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A NOTE ON A GENERALIZED SHISHKIN-TYPE MESH

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Abstract. The one-dimensional linear singularly perturbed convection-diffusion problem is discretized using the upwind scheme on a mesh which is a mild generalization of Shishkin-type meshes. The generalized mesh uses the transition point of the Shishkin mesh, but it does not require any structure of its fine and coarse parts. Convergence uniform in the perturbation parameter is proved by the barrier-function technique, which, because of the unstructured mesh, does not rely on any mesh-generating function. In this way, the technical requirements needed in the existing barrier-function approaches are simplified.

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1. Introduction

Let us consider a linear singularly perturbed convection-diffusion problem in one dimension,

$$\mathcal{L}u := -\varepsilon u'' - b(x)u' + c(x)u = f(x),\ x \in (0,1),\ u(0) = u(1) = 0, \tag{1.1}$$

with a small positive perturbation parameter ε, and $C^1[0,1]$-functions b, c, and f, where b and c satisfy

$$b(x) > \beta > 0, \quad c(x) \geq 0 \ \text{ for } x \in I := [0,1].$$

We are interested in a numerical method which yields an ε-uniform pointwise approximation to the $C^3(I)$-solution u of the problem (1.1). In order to obtain ε-uniform pointwise convergence of the numerical solution, the most commonly used numerical method is to apply an appropriate finite-difference scheme on layer-adapted meshes which are condensed in the layer region near $x = 0$. The most studied meshes of this type are Bakhvalov and Shishkin meshes. The latter has gained much attention because of its simplicity. In [9],

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Roos and Linß introduce a class of *Shishkin-type* meshes which can be summarized as follows.

Let I^N denote an arbitrary mesh with mesh points x_i, $i = 0, 1, \dots, N$, such that $0 = x_0 < x_1 < \cdots < x_N = 1$. The Shishkin-type meshes are characterized by the choice of a transition point, which is defined as

$$x_J = \sigma := \min\left\{q, \frac{2}{\beta}\varepsilon \ln N\right\}, \tag{1.2}$$

where $J := qN$ is a positive integer and q is fixed in $(0, 1)$, $q = 1/2$ being a frequent choice. The mesh is fine and may be graded in the interval $[0, \sigma]$, whereas it is coarse and equally spaced in $[\sigma, 1]$. The mesh in the fine region $[0, \sigma]$ is defined by $x_i = \frac{2}{\beta}\varepsilon\phi(i/N)$, $i = 0, 1, \dots, J$, where ϕ is a mesh generating function satisfying $\phi' > 0$.

The consistency error in the layer region is not convergent uniformly in ε (see, e.g., [8, 15]) and this is why the techniques required to prove ε-uniform convergence for finite-difference discretizations on Shishkin-type meshes are not simple. Special methods are devised, including the use of the hybrid stability, presented in [1, 4, 14], or of discrete barrier functions, see [9, 12] and also the monographs [2, 3, 5, 11]. In particular, the ε-uniform convergence proofs using barrier functions on Shishkin-type meshes rely on some technical conditions on the mesh-generating function ϕ. Even when these conditions are not met, it is expected that the same ε-uniform convergence result is true (cf. [9, Remark 3]) because other techniques can be applied, like the hybrid-stability technique of Andreev and Savin [1], or the preconditioning approach of Vulanović and Nhan [8, 7, 6, 15].

Proofs of the pointwise ε-uniform accuracy of the numerical solution obtained on a Shishkin-type mesh are often accompanied with the proof that the corresponding linear interpolant preserves the accuracy of the numerical solution, [2, 3, 5, 9, 16]. We show in the present paper that the need for interpolation, when it is used to approximate the value of the solution at a point other than one of the original mesh points, can practically be eliminated. This is so because Shishkin-type meshes can be generalized by only keeping the transition point (1.2) and not requiring any special structure of the mesh points in the fine and coarse parts of the mesh. Therefore, if it is known in advance that the value of the solution needs to be approximated at a particular point, this point can be easily made a point of the mesh, which makes interpolation unnecessary. Because the structure of the fine and coarse parts of the mesh is relaxed, we have to abandon the assumption that the mesh is generated by some function ϕ. Nevertheless, we prove ε-uniform convergence using the barrier-function technique. In this way, we introduce a new and simpler error analysis, as compared to that of [9], thus making the barrier-function approach equal, with respect to the technical requirements, to the approaches in [1] and [6].

The paper is organized as follows. We introduce the relaxed Shishkin-type meshes in the next section, where we also present the upwind discretization scheme for the problem. Then, we prove ε-uniform convergence in Section 3. Finally, several concluding remarks are made in Section 4.

2. Generalized Shishkin-type mesh and discretization

We first introduce a generalization of the Shishkin-type meshes described in the introduction. Referring to the mesh I^N from the introduction, we define the mesh steps $h_i = x_i - x_{i-1}$, $i = 1, 2, \ldots, N$. Like in the standard Shishkin-type meshes, we use the point $x_J = \sigma$, given in (1.2), as the transition point between the fine and coarse parts of the mesh. For simplicity, we consider

$$\sigma = \frac{2}{\beta}\varepsilon \ln N, \tag{2.1}$$

because N is unreasonably large otherwise.

As opposed to the standard Shishkin-type meshes, we only require that

$$h := \max_{1\le i\le J} h_i \le C\sigma N^{-1} \tag{2.2}$$

and

$$H := \max_{J+1\le i\le N} h_i \le CN^{-1}. \tag{2.3}$$

Here and throughout the paper, C denotes a generic positive constant which is independent of both ε and N.

Therefore, we do not assume any special structure of the fine mesh in $[0, \sigma]$ and the coarse mesh in $[\sigma, 1]$, other than the fulfillment of the natural conditions (2.2) and (2.3). It is easy to create a mesh satisfying (2.2) and (2.3) and containing any point of the interval I. This, as stated in the introduction, can eliminate the need for interpolating the numerical results.

Remark 2.1. The improved modifications of Shishkin's transition point due to Vulanović [13, 14] can be applied here. For example, in the spirit of [13], the factor $\ln N$ in (2.1) can be replaced with a more general quantity $L = L(N)$ which satisfies

$$e^{-L} \le \frac{L}{N}.$$

Nevertheless, for the simplicity of exposition, we consider Shishkin's original definition of the transition point.

Mesh functions on I^N are denoted by $W^N = \left(W_i^N\right)$, $U^N = \left(U_i^N\right)$, etc. If g is a function defined on I, we write g_i instead of $g(x_i)$ and g^N for the corresponding mesh function.

We discretize the problem (1.1) on I^N using the upwind finite-difference scheme:

$$U_0^N = 0,$$

$$\mathcal{L}^N U_i^N := -\varepsilon D'' U_i^N - b_i D^+ U_i^N + c_i U_i^N = f_i, \quad i = 1, 2, \ldots, N-1, \tag{2.4}$$

$$U_N^N = 0,$$

where

$$D''W_i^N = \frac{1}{\hbar_i}\left(D^+W_i^N - D^-W_i^N\right), \quad \hbar_i = (h_i + h_{i+1})/2,$$

and

$$D^+W_i^N = \frac{W_{i+1}^N - W_i^N}{h_{i+1}}, \quad D^-W_i^N = \frac{W_i^N - W_{i-1}^N}{h_i}.$$

It is easy to see that the operator $\mathcal{L}^N$ satisfies the discrete maximum principle. Therefore, the discrete problem (2.4) has a unique solution U^N.

3. Uniform convergence

Before proceeding to the main result, we briefly recall the facts from [3, Theorem 3.48] about a decomposition of u:

$$u(x) = s(x) + y(x),$$

(3.1) $$|s^{(k)}(x)| \le C\left(1 + \varepsilon^{2-k}\right), \quad |y^{(k)}(x)| \le C\varepsilon^{-k}e^{-\beta x/\varepsilon},$$

$$x \in I, \quad k = 0, 1, 2, 3.$$

In addition, the layer component, y, satisfies a homogeneous differential equation,

(3.2) $$\mathcal{L}y(x) = 0, \quad x \in (0,1).$$

Theorem 3.1. *Let u be the solution of the continuous problem* (1.1) *and let U^N be the solution of the discrete problem* (2.4) *on the generalized Shishkin mesh given in Section 2. Then the following error estimate is satisfied:*

(3.3) $$|u_i - U_i^N| \le CN^{-1}\ln N, \quad i = 0, 1, \dots, N.$$

Proof. As usual, the discrete solution U^N is split into $U^N = S^N + Y^N$, where S^N and Y^N are defined by

$$\begin{aligned} &\mathcal{L}^N S_i^N = (\mathcal{L}S)_i, && \text{for } i = 1, 2, \dots, N-1, \quad S_0^N = s_0,\ S_N^N = s_N,\\ &\mathcal{L}^N Y_i^N = 0, && \text{for } i = 1, 2, \dots, N-1, \quad Y_0^N = y_0,\ Y_N^N = y_N. \end{aligned}$$

Then for all i,

$$|u_i - U_i^N| \le |s_i - S_i^N| + |y_i - Y_i^N|,$$

so we can estimate the errors $|s_i - S_i^N|$ and $|y_i - Y_i^N|$ separately. For the regular part s, we use the standard truncation error, the estimates of $|s^{(k)}(x)|$ in (3.1), and the stability property of $\mathcal{L}^N$ to get that

(3.4) $$|s_i - S_i^N| \le C(h + H) \le CN^{-1}, \quad i = 0, 1, \dots, N.$$

We are left with the proof of

$$|y_i - Y_i^N| \le CN^{-1}\ln N, \quad i = 0, 1, \dots, N.$$

Let $\tau_i[y] = \mathcal{L}^N y_i - (\mathcal{L}y)_i$, $i = 1, 2, \dots, N-1$, be the consistency error of the layer component. Then we have

$$|\tau_i[y]| \le 2\varepsilon \int_{x_{i-1}}^{x_{i+1}} |y'''(x)|dx + b_i \int_{x_i}^{x_{i+1}} |y''(x)|dx. \tag{3.5}$$

The following estimate follows from (3.5) for $i = J+2, J+3, \dots, N-1$:

$$\begin{aligned} |\tau_i[y]| &\le CH\varepsilon^{-2}e^{-\beta x_{i-1}/\varepsilon} \le CH\varepsilon^{-2}e^{-\beta(\sigma+H)/\varepsilon} \\ &\le CN\left[(H\varepsilon^{-1})^2 e^{-\beta H/\varepsilon}\right] e^{-\beta\sigma/\varepsilon} \le CN^{-1}, \end{aligned}$$

where we have used (2.1) and the fact that $(H\varepsilon^{-1})^2 e^{-\beta H/\varepsilon} \le C$.

For $i = J, J+1$, we use (3.2) and an argument similar to the one in [13, Lemma 5] (see also [8, 15]) to bound $|\tau_i[y]|$ as follows:

$$|\tau_i[y]| \le P_i + Q_i + c_i|y_i|,$$

where $P_i = \varepsilon|D''y_i|$ and $Q_i = b_i|D'y_i|$. It is easy to see that

$$c_i|y_i| \le Ce^{-\beta x_i/\varepsilon} \le Ce^{-\beta\sigma/\varepsilon} \le CN^{-1}.$$

As for P_i and Q_i, we use the bounds

$$P_i \le C\hbar_i^{-1}e^{-\beta x_{i-1}/\varepsilon} \le CNe^{-\beta x_{J-1}/\varepsilon} \le CNe^{-\beta(\sigma-h_J)/\varepsilon} \le CN^{-1},$$

and

$$Q_i \le CH^{-1}e^{-\beta x_{i-1}/\varepsilon} \le CNe^{-\beta x_{J-1}/\varepsilon} \le CNe^{-\beta(\sigma-h_J)/\varepsilon} \le CN^{-1}.$$

The above estimates are true because the condition (2.2) yields

$$h_i \le C\varepsilon, \quad i = 1, 2, \dots, J, \tag{3.6}$$

which gives $e^{\beta h_J/\varepsilon} \le C$. Therefore, we have

$$|\tau_i[y]| \le CN^{-1}, \;\; i = J, J+1, \dots, N.$$

For the consistency error in the layer region, we again use (3.5) to get that this time

$$\begin{aligned} |\tau_i[y]| &\le Ch_{i+1}\varepsilon^{-2}e^{-\beta x_{i-1}/\varepsilon} \\ &\le C\varepsilon^{-1}N^{-1}(\ln N)e^{-\beta x_{i-1}/\varepsilon} \\ &\le C\varepsilon^{-1}N^{-1}(\ln N)e^{-\beta x_{i-1}/(2\varepsilon)} \\ &\le C\varepsilon^{-1}N^{-1}(\ln N)e^{-\beta x_i/(2\varepsilon)}e^{\beta h_i/(2\varepsilon)} \\ &\le C\varepsilon^{-1}N^{-1}(\ln N)e^{-\beta x_i/(2\varepsilon)}, \quad i = 1, 2, \dots, J-1, \end{aligned}$$

where we have used (3.6) in the second line from the bottom. We next note the inequality $e^t \geq 1 + t$, for all $t \geq 0$ and apply it to get

$$e^{-\beta x_i/(2\varepsilon)} = \prod_{j=1}^{i} \left(e^{-\beta h_j/(2\varepsilon)} \right) \leq \prod_{j=1}^{i} \left(1 + \frac{\beta h_j}{2\varepsilon} \right)^{-1} =: \bar{y}_i^N \tag{3.7}$$

for all $i = 0, 1, \ldots, N$. Therefore, we have the following truncation-error estimates:

$$|\tau_i[y]| \leq \begin{cases} C\varepsilon^{-1}\bar{y}_i^N N^{-1} \ln N, & i = 1, 2, \ldots, J-1, \\ CN^{-1}, & i = J, J+1, \ldots, N-1. \end{cases} \tag{3.8}$$

We now construct a barrier function $\{\gamma_i\}$ such that

$$\gamma_i = \gamma_i^{(1)} + \gamma_i^{(2)},$$

with

$$\gamma_i^{(1)} = C_1 N^{-1}(1 - x_i), \quad i = 0, 1, \ldots, N,$$

and

$$\gamma_i^{(2)} = C_2 \bar{y}_i^N N^{-1} \ln N, \quad i = 0, 1, \ldots, N,$$

where C_1 and C_2 are two appropriately chosen positive constants which are independent of both ε and N. It is obvious that

$$\gamma_i \leq CN^{-1} \ln N, \quad i = 0, 1, \ldots, N. \tag{3.9}$$

Since $\gamma_0 \geq y_0 - Y_0^N = 0$ and $\gamma_N \geq y_N - Y_N^N = 0$, we need to prove

$$\mathcal{L}^N \gamma_i \geq |\tau_i[y]|, \quad i = 1, 2, \ldots, N-1, \tag{3.10}$$

in order to get

$$|y_i - Y_i^N| \leq \gamma_i.$$

The assertion will then follow when the above inequality and (3.9) are combined with (3.4).

The rest of the proof is about showing that (3.10) is satisfied.

For $\gamma^{(1)}$, we have

$$D^+\gamma_i^{(1)} = D^-\gamma_i^{(1)} = -C_1 N^{-1}, \quad i = 1, 2, \ldots, N-1.$$

Hence, for $i = 1, 2, \ldots, N-1$,

$$\begin{aligned} \mathcal{L}^N \gamma_i^{(1)} &= \frac{-2\varepsilon}{h_i + h_{i+1}} \left(D^+\gamma_i^{(1)} - D^-\gamma_i^{(1)} \right) - b(x_i) D^+\gamma_i^{(1)} + c(x_i)\gamma_i^{(1)} \\ &= C_1 N^{-1} b(x_i) + c(x_i)\gamma_i^{(1)} \end{aligned}$$

and

$$\mathcal{L}^N \gamma_i^{(1)} \geq C_1 N^{-1} \beta. \tag{3.11}$$

As for $\gamma_i^{(2)}$, it is easy to see that

$$D^+\gamma_i^{(2)} = \frac{\frac{1}{1+\frac{\beta h_{i+1}}{2\varepsilon}} - 1}{h_{i+1}}\gamma_i^{(2)} = \frac{-\beta}{2\varepsilon + \beta h_{i+1}}\gamma_i^{(2)}, \quad i = 1, 2, \ldots, N-1,$$

$$D^-\gamma_i^{(2)} = \frac{1 - (1 + \frac{\beta h_i}{2\varepsilon})}{h_i}\gamma_i^{(2)} = \frac{-\beta}{2\varepsilon}\gamma_i^{(2)}, \quad i = 1, 2, \ldots, N-1,$$

and therefore,

$$D^+\gamma_i^{(2)} - D^-\gamma_i^{(2)} = \frac{\beta^2 h_{i+1}}{2\varepsilon(2\varepsilon + \beta h_{i+1})}\gamma_i^{(2)}, \quad i = 1, 2, \ldots, N-1.$$

We use the above to get

$$\begin{aligned}
\mathcal{L}^N\gamma_i^{(2)} &= -\frac{\varepsilon}{\hbar_i}\left(D^+\gamma_i^{(2)} - D^-\gamma_i^{(2)}\right) - b(x_i)D^+\gamma_i^{(2)} + c(x_i)\gamma_i^{(2)} \\
&\geq -\frac{\varepsilon}{\hbar_i}\frac{\beta^2 h_{i+1}}{2\varepsilon(2\varepsilon + \beta h_{i+1})}\gamma_i^{(2)} + \frac{\beta b(x_i)}{2\varepsilon + \beta h_{i+1}}\gamma_i^{(2)} \\
&\geq \frac{\beta}{2\varepsilon + \beta h_{i+1}}\left(b(x_i) - \beta\frac{h_{i+1}}{h_i + h_{i+1}}\right)\gamma_i^{(2)} \\
&\geq \frac{\beta}{2\varepsilon + \beta h_{i+1}}(b(x_i) - \beta)\gamma_i^{(2)} \\
&\geq \bar{C}[\max\{\varepsilon, h_{i+1}\}]^{-1}\gamma_i^{(2)}, \quad i = 1, 2, \ldots N-1,
\end{aligned}$$

where $\bar{C} = \beta\delta/(2+\beta)$ and $b(x) - \beta \geq \delta > 0$, $x \in I$. In particular, when $i = 1, 2, \ldots, J-1$, (3.6) implies that

$$\mathcal{L}^N\gamma_i^{(2)} \geq \varepsilon^{-1}\tilde{C}\gamma_i^{(2)},$$

with some appropriate positive constant $\tilde{C}$, which is independent of both ε and N.

We combine the above lower estimates of $\mathcal{L}^N\gamma_i^{(2)}$ with (3.11) and get

$$\mathcal{L}^N\gamma_i \geq \begin{cases} C_1\beta N^{-1} + \tilde{C}C_2\varepsilon^{-1}\bar{y}_i^N N^{-1}\ln N, & i = 1, 2, \ldots, J-1, \\ C_1\beta N^{-1} + \bar{C}C_2\dfrac{\bar{y}_i^N N^{-1}\ln N}{\max\{\varepsilon, h_{i+1}\}}, & i = J, J+1, \ldots, N-1. \end{cases}$$

We then recall (3.8) to conclude that we have (3.10) provided C_1 and C_2 are chosen appropriately. □

Remark 3.2. In the proof of Theorem 1 in [9], the assumption

$$\int_0^q [\phi'(t)]^2 dt \leq CN \tag{3.12}$$

is used to bound $|y_i - Y_i^N|$ on the coarse part of the mesh. More specifically, the argument there uses separate estimates of $|y_i|$ and $|Y_i^N|$ by means of barrier functions and invokes (3.12). This is the most challenging part of the proof because the derivatives of y remain large on the first coarse mesh interval $[x_J, x_{J+1}]$. This approach is a typical barrier-function technique that has been used in literature for many years for the error analysis of the problem (1.1) discretized by a finite-difference scheme on Shishkin-type meshes (cf. [9, 12] and monographs [5, 2, 11], for instance). We present here a direct approach which shows that the truncation error is actually small on the interval $[x_J, x_{J+1}]$. For this, we exploit the fact that the solution decomposition satisfies (3.2) and apply a barrier function on the entire interval $[0, 1]$ to bound $|y_i - Y_i^N|$.

Remark 3.3. When it comes to improving the error estimate in Theorem 3.1 by removing the locking factor $\ln N$ in (3.3), more structured mesh points are required in the layer region $[0, \sigma]$. This is enabled in [9] by some of the functions ϕ which generate the mesh in $[0, \sigma]$. In general, if ϕ is piecewise differentiable and satisfies both (3.12) and

$$\max \phi' \leq CN, \tag{3.13}$$

then the error estimate obtained in [9] is

$$|u_i - U_i^N| \leq CN^{-1} \max |\psi'|, \quad i = 0, 1, \ldots, N, \tag{3.14}$$

where ψ is the so-called mesh-characterizing function, $\psi = e^{-\phi}$. The condition (3.13) immediately yields (3.6). Moreover, for the Shishkin-type meshes defined in [9], it can be shown that (see, for example, [8, page 3])

$$h_{i+1} \leq \frac{2}{\beta} \varepsilon N^{-1} \max |\psi'| e^{\beta x_{i+1}/(2\varepsilon)}, \quad i = 1, 2, \ldots, J-1.$$

Therefore, in the proof of Theorem 3.1, the $\ln N$ factor can be replaced with $\max |\psi'|$ and we can obtain (3.14) using our technique. This means that any function ϕ from [9], which eliminates the locking factor from the error estimate, does the same in our approach. At the same time, no special structure is required of the coarse part of the mesh.

4. Concluding remarks

Motivated by interpolation-related needs, we introduced a generalization of the Shishkin-type meshes which do not require any structure of the fine and coarse parts of the mesh. No part of the mesh needs to be uniform, nor is a mesh-generating function needed for the fine part. In other words, the only thing that matters in a Shishkin-type mesh is the transition point. Therefore, the key property of a Shishkin-type mesh is its explicitly defined transition point. This sets Shishkin-type meshes apart from Bakhvalov-type meshes.

Such relaxed meshes call for a new barrier-function technique to prove parameter-uniform pointwise convergence for convection-dominated problems. In the proof of Theorem 3.1, we presented the new approach considering the upwind discretization scheme on the generalized Shishkin-type mesh. The new proof technique may be helpful when trying to solve a number of open problems, see [10]. In particular, while the techniques of [1] and [6, 8] can be extended to Bakhvalov-type meshes (see [7]), it is not known whether the same can be done with the barrier-function technique of the kind used in [9] (see [3, Remark 4.21]). Ultimately, we expect that it may be possible to do so with our barrier-function approach. This is the subject of an ongoing work.

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