the hyponormality of the adjoint of a weighted Toep-Hank operator on $H^2(\beta)$, *induced by the symbol*

$$\phi(z) = \sum_{n=-p}^{-1} a_n z^n \in L^\infty(\beta)$$

and obtain the following.

Theorem 3.6. *A necessary condition for the adjoint of a weighted Toep-Hank operator, induced by the symbol* $\phi(z) = \sum_{n=-p}^{-1} a_n z^n \in L^\infty(\beta)$, $a_{-p} \neq 0$ *for* $p \geq 1$, *to be hyponormal is that* $\beta_n = 1$ *for* $0 \leq n \leq 2p$.

Proof. Let $G_\phi^{\beta^*}$ be hyponormal. Then for all $j \geq 0$, $\|G_\phi^{\beta^*} e_{2j}\|^2 \geq \|G_\phi^\beta e_{2j}\|^2$. For $j = 0$, this gives that

$$|a_{-1}|^2\Big(\frac{1}{\beta_1^2} - \beta_1^2\Big) + |a_{-2}|^2\Big(\frac{1}{\beta_2^2} - \beta_2^2\Big) + \cdots + |a_{-p}|^2\Big(\frac{1}{\beta_p^2} - \beta_p^2\Big) \geq 0.$$

Since $\frac{1}{\beta_i^2} - \beta_i^2 \leq 0$ for $1 \leq i \leq p$, hence above inequality implies $\beta_n = 1$ for each $0 \leq n \leq p$. Now on applying $\|G_\phi^{\beta^*} e_{2j+1}\|^2 \geq \|G_\phi^\beta e_{2j+1}\|^2$ for $j = 0, 1, 2, \cdots, p-1$ successively to conclude that $\beta_n = 1$ for each $p+1 \leq n \leq 2p$. This proves the result. □

Along the lines of computations in Theorem 3.6, one can immediately conclude the following.

Corollary 3.7. *If $\phi \in L^\infty(\beta)$ is such that $\phi(z) = \sum_{n=-\infty}^{-1} a_n z^n$, then a necessary condition for the operator $G_\phi^{\beta^*}$ to be hyponormal is that $\beta_n = 1$ for each $n \in \mathbb{Z}$.*

The condition obtained in Theorem 3.6 is just necessary. It is not sufficient for the adjoint $G_\phi^{\beta^*}$ to be hyponormal. For, let $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ be a semi-dual sequence defined as

$$\beta_n = \begin{cases} 1 & if\ -2 \leq n \leq 2 \\ 2^{|n|} & otherwise \end{cases}$$

and let $\phi(z) = a_{-1}z^{-1} \in L^\infty(\beta)$. Here, $p = 1$. Then $\|G_\phi^{\beta^*} e_5\|^2 = |a_{-1}|^2 2^4 < |a_{-1}|^2 2^8 = \|G_\phi^\beta e_5\|^2$.

Example 3.8. Consider the space $L^\infty(\beta)$, where the sequence $\beta = \{\beta_n\}$ is given by

$$\beta_n = \begin{cases} 1 & if\ n = 0, 1, -1 \\ 2 & otherwise \end{cases}.$$

Let $\phi(z) = 2z^{-2} + z^{-1}$. Then, $\phi \in L^\infty(\beta)$. Consider $G_\phi^{\beta^*}$, the adjoint of a weighted Toep-Hank operator induced by the above defined ϕ. Then, it is clearly evident from Theorems 2.12 and 3.6 that the operator $G_\phi^{\beta^*}$ is neither an isometry (as $\beta_n \neq 1$ for each $n \geq 2$) nor a hyponormal operator (as $\beta_n \neq 1$ for $2 \leq n \leq 4$).

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