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ON CHARACTERIZATION OF MINIMAL k-BI-IDEALS IN k-REGULAR AND COMPLETELY k-REGULAR SEMIRINGS

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Abstract. In this paper, we study k-regular and completely k-regular semirings. We characterize the minimal k-bi-ideals in k-regular semirings via principal k-bi-ideals and also in completely k-regular semirings via k-bi-ideals generated by k-idempotent elements. Finally we characterize the completely k-regular semirings by k-bi-ideals generated via k-idempotents.

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1. Introduction

The notion of a semiring was introduced by Vandiver [15]. In 1951, Bourne defined a regular semiring as a semiring S in which for all $a \in S$ there exist $x, y \in S$ such that $a + axa = aya$. In [1], Adhikari, Sen and Weinert renamed it as a k-regular semiring. In [14], Sen and Bhuniya studied k-regular semirings with a semilattice additive reduct, and constructed k-regular semirings. If F is any semigroup, then the set $P(F)$ of all subsets of F is a semiring in $\mathcal{SL}^+$, where addition and multiplication are defined by the set union and the usual product of subsets of a semigroup, respectively. In [14], it is shown that $P(F)$ is a k-regular semiring if and only if F is a regular semigroup [Theorem 3.1], and if $(F, \cdot)$ is a regular semigroup, then the k-idempotents of $P(F)$ commute if and only if $P(F)$ is a commutative semiring [Theorem 3.4]. Sen and Bhuniya defined k-idempotents to characterize the k-regular semirings which are distributive lattices of k-semifields [13]. Bhuniya and Jana introduced the notion of k-bi-ideals in a semiring, characterized the k-regular semirings by k-bi-ideals, and gave the description of the principal k-bi-ideals in a semiring with semilattice additive reduct [2]. In [9], Jana studied quasi k-ideals in k-regular semirings and characterized the k-regular semirings via their quasi k-ideals. In [12], Sen and Bhuniya defined completely k-regular semirings and presented various interesting properties of classes of such semirings. They characterized

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completely k-regular semirings. A semiring S is completely k-regular if and only if S is k-regular and $a \in \overline{a^2S} \cap \overline{Sa^2}$ for all $a \in S$. The structure of such semirings was also given. A semiring is completely k-regular if and only if S is a union of k-semifields.

In this paper we study the semirings with semilattice additive reduct. Such semirings have been studied by Bhuniya and Mondal [3, 4, 5, 10, 11] to give the decompositions of underlying semirings through distributive lattice congruence into simpler components. Here we study k-regular and completely k-regular semirings with semilattice additive reduct and their k-bi-ideals. In Section 2, the preliminaries have been provided. In Section 3, we study completely k-regular semirings and k-regular semirings. We show that in a completely k-regular semiring, for any element $a \in S$ there exist two $\overline{\mathcal{H}}$-related k-idempotent elements. In Section 4, our main intention is to characterize minimal k-bi-ideals in a k-regular and completely k-regular semiring by principal k-bi-ideals generated by k-idempotents. We show that a k-bi-ideal B in a k-regular semiring is minimal if and only if for all $a, b \in B$, the principal k-b-ideals generated by a and b are the same, while a k-bi-ideal B in a completely k-regular semiring S is minimal if and only if the principal k-bi-ideals generated by k-idempotents in B coincide. We define k-bi-simple semirings, and characterize the minimal k-bi-ideals by k-bi-simplicity of the semirings. Finally, we characterize completely k-regular semirings by the principal k-bi-ideals generated by k-idempotents of S.

2. Preliminaries

A *semiring* $(S, +, \cdot)$ is an algebra with two binary operations $+$ and $\cdot$ such that both the *additive reduct* $(S, +)$ and the *multiplicative reduct* $(S, \cdot)$ are semigroups and such that the following distributive laws hold:

$$x(y + z) = xy + xz \text{ and } (x + y)z = xz + yz.$$

Thus the semirings can be regarded as a common generalization of both rings and distributive lattices. By $\mathcal{SL}^+$ we denote the category of all semirings $(S, +, \cdot)$ such that $(S, +)$ is a semilattice, i.e. a commutative and idempotent semigroup. Throughout this paper, unless otherwise stated, S is always a semiring in $\mathcal{SL}^+$.

Let A be a nonempty subset of S. The *k-closure* of A is defined by

$$\overline{A} = \{x \in S \mid x + a_1 = a_2 \text{ for some } a_1, a_2 \in A\}.$$

We assume that $x + a_1 = a_2$. Hence $x + a_2 = x + x + a_1 = a_2$. So $\overline{A}$ is also described by

$$\overline{A} = \{x \in S \mid x + a = a \text{ for some } a \in A\}.$$

Then we have $A \subseteq \overline{A}$ and $\overline{A} = \overline{\overline{A}}$, since $(S, +)$ is a semilattice, A is called a *k-set* if $\overline{A} \subseteq A$. An ideal (left, right) A of S is called a k-ideal (left, right) if it is a *k-set*, i.e. $\overline{A} = A$.

A semiring S is called a k-regular semiring [6] if for every $a \in S$, there exists an $s \in S$ such that $a + asa = asa$. A semiring S is called a completely k-regular semiring if for every $a \in S$, there exists an $s \in S$ such that $a + asa = asa, as + as^2a = as^2a$ and $sa + as^2a = as^2a$, equivalently, $a + a^2sa^2 = a^2sa^2$ [Theorem 5.1 [12]].

For $a \in S$, the principal left k-ideal (resp. principal right k-ideal) generated by a is the least left k-ideal (resp. least right k-ideal) of S containing a. Bhuniya and Jana [2] introduced k-bi-ideals in a semiring in $\mathcal{SL}^+$. A non-empty subset B of S is said to be a k-bi-ideal of S if $BSB \subseteq B$ and B is a k-subsemiring of S. The structures of the principal left k-ideal (resp. principal right k-ideal and principal k-bi-ideal) are given, respectively, by

$$L_k(a) = \{x \in S \mid x + a + sa = a + sa, \text{ for some } s \in S\},$$

$$R_k(a) = \{x \in S \mid x + a + as = a + as, \text{ for some } s \in S\}$$

and

$$B_k(a) = \{x \in S \mid x + a + a^2 + asa = a + a^2 + asa, \text{ for some } s \in S\}.$$

Sen and Bhuniya [12] defined four equivalence relations namely $\overline{\mathcal{L}}$, $\overline{\mathcal{R}}$, $\overline{\mathcal{J}}$and $\overline{\mathcal{H}}$ analogous to the Green's relations, on a k-regular semiring S in $\mathcal{SL}^+$. If $a \in S$ be a k-regular element, then one has $L_k(a) = \overline{Sa}, R_k(a) = \overline{aS}, B_k(a) = \overline{aSa}$. Bhuniya and Mondal [4], [11] generalized the Green's relations $\overline{\mathcal{L}}$, $\overline{\mathcal{R}}$, and $\overline{\mathcal{H}}$ on a semiring S in $\mathcal{SL}^+$ and they are

$$\overline{\mathcal{L}} = \{(x, y) \in S \times S \mid L_k(x) = L_k(y)\},$$

$$\overline{\mathcal{R}} = \{(x, y) \in S \times S \mid R_k(x) = R_k(y)\}$$

and

$$\overline{\mathcal{H}} = \overline{\mathcal{L}} \cap \overline{\mathcal{R}}.$$

Mondal and Bhuniya also defined an equivalence relation $\overline{\mathcal{B}}$ [11] by: for $a, b \in S$,

$$a\overline{\mathcal{B}}b \Leftrightarrow B_k(a) = B_k(b).$$

If $S \in \mathcal{SL}^+$, then both $\overline{\mathcal{L}}$ and $\overline{\mathcal{R}}$ are additive congruences on S and $\overline{\mathcal{L}}$ is a right congruence and $\overline{\mathcal{R}}$ is a left congruence on S.

An element $e \in S$ is said to be k-idempotent if $e + e^2 = e^2$ [12]. If A is a subsemiring of S, then let $E_k(A)$ denote the set of all k-idempotents of A.

For undefined concepts in semigroup theory we refer to [8], for undefined concepts in semiring theory cf. [7].

3. Completely k-regular semirings

In this section we study completely k-regular semirings and k-regular semirings. In completely k-regular semirings we show that for any given element $a \in S$ we can always find two k-idempotents, depending on a, such that they are $\overline{\mathcal{H}}$-related. We also characterize the k-regular semirings by the product of a principal right k-ideal and a principal left k-ideal of the semirings.

Lemma 3.1. *Let S be a completely k-regular semiring. Then*

1. *for every $a \in S$, there exists a $z \in S$ such that $a+aza = asa, a+a^2z = a^2z$ and $a + za^2 = za^2$.*

2. *for every $a \in S$, there exists a $u \in S$ such that $a\overline{\mathcal{H}}au$ and $a\overline{\mathcal{H}}ua$.*

3. *for every $a \in S$, there exist $e, f \in E_k(S)$ depending on a such that $e\overline{\mathcal{H}}f$.*

Proof. (1) Since S is completely k-regular, for $a \in S$, there exists a $x \in S$ such that $a + a^2xa^2 = a^2xa^2$. Adding $a^3xa^2xa^2$ on both sides one gets $a+a(a+a^2xa^2)xa^2 = a(a+a^2xa^2)xa^2$. This implies $a+a^3xa^2xa^2 = a^3xa^2xa^2$. Again adding $a^3xa^2xa^2xa^3$ on both sides we get of $a + a^3xa^2x(a + a^2xa^2)a = a^3xa^2x(a + a^2xa^2)a$ so that $a + a^3xa^2xa^2xa^3 = a^3xa^2xa^2xa^3$. For $z = a^2xa^2xa^2xa^2$, we get $a + aza = aza$. Again adding $a^4xa^2xa^2xa^2$ on both sides of $a + a^3xa^2xa^2 = a^3xa^2xa^2$ we get $a + a^2(a + a^2xa^2)xa^2xa^2 = a^2(a + a^2xa^2)xa^2xa^2$. This yields $a + a^2a^2xa^2xa^2xa^2 = a^2a^2xa^2xa^2xa^2$, i.e. $a + a^2z = a^2z$. Similarly one can show that $a + za^2 = za^2$.
(2) For $a \in S$, there exists a $x \in S$ such that $a + axa = axa, ax + ax^2a = ax^2a$ and $xa + ax^2a = ax^2a$. Now we can write $a + axa = axa$ as $a + as = as$ and $a + ta = ta$, where $s = xa, t = ax$. Then we have $a + au = au$ and $a + ua = ua$, where $u = s + t$. Adding xa^2 on both sides of $a + axa = axa$, one gets $a + xa^2 + axa = xa^2 + axa$. Now adding ax^2a^2 on both sides we get $a + xa^2 + (ax + ax^2a)a = (ax + ax^2a)a$, giving $a + xaa + axxaa = axxaa$ which yields $a + sa + ax(sa) = sa + ax(sa)$. Similarly adding a^2x on both sides of $a + axa = axa$ and proceeding as above one gets $a + at + (at)xa = at + (at)xa$. Now adding $ta+axta$ and $as+asxa$, respectively on $a+sa+ax(sa) = sa+ax(sa)$ and $a + at + (at)xa = at + (at)xa$, we have $a + ua + ax(ua) = ua + ax(ua)$ and $a + au + (au)xa = au + (au)xa$. These two relations yield $a \in L_k(ua) \cap R_k(au)$. Also $ua \in L_k(a)$ and $au \in R_k(a)$. Thus $L_k(a) = L_k(ua)$ and $R_k(a) = R_k(au)$. The relation $xa + ax^2a = ax^2a$ can be written as $s + ts = ts$, and $ax + ax^2a = ax^2a$ as $t + ts = ts$. Then we get $(s + t) + ts = ts$, i.e. $u + ts = ts$. Now $au + ats = ats$, and $ua + tsa = tsa$, i.e. $au + (atx)a = (atx)a$, and $ua + a(xsa) = a(xsa)$. These two yield $au \in \overline{Sa} = L_k(a), ua \in \overline{aS} = R_k(a)$, since S is k-regular. Therefore, $L_k(au) \subseteq L_k(a)$ and $R_k(ua) \subseteq R_k(a)$. Also from $a + axa = axa$, we have $a + a(xa + ax) = a(xa + ax), a + (ax + xa)a = (ax + xa)a$, i.e. $a + au = au \in L_k(au), a + ua = ua \in R_k(ua)$ so that $a \in L_k(au), a \in R_k(ua)$. Therefore, $L_k(a) \subseteq L_k(au)$ and $R_k(a) \subseteq R_k(ua)$. Consequently, $L_k(a) = L_k(au) = L_k(ua)$, and $R_k(a) = R_k(au) = R_k(ua)$. Finally, we get $a\overline{\mathcal{H}}au$ and $a\overline{\mathcal{H}}ua$.
(3) Let $a \in S$. Then from the proof of (1), one has $a + aza = aza$. Then $az + (az)^2 = (az)^2$ and $za + (za)^2 = (za)^2$ yield $e(= za), f(= az) \in E_k(S)$. Now $a + aza = aza$, i.e $a + ae = ae \in \overline{Se} \subseteq L_k(e)$, whence $a \in L_k(e)$. Also $e = za \in L_k(a)$. Therefore, $a\overline{\mathcal{L}}e$. Now $a + za^2 = za^2$, i.e $a + ea = ea \in R_k(e)$ so that $a \in R_k(e)$. Again $za + zaza = zaza$, i.e. $e + a^2xa^2xa^2xa^2za = a^2xa^2xa^2xa^2za \in R_k(a)$ so that $e \in R_k(a)$. Thus $a\overline{\mathcal{R}}e$. Consequently, $a\overline{\mathcal{H}}e$. Similarly, one can get $a\overline{\mathcal{H}}f$, whence $e\overline{\mathcal{H}}f$, since $\overline{\mathcal{H}}$ is an equivalence relation on S.

□

Lemma 3.2. *If A, B are two subsemirings of a semiring S, then*

1. *for $x \in S, a_1, a_2 \in A, b_1, b_2 \in B$ with $x + a_1b_1 = a_2b_2$, there exist $u \in A, v \in B$ such that $x + uv = uv$.*
2. *if $a, b, u, v, s, t \in S$ satisfying $u + as = as$ and $v + ta = ta$, then there exists a $w \in S$ such that $u + aw = aw$ and $v + wa = wa$.*

Proof. (1) Follows if we take $u = a_1 + a_2, v = b_1 + b_2$.
(2) $w = s + t$ serves our purpose. □

Lemma 3.3. *[Theorem 3.2 [9]] A semiring S is k-regular if and only if for every right k-ideal R and left k-ideal L of S, $\overline{RL} = R \cap L$.*

Lemma 3.4. *Let S be a k-regular semiring. Then*

1. *for every $a \in S, B_k(a) = \overline{R_k(a)L_k(a)}$.*
2. *for any subset A of $S, \overline{SA} \cap \overline{AS} = \overline{SA \cap AS}$.*

Proof. (1) Let $a \in S$ and $x \in B_k(a)$. Then there exists an $s \in S$ such that $x + asa = asa$. Since S is a k-regular, there exists a $u \in S$ such that $a + aua = aua$. Adding $asaua + auasa + auasaua$ on both sides of $x + asa = asa$, we get $x + (a + aua)s(a + aua) = (a + aua)s(a + aua)$, i.e., $x + (auas)(aua) = (auas)(aua) \in R_k(a)L_k(a)$. This implies that $x \in \overline{R_k(a)L_k(a)}$. Therefore $B_k(a) \subseteq \overline{R_k(a)L_k(a)}$. Conversely, suppose that $x \in \overline{R_k(a)L_k(a)}$. Then by Lemma 3.2, there are $u \in R_k(a), v \in L_k(a)$ such that $x + uv = uv$. Also there is a $w \in S$ such that $u + aw = aw, v + wa = wa$. Adding $uwa + awv + awwa$ on both sides of $x + uv = uv$, one gets $x + (u + aw)(v + wa) = (u + aw)(v + wa)$ so that $x + awwa = awwa \in \overline{aSa} = B_k(a)$. This implies $x \in B_k(a)$. Thus $\overline{R_k(a)L_k(a)} \subseteq B_k(a)$. Consequently, $B_k(a) = \overline{R_k(a)L_k(a)}$.
(2) Let $x \in \overline{SA} \cap \overline{AS}$. Then using Lemma 3.2, we get $x + sa = sa, x + as = as$ for some $s \in S$. Since S is k-regular, there exists a $z \in S$ such that $x + xzx = xzx$. Adding $xzsa + aszx + aszsa$ on both sides we get $x + (x + as)z(x + sa) = (x + as)z(x + sa)$, i.e., $x + aszsa = aszsa \in SA \cap AS$ yielding $x \in \overline{SA \cap AS}$. Conversely, for $x \in \overline{SA \cap AS}$, there are $u, v \in SA \cap AS$ such that $x + u = v$. Now there are $s_1, s_2, s_3, s_4 \in S, a_1, a_2, a_3, a_4 \in A$ such that $u = s_1a_1 = a_2s_2, v = s_3a_3 = a_4s_4$. Then one gets $x + s_1a_1 = s_3a_3, x + a_2s_2 = a_4s_4$ so that $x \in \overline{SA} \cap \overline{AS}$. Therefore $\overline{SA \cap AS} \subseteq \overline{SA} \cap \overline{AS}$. Consequently, $\overline{SA \cap AS} = \overline{SA} \cap \overline{AS}$. □

4. Characterization of minimal k-bi-ideals

In this section we find a necessary and sufficient condition for a k-bi-ideal to be minimal in a semiring, k-regular semiring as well as in completely k-regular semiring.

Lemma 4.1. *Let S be a k-regular semiring. A k-bi-ideal B of S is minimal if and only if $B_k(a) = B_k(b)$ for all $a, b \in B$.*

Proof. Let B be a minimal k-bi-ideal. Then for $a, b \in B$ one has $B_k(a) \subseteq B, B_k(b) \subseteq B$ so that $B_k(a) = B = B_k(b)$. Conversely, suppose that the given condition holds. Let C be a k-bi-ideal of S with $C \subseteq B$. For $x \in C, y \in B$, we have $x, y \in B$. This implies $B_k(x) = B_k(y)$ so that $y \in B_k(x) \subseteq C$, i.e., $B \subseteq C$. Consequently, B is minimal. □

In the following lemma we find that in a k-regular semiring the relation $\overline{\mathcal{B}}$ coincides with the relation $\overline{\mathcal{H}}$.

Lemma 4.2. *The following results hold in a semiring S:*

1. $\overline{\mathcal{B}} \subseteq \overline{\mathcal{H}}$.
2. *If S is k-regular, then $\overline{\mathcal{B}} = \overline{\mathcal{H}}$.*

Proof. (1) Let $a, b \in S$ with $a\overline{\mathcal{B}}b$. Then there are $s, t \in S$ such that $a + b + b^2 + bsb = b + b^2 + bsb$ and $b + a + a^2 + ata = a + a^2 + ata$. Then we have $a + b + (b + bs)b = b + (b + bs)b$ and $b + a + (a + at)a = a + (a + at)a$ yielding $a \in L_k(b), b \in L_k(a)$ so that $L_k(a) = L_k(b)$, i.e., $a\overline{\mathcal{L}}b$. Again we can write $a + b + b(b + sb) = b + b(b + sb)$ and $b + a + a(a + ta) = a + a(a + ta)$ yielding $a \in R_k(b), b \in R_k(a)$ so that $R_k(a) = R_k(b)$. Thus $a\overline{\mathcal{R}}b$. Consequently, $\overline{\mathcal{B}} \subseteq \overline{\mathcal{H}}$.
(2) Let S be k-regular, and $x\overline{\mathcal{H}}y$. Then one has $L_k(a) = L_k(b), R_k(a) = R_k(b)$. Since S is k-regular, by Lemmas 3.3 and 3.4, we get $B_k(a) = \overline{R_k(a)L_k(a)} = R_k(a) \cap L_k(a)$. Then $B_k(a) = R_k(a) \cap L_k(a) = R_k(b) \cap L_k(b) = B_k(b)$ yielding $a\overline{\mathcal{B}}b$. Consequently, $\overline{\mathcal{B}} = \overline{\mathcal{H}}$. □

In the following theorem we characterize the minimal k-bi-ideals in a semiring via the relation $\overline{\mathcal{B}}$.

Theorem 4.3. *A k-bi-ideal B of a semiring S is minimal if and only if it is a $\overline{\mathcal{B}}$-class.*

Proof. Let B be a minimal k-bi-ideal of a semiring S, and $a, b \in B$. Then by Lemma 4.1, one gets $B_k(a) = B_k(b)$. This implies that $a\overline{\mathcal{B}}b$. Thus B is a $\overline{\mathcal{B}}$-class. Conversely, suppose that B is a $\overline{\mathcal{B}}$-class, and K a k-bi-ideal of S such that $K \subseteq B$. Let $x \in B, y \in K$. Then $x, y \in B$ giving that $x\overline{\mathcal{B}}y$, i.e., $B_k(x) = B_k(y)$. Then $x \in B_k(x) = B_k(y) \subseteq K$. Therefore $x \in K$. Thus $B \subseteq K$. Consequently, B is minimal. □

Theorem 4.4. *A k-bi-ideal B in a k-regular semiring S is minimal if and only if B is an $\overline{\mathcal{H}}$-class of S.*

Proof. Let B be a minimal k-ideal of S. Then by Lemma 4.1 and Theorem 4.3, we find that B is an $\overline{\mathcal{H}}$-class of S. Converse part follows from the Lemma 4.2 and Theorem 4.3. □

In the following theorem we characterize the minimal k-bi-ideals in a completely k-regular semiring via k-bi-ideals generated by k-idempotent elements.

Theorem 4.5. *A k-bi-ideal B of a completely k-regular semiring S is minimal if and only if $B_k(a) = B_k(e)$ for all $a \in B$ and for all $e \in E_k(B)$.*

Proof. Let B be a minimal k-bi-ideal of S, and $a \in B, e \in E_k(B)$. Then $a, e \in B$, and so by Lemma 4.1, one gets $B_k(a) = B_k(e)$. Conversely suppose that the given conditions hold, and let K be a k-bi-ideal of S such that $K \subseteq B$. Let $x \in K, b \in B$. Since S is completely k-regular, there exists an $s \in S$ such that $x + xsx = xsx, xs + xs^2x = xs^2x, sx + xs^2x = xs^2x$. Then $xs, sx \in E_k(S)$. Now $xs + xs^2x = xs^2x \in \overline{xSx} = B_k(x)$ implies $xs \in B_k(x) \subseteq B$ so that $xs \in E_k(B)$. By hypothesis, $b, xs \in B$ implies $B_k(b) = B_k(xs)$. Similarly, $B_k(b) = B_k(sx)$. Now $b \in B_k(xs)$ ensures the existence of some $w \in S$ such that $b + xswxs = xswxs$. Adding $xswxs^2x$ on both sides we get $b + xsw(xs + xs^2x) = xsw(xs + xs^2x)$. This implies $b + xswxs^2x = xswxs^2x \in \overline{xSx} = B_k(x)$ so that $b \in B_k(x) \subseteq K$, i.e., $b \in K$. Therefore $B \subseteq K$. Consequently, B is minimal. □

Corollary 4.6. *A k-bi-ideal B of a completely k-regular semiring S is minimal if and only if $B_k(e) = B_k(f)$ for all $e, f \in E_k(B)$.*

Proof. Let B be a minimal k-bi-ideal of S, and $e, f \in E_k(B)$. Then by Lemma 4.1, one gets $B_k(e) = B_k(f)$. Conversely, suppose that the given conditions hold, and $a \in B, e \in E_k(B)$. Then as in the proof of (1) of Lemma 3.1, for this $a \in B$, there is $z = a^2xa^2xa^2xa^2$ such that $a + aza = aza, a + a^2z = a^2z$ and $a + za^2 = za^2$. Since B is a k-bi-ideal of $S, z = a(axa^2xa^2xa)a \in BSB \subseteq B$. Consequently, $az \in E_k(B)$. Since $e, az \in E_k(B)$, one gets $B_k(az) = B_k(e)$. Now adding aza^2z on both sides of $a + aza = aza$, we get $a + az(a + a^2z) = az(a + a^2z)$, that is, $a + az(a)az = az(a)az$. This yields $a \in B_k(az)$. Also $az + azaz = azaz$ gives $az + a(zaa^2xa^2xa^2xa)a = a(zaa^2xa^2xa^2xa)a$, and so $az \in B_k(a)$. Now $B_k(a) = B_k(az) = B_k(e)$, and then by Theorem 4.5, B is a minimal k-bi-ideal of S. □

In [11], Mondal and Bhuniya defined $\overline{\mathcal{B}}$-simple semirings. In this paper we rename it by k-bi-simple semirings. Then a semiring S is called k-bi-simple if it has no non-trivial proper k-bi-ideal.

Example 4.7. Let $\mathbb{R}^+$ denote the set of all positive real numbers, and consider the group $(\mathbb{R}^+, \cdot)$. Let $P_f(\mathbb{R}^+)$ be the set of all finite subsets of $\mathbb{R}^+$. Define $+$ and $\cdot$ on $P_f(\mathbb{R}^+)$ by: $A + B = A \cup B$ and $A \cdot B = \{ab \mid a \in A, b \in B\}$ for all $A, B \in P_f(\mathbb{R}^+)$. Then $(P_f(\mathbb{R}^+), +, \cdot)$ is a k-bi-simple semiring.

Then we have the following lemma:

Lemma 4.8. *[Lemma 3.1 [11]] In a semiring S the following conditions are equivalent:*

1. *S is a t-k-simple semiring;*
2. *S is a k-bi-simple($\overline{\mathcal{B}}$-simple) semiring;*
3. *S is a $\overline{\mathcal{H}}$-simple semiring.*

Now as in Remark 2.6 [11], we find that that every k-bi-simple semiring is a k-regular semiring. Now we are in a position to characterize the k-bi-simple semirings via k-bi-ideals generated by k-idempotent elements.

Theorem 4.9. *A semiring S is k-bi-simple if and only if S is k-regular and for all $e, f \in E_k(S), B_k(e) = B_k(f)$.*

Proof. Let S be a k-bi-simple semiring and $e, f \in E_k(S)$. Then by hypothesis, we have $B_k(e) = B_k(f)$. Also, every k-bi-simple semiring is k-regular. Conversely, suppose that the given conditions hold, and let B be a k-bi-ideal of S. We are interested to show $B = S$. For, let $s \in S, b \in B$. Since S is k-regular, there are $x, y \in S$ such that $s + sxs = sxs$ and $b + byb = byb$. Then sx, xs, by, yb are all in $E_k(S)$. By hypothesis, we get $B_k(sx) = B_k(by)$ and $B_k(xs) = B_k(yb)$. This implies $sx\overline{\mathcal{B}}by$ and $xs\overline{\mathcal{B}}yb$. Then by Lemma 4.2, one gets $sx\overline{\mathcal{H}}by$ and $xs\overline{\mathcal{H}}yb$ so that $sx\overline{\mathcal{R}}by$ and $xs\overline{\mathcal{L}}yb$. Now there exist $u, v \in S$ such that $sx + byu = byu$ and $xs + vyb = vyb$. Now adding $sxsxs$ on both sides of $s + sxs = sxs$ we have $s + sx(s + sxs) = sx(s + sxs)$. This implies $s + sxsxs = sxsxs$. Again adding $sxsvyb + byusxs + byusvyb$ on both sides one can write $s + (sx + byu)s(xs + vyb) = (sx + byu)s(xs + vyb)$, i.e., $s + byusvyb = byusvyb \in \overline{bSb} = B_k(b)$ so that $s \in B_k(b) \subseteq B$ yielding $s \in B$. Thus $S \subseteq B$ so that $S = B$, whence B is k-bi-simple. □

In the following theorem we characterize the minimal k-bi-ideals by its k-bi-simplicity. Before that we have the lemma:

Lemma 4.10. *If B is a k-bi-ideal of a semiring S, then for $a \in S$, $\overline{aBa}$ is a k-bi-ideal of S.*

Proof. Let $x, y \in \overline{aBa}$. Then there exist $b_1, b_2 \in B$ such that $x + ab_1a = ab_1a, y + ab_2a = ab_2a$. These yield $x + aba = aba, y + aba = aba$, where $b = b_1 + b_2 \in B$. Then $(x + y) + aba = aba \in \overline{aBa}$ implies that $x + y \in \overline{aBa}$. If $s \in S, b \in \overline{aBa}$, then there exists a $z \in B$ such that $b + aza = aza$. Now $zasaz \in B$, since B is a k-bi-ideal of S. Multiplying both sides of $b+aza = aza$ by sb on the right, we have $bsb + azasb = azasb$. Adding $azasaza$ on both sides we get $bsb+azas(b+aza) = azas(b+aza)$. This implies $bsb+azasaza = azasaza \in aBa$ so that $bsb \in \overline{aBa}$. Thus $\overline{aBa}S\overline{aBa} \subseteq \overline{aBa}$. Now to show that $\overline{aBa}$ is a k-set, suppose that $x \in S, y \in \overline{aBa}$ satisfying $x + y = y$. Now there exists a $b \in B$ such that $y + aba = aba$. This implies $x + y + aba = aba$, that is, $x + aba = aba \in aBa$ yielding $x \in \overline{aBa}$. Consequently, $\overline{aBa}$ is a k-bi-ideal of S. □

Theorem 4.11. *Let S be a semiring. Then a k-bi-ideal B is minimal if and only if it is k-bi-simple.*

Proof. Let B be a minimal k-bi-ideal of S, and T a k-bi-ideal of B. Let $t \in T$. then one gets $\overline{tBt} \subseteq \overline{TBT} \subseteq \overline{T} = T \subseteq B$. By Lemma 4.10, $\overline{tBt}$ is a k-bi-ideal of S, and B is minimal in S, we get $\overline{tBt} = B$. This implies $T = B$, whence B is k-bi-simple. Conversely, suppose that B is a k-bi-simple, and C a k-bi-ideal of S with $C \subseteq B$. Let $c \in C$. Then $\overline{cBc}$ is a k-bi-ideal of B. Since B is a

k-bi-simple, $\overline{cBc} = B$. Then $B = \overline{cBc} \subseteq \overline{CBC} \subseteq \overline{CSC} \subseteq \overline{C} = C$ yielding $B = C$. Consequently, B is minimal. □

Finally we characterize the completely k-regular semirings via k-bi-ideals generated by k-idempotent elements.

Theorem 4.12. *A semiring S is completely k-regular semiring if and only if*
(1) *for every k-bi-ideal B of S, there is some $e \in E_k(S)$ such that $B = B_k(e)$, and*
(2) *for every $x \in B, B_k(x^2) = B_k(e)$.*

Proof. Let S be a completely k-regular semiring.
(1): Let B be a k-bi-ideal of S, and $a \in B$. Now by the proof of (1) of Lemma 3.1, one has $a + aza = aza, a + a^2z = a^2z$ and $a + za^2 = za^2$, where $z = a^2xa^2xa^2xa^2$. Now adding aza^2z on both sides of $a + aza = aza$ one gets $a + az(a + a^2z) = az(a + a^2z)$, i.e. $a + azaaz = azaaz = eae \in \overline{eSe} = B_k(e)$ so that $a \in B_k(e)$ yielding $B \subseteq B_k(e)$. Now suppose that $y \in B_k(e) = \overline{eSe}$. Then there exists a $u \in S$ such that $y + azuaz = azuaz$, i.e. $y + azua^3xa^2xa^2xa^2 = azua^3xa^2xa^2xa^2 \in \overline{aSa} = B_k(a) \subseteq B$ yielding $B_k(a) = B = B_k(e)$.
(2): Let $x \in B$. Then $x^2 \in B$, whence by (1), there exists an $f \in E_k(S)$ such that $B = B_k(x^2) = B_k(f)$. Consequently, $B_k(x^2) = B_k(f) = B = B_k(e)$, by (1).
Conversely, suppose that the conditions hold, and $a \in S$. Consider the k-bi-ideal $B_k(a)$ of S. Then there exists an $e \in E_k(S)$ such that $B_k(a) = B_k(e)$. Since $a^2 \in B_k(a)$, by condition (2), it follows that $B_k(e) = B_k(a^2)$. Then $a + a^2sa^2 = a^2sa^2$ yielding a is completely k-regular element. Consequently, S is completely k-regular. □

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References

[1] Adhikari, M. R., Sen, M. K., and Weinert, H. J. On k-regular semirings. *Bull. Calcutta Math. Soc. 88*, 2 (1996), 141–144.

[2] Bhuniya, A. K., and Jana, K. Bi-ideals in k-regular and intra k-regular semirings. *Discuss. Math. Gen. Algebra Appl. 31*, 1 (2011), 5–25.

[3] Bhuniya, A. K., and Mondal, T. K. Distributive lattice decompositions of semirings with a semilattice additive reduct. *Semigroup Forum 80*, 2 (2010), 293–301.

[4] Bhuniya, A. K., and Mondal, T. K. Semirings which are distributive lattices of k-simple semirings. *Southeast Asian Bull. Math. 36*, 3 (2012), 309–318.

[5] Bhuniya, A. K., and Mondal, T. K. On the least distributive lattice congruence on a semiring with a semilattice additive reduct. *Acta Math. Hungar. 147*, 1 (2015), 189–204.

[6] Bourne, S. The Jacobson radical of a semiring. *Proc. Nat. Acad. Sci. U. S. A. 37* (1951), 163–170.

[7] Hebisch, U., and Weinert, H. J. *Semirings: algebraic theory and applications in computer science*, vol. 5 of *Series in Algebra*. World Scientific Publishing Co., Inc., River Edge, NJ, 1998. Translated from the 1993 German original.

[8] Howie, J. M. *Fundamentals of semigroup theory*, vol. 12 of *London Mathematical Society Monographs. New Series*. The Clarendon Press, Oxford University Press, New York, 1995. Oxford Science Publications.

[9] Jana, K. Quasi k-ideals in k-regular and intra k-regular semirings. *Pure Math. Appl. (PU.M.A.) 22*, 1 (2011), 65–74.

[10] Mondal, T. K. Distributive lattices of t-k-Archimedean semirings. *Discuss. Math. Gen. Algebra Appl. 31*, 2 (2011), 147–158.

[11] Mondal, T. K., and Bhuniya, A. K. Semirings which are distributive lattices of t-k-simple semirings. *Tbilisi Math. J. 8*, 2 (2015), 149–157.

[12] Sen, M. K., and Bhuniya, A. K. On additive idempotent k-regular semirings. *Bull. Calcutta Math. Soc. 93*, 5 (2001), 371–384.

[13] Sen, M. K., and Bhuniya, A. K. On additive idempotent k-Clifford semirings. *Southeast Asian Bull. Math. 32*, 6 (2008), 1149–1159.

[14] Sen, M. K., and Bhuniya, A. K. On semirings whose additive reduct is a semilattice. *Semigroup Forum 82*, 1 (2011), 131–140.

[15] Vandiver, H. S. Note on a simple type of algebra in which the cancellation law of addition does not hold. *Bull. Amer. Math. Soc. 40*, 12 (1934), 914–920.

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