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ONE CLASS OF SPECIAL POLYNOMIALS AND SPECIAL FUNCTIONS IN $L^2(\mathbb{R})$ SPACE

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Abstract. We construct one class of special polynomials and special functions and give some their interesting propreties. The aim of this paper is to prove that that these functions form a basis of $L^2(\mathbb{R})$ space. In the end we give some interesting sumation formulas.

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1. Introduction

Spaces L^p, $1 \leq p \leq \infty$, and their various subspaces, for example Hardy spaces H^p, $1 \leq p \leq \infty$, are investigated in many papers and books (see [3], [8], [9], [10]). Using these subspaces, various very useful bases of L^p spaces, $1 < p < \infty$, were constructed (see [2], [6], [11]). In this paper we focus only on $L^2(\mathbb{R})$ space. It is proven in [3] that the Hardy space $H^2(\mathbb{R})$ is a subspace of $L^2(\mathbb{R})$ and its basis consists of the functions

$$\left\{ \frac{1}{\sqrt{\pi}} \frac{(x-i)^n}{(x+i)^{n+1}} \right\}_{n=0}^{\infty}. \tag{1.1}$$

Our motivation for this paper is to find an ortonormal basis $\{\psi_n(x)\}_{n=0}^{\infty}$ of $L^2(\mathbb{R})$ space which consists of real and imaginary parts of functions in (1.1) (multiplied by a constant) which we call special functions and denote them by d_n, $n \in \mathbb{N}_0$. In order to construct special functions, d_n, $n \in \mathbb{N}_0$, we use polynomials D_n, $n \in \mathbb{N}_0$, which we call special polynomials. We proved that special polynomials D_n, $n \in \mathbb{N}_0$, are solutions of the Sturm-Liouville differential equation (see [1],[4],[5])

$$(x^2+1)y''(x) - 4nxy'(x) + 2n(2n+1)y(x) = 0 \tag{1.2}$$

and special functions d_n, $n \in \mathbb{N}_0$, are solutions of the Sturm-Liouville differential equation (see [7])

$$(x^2+1)^2y''(x) + 4x(x^2+1)y'(x) + (2x^2+1+(2n+1)^2)y(x) = 0. \tag{1.3}$$

Using these, we obtain some interesting results and sumation formulas.

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2. Preliminaries

We employ the notation $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ for the sets of positive integers, real and complex numbers, respectively; $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{C}^+ = \{z \in \mathbb{C} : Im(z) > 0\}$. For the Fourier transform of $f \in L^2(\mathbb{R})$ we use the symbol $\mathcal{F}(f) := \int_{\mathbb{R}} f(x)e^{-ix\cdot}dx$. We use the folowing notation: $\mathbb{D} = \{w \in \mathbb{C} : |w| < 1\}$ for the open unit disc, $\mathbb{T} = \{w \in \mathbb{C} : |w| = 1\}$ for the unit circle and $\chi_{(0,1)} = 1$ on $(0,1)$, $\chi_{(0,1)} = 0$ otherwise.

2.1. The Hardy space

Following the approach of [3], we introduce Hardy spaces in the following way: The Hardy space $H^p(\mathbb{D})$, $1 \leq p \leq \infty$, is the space of all analytic functions $f : \mathbb{D} \to \mathbb{C}$ such that

$$\|f\|_{H^p}^p := \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta, \quad 1 \leq p \leq \infty.$$

Any function in $H^p(D)$ has the radial and also the non-tangetial limit on $\mathbb{T}$ and, moreover, the space $H^p(\mathbb{D})$ can be identified with the corresponding subspace $H^p(\mathbb{T}) \subset L^p(\mathbb{T})$, $1 < p < \infty$. By [8] the space $H^2(\mathbb{D})$ is a Hilbert space with the orthogonal basis $\{z^n\}_{n=0}^{\infty}$ and $H^2(\mathbb{D}) = \{f : f = \sum_{n=0}^{\infty} a_n z^n, a_n \in l^2\}$ with the norm $\|f\|_{H^2(\mathbb{D})} = (\sum_{n=0}^{\infty} |a_n|^2)^{1/2}$.

Definition 2.1. The Hardy space $H^p(\mathbb{C}^+)$, $1 < p < \infty$, is the space of all analytic function $F : \mathbb{C}^+ \to \mathbb{C}$ such that

$$\|F\|_{H^p(\mathbb{C}^+)} = \sup_{y>0} \left(\int_{\mathbb{R}} |F(x+iy)|^p \right)^{1/p} < \infty.$$

Spaces $H^p(\mathbb{C}^+)$, $1 < p < \infty$, are Banach spaces and $H^2(\mathbb{C}^+)$ is the Hilbert space. An isometric isomorphism between $H^2(\mathbb{D})$ and $H^2(\mathbb{C}^+)$ is given by

$$\Phi(f)(z) = \frac{1}{\sqrt{\pi}(i+z)} f\left(\frac{z-i}{i+z}\right)$$

and $\|f\|_{H^2(\mathbb{D})} = \|\Psi(f)\|_{H^2(\mathbb{C}^+)}$. The orthogonal basis of the Hilbert space $H^2(\mathbb{C}^+)$ is given by

$$\left\{ \frac{1}{\sqrt{\pi}} \frac{(z-i)^n}{(i+z)^{n+1}} \right\}_{n=0}^{\infty}.$$

Definition 2.2. The Hardy space $H^p(\mathbb{R})$, $1 < p < \infty$, is defined by

$$H^p(\mathbb{R}) := \{f \in L^p(\mathbb{R}) : \mathcal{F}(f)(w) = 0, \forall w < 0\}.$$

By [3] each function in $H^p(\mathbb{C}^+)$, $1 < p < \infty$, has non-tangential limits on the real line and, moreover, spaces $H^p(\mathbb{C}^+)$, $1 < p < \infty$, can be identified with the corresponding subspaces $H^p(\mathbb{R})$ of $L^p(\mathbb{R})$, $1 < p < \infty$. The space $H^2(\mathbb{R})$ is the Hilbert space with the orthogonal basis $\{e_n(x)\}_{n=0}^{\infty}$ given by (1.1).

3. Special polynomials and special functions

3.1. Special polynomials

Definition 3.1. We define polynomials $D_{2n}(x)$ and $D_{2n+1}(x)$, $n \in \mathbb{N}_0$, in the following way:

$$D_{2n}(x) := \sum_{k=0}^{n} (-1)^{n+k+1} \binom{2n+1}{2k} x^{2k},$$

$$D_{2n+1}(x) := \sum_{k=0}^{n} (-1)^{n+k} \binom{2n+1}{2k+1} x^{2k+1}.$$

First several polynomials are:

$$D_0(x) = -1, \quad D_1(x) = x, \quad D_2(x) = -3x^2 + 1,$$
$$D_3(x) = x^3 - 3x, \quad D_4(x) = -5x^4 + 10x^2 - 1, \quad D_5(x) = x^5 - 10x^3 + x, \ \ldots$$

Remark 3.2. Polynomials $D_{2n}(x)$ and $D_{2n+1}(x)$, $n \in \mathbb{N}_0$, we call special polynomials and they satisfy:

$$D_0(x) = -1, \quad D''_{2n}(x) = (2n+1)2nD_{2n-2}(x),$$

$$D_1(x) = x, \quad D''_{2n+1}(x) = (2n+1)2nD_{2n-1}(x).$$

In addition we give some interesting poperties of special polynomials $D_n(x)$, $n \in \mathbb{N}_0$.

Proposition 3.3. *Polynomials $D_{2n}(x)$ and $D_{2n+1}(x)$, $n \in \mathbb{N}_0$, satisfy:*

$$D_{2n+1}(x) = \frac{x^2+1}{2n+1} D'_{2n}(x) - xD_{2n}(x) \tag{3.1}$$

and

$$D_{2n}(x) = -\frac{x^2+1}{2n+1} D'_{2n+1}(x) + xD_{2n+1}(x). \tag{3.2}$$

Proof. Notice that

$$D_{2n+1}(x) + iD_{2n}(x) = (x-i)^{2n+1}, \tag{3.3}$$

from which we have

$$D'_{2n+1}(x) + iD'_{2n}(x) = (2n+1)(x-i)^{2n}. \tag{3.4}$$

Multiplying (3.4) by $x - i$ and taking real and imaginary parts we obtain

$$xD'_{2n+1}(x) + D'_{2n}(x) = (2n+1)D_{2n+1}(x) \tag{3.5}$$

and

$$-D'_{2n+1}(x) + xD'_{2n}(x) = (2n+1)D_{2n}(x). \tag{3.6}$$

From (3.5) and (3.6) we obtain (3.1) and (3.2). □

Corollary 3.4. *For $D_{2n}(x)$ and $D_{2n+1}(x)$, $n \in \mathbb{N}_0$, it holds that*

$$D_{2n+1}^2(x) + D_{2n}^2(x) = (x^2+1)^{2n+1}. \tag{3.7}$$

Proof. Using (3.3) we obtain the assertion. □

Theorem 3.5. *Polynomials $D_n(x)$, $n \in \mathbb{N}_0$, are solutions the Sturm-Liouville differential equation* (1.2)*. Moreover, $y_n(x) = C_1 D_{2n+1}(x) + C_2 D_{2n}(x)$ are the only solutions of* (1.2)*.*

Proof. We will prove the assertion only for polynomials $D_{2n}(x)$, since the proof for $D_{2n+1}(x)$, $n \in \mathbb{N}_0$, is the same. If we differentiate (3.1), we obtain

$$(x^2+1)D_{2n}''(x) + (-2n+1)xD_{2n}'(x) = (2n+1)D_{2n}(x) + (2n+1)D_{2n+1}(x).$$

From (3.5), (3.6) and (3.7) we obtain

$$(x^2+1)D_{2n}''(x) - 4nxD_{2n}'(x) + 2n(2n+1)D_{2n}(x) = 0.$$

Conversely, it is well known that $y_1(x) = D_{2n+1}(x)$ is the particular solution of (1.2). General solution is of the form $y(x) = C_1y_1(x) + C_2y_2(x)$, where

$$y_2(x) = D_{2n+1}(x) \int \frac{(x^2+1)^{2n}}{D_{2n+1}^2(x)} dx.$$

On the other hand, using (3.7) we obtain

$$\left(\frac{D_{2n}(x)}{D_{2n+1}(x)}\right)' = (2n+1)\frac{(x^2+1)^{2n}}{D_{2n+1}^2(x)}$$

from which it follows that

$$y_2(x) = \frac{D_{2n}(x)}{2n+1}.$$

So, $y(x) = C_1D_{2n}(x) + C_2D_{2n+1}(x)$. □

3.2. Special functions

Using Definition 3.1 we define special functions as follows:

Definition 3.6. Special functions d_{2n} and d_{2n+1}, $n \in \mathbb{N}_0$, are defined in the following way:

$$d_{2n}(x) = \frac{D_{2n}(x)}{(x^2+1)^{n+1}}, \qquad d_{2n+1}(x) = \frac{D_{2n+1}(x)}{(x^2+1)^{n+1}}.$$

Notice that

$$e_n(x) = \frac{d_{2n+1}(x) + id_{2n}(x)}{\sqrt{\pi}}. \tag{3.8}$$

Proposition 3.7. *Special functions $d_{2n}(x)$ and $d_{2n+1}(x)$, $n \in \mathbb{N}_0$, are given by*

$$d_{2n}(x) = (-1)^{n+1} \frac{\cos((2n+1)\arctan x)}{\sqrt{x^2+1}} \tag{3.9}$$

and

$$d_{2n+1}(x) = (-1)^{n} \frac{\sin((2n+1)\arctan x)}{\sqrt{x^2+1}}. \tag{3.10}$$

Proof. It is well known that

$$\cos(\arctan x) = \frac{1}{\sqrt{x^2+1}}, \quad \sin(\arctan x) = \frac{x}{\sqrt{x^2+1}},$$

so by the using of De Moivire's formula we obtain

$$\begin{aligned}\left(\frac{1+ix}{x^2+1}\right)^{2n+1} &= \left(\frac{\cos(\arctan x) + i\sin(\arctan x)}{\sqrt{x^2+1}}\right)^{2n+1} \\ &= \frac{\cos((2n+1)\arctan x) + i\sin((2n+1)\arctan x)}{\sqrt{x^2+1}^{2n+1}}.\end{aligned}$$

Taking real and imaginary parts of the previous equation we obtain the desired conclusion. □

Proposition 3.8. *Functions $d_{2n}(x)$ and $d_{2n+1}(x)$, $n \in \mathbb{N}_0$, satisfy:*

$$(x^2+1)d'_{2n+1}(x) + xd_{2n+1}(x) = -(2n+1)d_{2n}(x) \tag{3.11}$$

and

$$(x^2+1)d'_{2n}(x) + xd_{2n}(x) = (2n+1)d_{2n+1}(x). \tag{3.12}$$

Proof. From

$$e'_n(x) = ne_n(x)\frac{x+i}{x-i} - (n+1)e_n(x)$$

we obtain

$$\begin{aligned}d'_{2n+1}(x) + id'_{2n}(x) = n(d_{2n+1}(x) + id_{2n}(x))(x^2 - 1 + 2i) \\ - (x^2+1)(n+1)(d_{2n+1}(x) + id_{2n}(x)).\end{aligned}$$

Taking real and imaginary parts in the previous equation we obtain (3.11) and (3.12). □

From Corollary 3.4 we obtain the following result

Corollary 3.9. *For $d_{2n}(x)$ and $d_{2n+1}(x)$, $n \in \mathbb{N}_0$, it holds that*

$$d^2_{2n+1}(x) + d^2_{2n}(x) = \frac{1}{x^2+1}. \tag{3.13}$$

Theorem 3.10. *Functions $d_n(x)$, $n \in \mathbb{N}_0$, are solutions of the Sturm-Liouville differential equation* (1.3). *Moreover, $y_n(x) = C_1 d_{2n+1}(x) + C_2 d_{2n}(x)$ are the only solutions of* (1.3).

Proof. We will prove the assertion only for functions $d_{2n+1}(x)$, since the proof for functions $d_{2n}(x)$, $n \in \mathbb{N}_0$, is the same. If we derivate (3.11), we obtain

$$(x^2+1)d''_{2n+1}(x) + 3xd'_{2n+1}(x) = d_{2n+1}(x) - (2n+1)d_{2n}(x). \tag{3.14}$$

From (3.11), (3.12) and (3.14) it holds that

$$(x^2+1)^2 d''_{2n+1}(x) + 4x(x^2+1)d'_{2n+1}(x) + (2x^2+1+(2n+1)^2)d_{2n+1}(x) = 0.$$

Using a similar proof like in Theorem 3.5 we obtain the converse part. □

Lemma 3.11. *(Orthogonality) Special functions $d_n(x)$, $n \in \mathbb{N}_0$, satisfy*

$$\int_{-\infty}^{\infty} d_m(x)d_n(x)dx = \frac{\pi}{2}\delta_{mn},$$

where $\delta_{m,n}$ is the Kronecker delta.

Proof. Let $m \neq n$. Then fromm (3.9), (3.10) follows

$$\begin{aligned} &\int_{-\infty}^{\infty} (-1)^{n+m} d_{2n}(x)d_{2m}(x)dx \\ &= \int_{-\infty}^{\infty} \frac{\cos((2n+1)\arctan x)\cos((2m+1)\arctan x)}{x^2+1}dx \\ &= \int_{-\pi/2}^{\pi/2} \cos((2n+1)x)\cos((2m+1)x)dx = 0. \end{aligned}$$

Similarly

$$\int_{-\infty}^{\infty} d_{2n+1}(x)d_{2m+1}(x)dx = 0, \quad \int_{-\infty}^{\infty} d_{2n}(x)d_{2m+1}(x)dx = 0.$$

Also

$$\int_{-\infty}^{\infty} d^2_{2n+1}(x)dx = \int_{-\infty}^{\infty} d^2_{2n}(x)dx = \frac{\pi}{2}.$$

□

Theorem 3.12. *The set $\{d_n(x)\}_{n=0}^{\infty}$ is a complete orthogonal system in $L^2(\mathbb{R})$.*

Proof. It is enough to prove that from

$$\int_{-\infty}^{\infty} f(x)\frac{x^{2n+k}}{(x^2+1)^{n+1}} = 0, \qquad k = 0, 1,$$

it follows that $f = 0$ almost everywhere. Suppose that $f \in L^2(\mathbb{R})$ is an even function and

$$\int_{-\infty}^{\infty} f(x)\frac{x^{2n+k}}{(x^2+1)^{n+1}} = 0, \quad k \in \{0, 1\}.$$

From

$$F(z) = \int_{-\infty}^{\infty} \frac{f(x)}{x^2+1} e^{\frac{x^2}{x^2+1}z} dx = \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{-\infty}^{\infty} \frac{f(x)}{x^2+1} \left(\frac{x^2}{x^2+1}\right)^n dx = 0,$$

it is obvious that

$$\begin{aligned} 0 = F(-it) &= \int_{-\infty}^{\infty} \frac{f(x)}{x^2+1} e^{\frac{x^2}{x^2+1}(-it)} dx = 2\int_{0}^{\infty} \frac{f(x)}{x^2+1} e^{\frac{x^2}{x^2+1}(-it)} dx \\ &= \int_0^1 \frac{f(\sqrt{\frac{u}{1-u}})}{\sqrt{u(1-u)}} e^{-itu} du = \mathcal{F}(g)(t) \end{aligned}$$

where $g(u) = \frac{f(\sqrt{\frac{u}{1-u}})}{\sqrt{u(1-u)}} \chi_{(0,1)}(u)$. From $\mathcal{F}(g)(t) = 0$ follows that $g(u) = 0$ almost everywhere, so $f(\sqrt{\frac{u}{1-u}}) = 0$ almost everywhere for $u \in (0,1)$. Now, $f(x) = 0$ almost everywhere for $x \in (0,\infty)$, so $f = 0$ almost everywhere on $(-\infty,\infty)$, because f is an even function. The proof is similar when $f \in L^2(\mathbb{R})$ is an odd function. □

Corollary 3.13. *The set*

$$\Big\{\psi_{2n}(x), \psi_{2n+1}(x)\Big\}_{n=0}^{\infty}$$

is the orthonormal basis for $L^2(\mathbb{R})$, where

$$\psi_n(x) = \sqrt{\frac{2}{\pi}} d_n(x).$$

4. On summation of special polynomials and special functions

Theorem 4.1. *Special polynomials $D_{2n}(x)$ and $D_{2n+1}(x)$, $n \in \mathbb{N}_0$, are given by the exponential generating functions:*

$$\sum_{n=0}^{\infty} \frac{D_{2n+1}(x)t^{2n+1}}{(2n+1)!} = \cos(t)\sinh(xt) \tag{4.1}$$

and

$$\sum_{n=0}^{\infty} \frac{D_{2n}(x)t^{2n+1}}{(2n+1)!} = -\sin(t)\cosh(xt). \tag{4.2}$$

Proof. If we take the real and the imaginary part in

$$\sinh(t(x-i)) = \sum_{n=0}^{\infty} \frac{(x-i)^{2n+1}}{(2n+1)!} t^{2n+1} = \sum_{n=0}^{\infty} \frac{D_{2n+1}(x)}{(2n+1)!} t^{2n+1} + i \frac{D_{2n}(x)}{(2n+1)!} t^{2n+1}$$

we obtain (4.1) and (4.2). □

Using Cauchy's integral formula on the closed contour C encircling the origin in (4.1) and (4.2) we have

Corollary 4.2. *Special polynomials $D_{2n+1}(x)$ and $D_{2n}(x)$, $n \in \mathbb{N}_0$, satisfy:*

$$D_{2n+1}(x) = \frac{(2n+1)!}{2\pi i} \oint_C \frac{\cos(z)\sinh(xz)}{z^{2n+2}} dz$$

and

$$D_{2n}(x) = \frac{(2n+1)!}{2\pi i} \oint_C \frac{-\sin(z)\cosh(xz)}{z^{2n+2}} dz.$$

Corollary 4.3. *Special polynomials $D_{2n+1}(x)$ and $D_{2n}(x)$, $n \in \mathbb{N}_0$, are given by:*

$$D_{2n+1}(x) = \frac{d^{2n+1}}{dt^{2n+1}} \Big(\cos(t)\sinh(xt) \Big) \Big|_{t=0}$$

and

$$D_{2n}(x) = \frac{d^{2n+1}}{dt^{2n+1}} \Big(-\sin(t)\cosh(xt) \Big) \Big|_{t=0}.$$

Proposition 4.4. *For $|t| < 1$, the special functions $d_{2n}(x)$ and $d_{2n+1}(x)$, $n \in \mathbb{N}_0$, are given by generating functions:*

$$\sum_{n=0}^{\infty} d_{2n+1}(x)t^n = \frac{x-xt}{(x-xt)^2+(1+t)^2}, \quad \sum_{n=0}^{\infty} d_{2n}(x)t^n = \frac{-(1+t)}{(x-xt)^2+(1+t)^2}.$$

Proof. The assertion follows by thaking real and imaginary parts in

$$\frac{1}{x+i}\sum_{n=0}^{\infty}\left(\frac{x-i}{x+i}\right)^n t^n = \frac{x-xt}{(x-xt)^2+(1+t)^2} - i\frac{(1+t)}{(x-xt)^2+(1+t)^2}.$$

□

5. Appendix

Theorem 5.1. *The mapping $\Lambda : L^2(-\pi/2, \pi/2) \to L^2(\mathbb{R})$ given by*

$$\Lambda(f)(x) = \frac{f(\arctan(x))}{\sqrt{x^2+1}}$$

is an isometric isomorphism.

Theorem 5.2. *Let $f_0(x) = \frac{1}{\sqrt{x^2+1}}$ and, for $n \in \mathbb{N}$,*

$$\begin{aligned} f_{2n}(x) &:= (-1)^n \frac{\cos((2n)\arctan x)}{\sqrt{x^2+1}}, \\ f_{2n-1}(x) &:= (-1)^{n+1} \frac{\sin((2n)\arctan x)}{\sqrt{x^2+1}}. \end{aligned}$$

The set $\left\{ \sqrt{\frac{2}{\pi}} f_0(x), \sqrt{\frac{2}{\pi}} f_{2n}(x), \sqrt{\frac{2}{\pi}} f_{2n-1}(x) \right\}_{n=1}^{\infty}$ is the orthonormal basis in $L^2(\mathbb{R})$.

Proof. The set $\left\{\sqrt{\frac{2}{\pi}}\cos((2n)x), \sqrt{\frac{2}{\pi}}\sin((2n)x)\right\}_{n=0}^{\infty}$ is the orthonormal basis in $L^2(-\pi/2, \pi/2)$. Using Theorem 5.1 we obtain the assertion. □

Remark 5.3. Notice that

$$f_{2n-1}(x) = \frac{F_{2n-1}(x)}{(x^2+1)^{n+1/2}}, \quad f_{2n}(x) = \frac{F_{2n}(x)}{(x^2+1)^{n+1/2}}, \quad n \in \mathbb{N},$$

where

$$F_{2n}(x) = \Re((x-i)^{2n}) = \sum_{k=0}^{n}(-1)^{n+k}\binom{2n}{2k}x^{2k}$$

and

$$F_{2n-1}(x) = \Im((x-i)^{2n}) = \sum_{k=1}^{n}(-1)^{n+k+1}\binom{2n}{2k-1}x^{2k-1}.$$

The following theorems are given without the proof. We are refering same methods as in Section 3 and Section 4.

Theorem 5.4. *Special functions $f_n(x)$, $n \in \mathbb{N}_0$, are solutions of the Sturm-Liouville differential equation*

$$(x^2+1)^2y''(x) + 4x(x^2+1)y'(x) + (2x^2+1+4n^2)y(x) = 0. \tag{5.1}$$

Moreover, $y_n(x) = C_1f_{2n-1}(x) + C_2f_{2n}(x)$, $n \in \mathbb{N}$, are the only solutions of (5.1).

Theorem 5.5. *Special polynomials $F_{2n-1}(x)$ and $F_{2n}(x)$, $n \in \mathbb{N}$, are given by the exponential generating functions:*

$$\sum_{n=1}^{\infty}\frac{F_{2n-1}(x)t^{2n}}{(2n)!} = -\sin(t)\sinh(xt) \tag{5.2}$$

and

$$\sum_{n=0}^{\infty}\frac{F_{2n}(x)t^{2n}}{(2n)!} = \cos(t)\cosh(xt). \tag{5.3}$$

Using Cauchy's integral formula on the closed contour C encircling the origin in (5.2) and (5.3) we have

Corollary 5.6. *Special polynomials $D_{2n+1}(x)$ and $D_{2n}(x)$, $n \in \mathbb{N}_0$, satisfy:*

$$F_{2n+1}(x) = \frac{(2n)!}{2\pi i}\oint_C \frac{-\sin(z)\sinh(xz)}{z^{2n+2}}dz$$

and

$$F_{2n}(x) = \frac{(2n)!}{2\pi i}\oint_C \frac{\cos(z)\cosh(xz)}{z^{2n+2}}dz.$$

Corollary 5.7. *Special polynomials $F_{2n+1}(x)$ and $F_{2n}(x)$, $n \in \mathbb{N}_0$, are given by:*

$$F_{2n+1}(x) = \frac{d^{2n}}{dt^{2n}}\Big(-\sin(t)\sinh(xt)\Big)\Big|_{t=0}$$

and

$$F_{2n}(x) = \frac{d^{2n}}{dt^{2n}}\Big(\cos(t)\cosh(xt)\Big)\Big|_{t=0}.$$

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