STUDY OF GLOBAL ASYMPTOTIC STABILITY IN NONLINEAR NEUTRAL DYNAMIC EQUATIONS WITH VARIABLE DELAYS

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Abstract. In this paper, we consider a class of nonlinear neutral dynamic equations. By using the Banach contraction mapping principle, we give some new conditions to ensure that the zero solution of the considered equation is globally asymptotically stable in C_{rd}^1. Our work extends previous results in the literature.

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1. Introduction

The concept of time scale analysis is a fairly new idea. In 1988, it was introduced by the German mathematician Stefan Hilger in his Ph.D. thesis [14]. It combines the traditional areas of continuous and discrete analysis into one theory. After the publication of two textbooks in this area by Bohner and Peterson [8] and [9], more and more researchers were getting involved in this fast-growing field of mathematics. The study of dynamic equations brings together the traditional research areas of differential and difference equations. It allows one to handle these two research areas at the same time, hence shedding light on the reasons for their seeming discrepancies. In fact, many new results for the continuous and discrete cases have been obtained by studying the more general time scales case (see [1, 4, 5, 7, 15] and the references therein).

There is no doubt that the Lyapunov method has been used successfully to investigate stability properties of wide variety of ordinary, functional and partial equations. Nevertheless, the application of this method to the problem of stability in differential equations with delay has encountered serious difficulties if the delay is unbounded or if the equation has an unbounded term. It has been noticed that some of theses difficulties vanish by using the fixed point technique. Other advantages of fixed point theory over Lyapunov's method is that the conditions of the former are average while those of the latter are pointwise (see [2, 3, 6, 10, 11, 12, 13, 16] and references therein).

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In this paper, we consider the following neutral nonlinear dynamic equations with variable delays given by

$$x^{\triangle}(t) = -a(t) x^{\sigma}(t) + g\left(t, x^{\widetilde{\triangle}}(t - \tau_1(t)), ..., x^{\widetilde{\triangle}}(t - \tau_N(t))\right)$$

(1.1) $$+ f(t, x(t), x(t - \tau_1(t)), ..., x(t - \tau_N(t))),$$

with the initial condition

$$x(t) = \varphi(t), \ t \in [d_{t_0}, t_0] \cap \mathbb{T},$$

where

$$d_j(t_0) = \inf_{t \in [t_0, \infty) \cap \mathbb{T}} \{t - \tau_j(t)\}, \ d_{t_0} = \min\{d_j(t_0), \ 1 \leq j \leq N\},$$

for each $t_0 \in [0, \infty) \cap \mathbb{T}$ and $\mathbb{T}$ is an unbounded above and below time scale and such that $t_0 \in \mathbb{T}$.

Our results are obtained with no need of further assumptions on the twice delta-differentiable τ_j with $\tau_j^{\triangle}(t) \neq 1$ for $t \in [0, \infty) \cap \mathbb{T}$. For a given initial function $\varphi \in \Phi_{t_0}$, a mapping P for (1.1) is constructed in such a way to map a, carefully chosen, closed convex nonempty subset D of a Banach space X into itself on which P is a contraction mapping possessing a fixed point. This procedure will enable us to establish and prove by means of the contraction mapping theorem ([17], p. 2) the global asymptotic stability in C_{rd}^1 for the zero solution of (1.1) with less restrictive conditions.

For $\mathbb{T} = \mathbb{R}$, special cases of equation (1.1) have been previously considered and studied under various conditions. Particularly, in [16] Liu and Yan investigated the global asymptotical stability in C^1 of the zero solution of the following nonlinear neutral differential equation

$$x'(t) = -a(t) x(t) + c(t) x'(t - \tau_1(t)) + q(t, x(t), x(t - \tau_2(t)))$$

by using fixed point theory. In [6], Ardjouni and Djoudi studied, by using the contraction mapping theorem, the globally asymptotically stability in C^1 of the zero solution, under appropriate conditions, of the following equation

$$x'(t) = -a(t) x(t) + g(t, x'(t - \tau_1(t)), ..., x'(t - \tau_N(t)))$$
$$+ f(t, x(t), x(t - \tau_1(t)), ..., x(t - \tau_N(t))),$$

and generalized the results of Liu and Yan. The results obtained here extend the works of Liu and Yan [16] and Ardjouni and Djoudi [6].

2.   Preliminaries

In this section, we consider some advanced topics in the theory of dynamic equations on time scales. Again, we remind that for a review of this topic we direct the reader to the monographs of Bohner and Peterson [8] and [9].

A time scale $\mathbb{T}$ is a closed nonempty subset of $\mathbb{R}$. For $t \in \mathbb{T}$ the forward jump operator σ, and the backward jump operator ρ, respectively, are defined as $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ and $\rho(t) = \sup\{t \in \mathbb{T} : s < t\}$. These operators allow elements in the time scale to be classified as follows. We say t is right scattered if $\sigma(t) > t$ and right dense if $\sigma(t) = t$. We say t is left scattered if $\rho(t) < t$ and left dense if $\rho(t) = t$. The graininess function $\mu : \mathbb{T} \to [0, \infty)$, is defined by $\mu(t) = \sigma(t) - t$ and gives the distance between an element and its successor. We set $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$. If $\mathbb{T}$ has a left scattered maximum M, we define $\mathbb{T}^k = \mathbb{T} \setminus \{M\}$. Otherwise, we define $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right scattered minimum m, we define $\mathbb{T}_k = \mathbb{T} \setminus \{m\}$. Otherwise, we define $\mathbb{T}_k = \mathbb{T}$.

Let $t \in \mathbb{T}^k$ and let $f : \mathbb{T} \to \mathbb{R}$. The delta derivative of $f(t)$, denoted $f^{\triangle}(t)$, is defined to be the number (when it exists), with the property that, for each $\varepsilon > 0$, there is a neighborhood U of t such that

$$\left| f(\sigma(t)) - f(s) - f^{\triangle}(t)[\sigma(t) - s] \right| \leq \varepsilon |\sigma(t) - s|,$$

for all $s \in U$. If $\mathbb{T} = \mathbb{R}$ then $f^{\triangle}(t) = f'(t)$ is the usual derivative. If $\mathbb{T} = \mathbb{Z}$ then $f^{\triangle}(t) = \triangle f(t) = f(t+1) - f(t)$ is the forward difference of f at t.

A function f is right dense continuous (rd-continuous), $f \in C_{rd} = C_{rd}(\mathbb{T}, \mathbb{R})$, if it is continuous at every right dense point $t \in \mathbb{T}$ and its left-hand limits exist at each left dense point $t \in \mathbb{T}$. The function $f : \mathbb{T} \to \mathbb{R}$ is differentiable on $\mathbb{T}^k$ provided $f^{\triangle}(t)$ exists for all $t \in \mathbb{T}^k$. $f \in C_{rd}^1 = C_{rd}^1(\mathbb{T}, \mathbb{R})$ if $f^{\triangle} \in C_{rd}(\mathbb{T}, \mathbb{R})$.

We are now ready to state some properties of the delta-derivative of f. Note $f^{\sigma}(t) = f(\sigma(t))$.

Theorem 2.1 ([8, Theorem 1.20]). *Assume $f, g : \mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}^k$ and let α be a scalar.*

(i) $(f + g)^{\triangle}(t) = g^{\triangle}(t) + f^{\triangle}(t)$.

(ii) $(\alpha f)^{\triangle}(t) = \alpha f^{\triangle}(t)$.

(iii) The product rules

$$\begin{aligned} (fg)^{\triangle}(t) &= f^{\triangle}(t) g(t) + f^{\sigma}(t) g^{\triangle}(t), \\ (fg)^{\triangle}(t) &= f(t) g^{\triangle}(t) + f^{\triangle}(t) g^{\sigma}(t). \end{aligned}$$

(iv) If $g(t) g^{\sigma}(t) \neq 0$ then

$$\left(\frac{f}{g} \right)^{\triangle}(t) = \frac{f^{\triangle}(t) g(t) - f(t) g^{\triangle}(t)}{g(t) g^{\sigma}(t)}.$$

The next theorem is the chain rule on time scales ([8, Theorem 1.93]).

Theorem 2.2 (Chain Rule). *Assume $\nu : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\widetilde{\mathbb{T}} := \nu(\mathbb{T})$ is a time scale. Let $\omega : \widetilde{\mathbb{T}} \to \mathbb{R}$. If $\nu^{\triangle}(t)$ and $\omega^{\widetilde{\triangle}}(\nu(t))$ exist for $t \in \mathbb{T}^k$, then $(\omega \circ \nu)^{\triangle} = \left(\omega^{\widetilde{\triangle}} \circ \nu \right) \nu^{\triangle}$.*

In the sequel we will need to differentiate and integrate functions of the form $f(t - \tau(t)) = f(\nu(t))$ where, $\nu(t) := t - \tau(t)$. Our next theorem is the substitution rule ([8, Theorem 1.98]).

Theorem 2.3 (Substitution)**.** *Assume* $\nu : \mathbb{T} \to \mathbb{R}$ *is strictly increasing and* $\widetilde{\mathbb{T}} := \nu(\mathbb{T})$ *is a time scale. If* $f : \mathbb{T} \to \mathbb{R}$ *is rd-continuous function and* ν *is differentiable with rd-continuous derivative, then for* $a, b \in \mathbb{T}$,

$$\int_a^b f(t) \nu^\triangle(t) \triangle t = \int_{\nu(a)}^{\nu(b)} \left(f \circ \nu^{-1}\right)(s) \widetilde{\triangle} s.$$

A function $p : \mathbb{T} \to \mathbb{R}$ is said to be regressive provided $1 + \mu(t) p(t) \neq 0$ for all $t \in \mathbb{T}^k$. The set of all regressive rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R}$. The set of all positively regressive functions $\mathcal{R}^+$, is given by $\mathcal{R}^+ = \{f \in \mathcal{R} : 1 + \mu(t) f(t) > 0 \text{ for all } t \in \mathbb{T}\}$.

Let $p \in \mathcal{R}$ and $\mu(t) \neq 0$ for all $t \in \mathbb{T}$. The exponential function on $\mathbb{T}$ is defined by

$$e_p(t, s) = \exp\left(\int_s^t \frac{1}{\mu(z)} \log(1 + \mu(z) p(z)) \triangle z\right).$$

It is well known that if $p \in \mathcal{R}^+$, then $e_p(t, s) > 0$ for all $t \in \mathbb{T}$. Also, the exponential function $y(t) = e_p(t, s)$ is the solution to the initial value problem $y^\triangle = p(t) y$, $y(s) = 1$. Other properties of the exponential function are given by the following lemma.

Lemma 2.4 ([8, Theorem 2.36])**.** *Let* $p, q \in \mathcal{R}$. *Then*
    (i) $e_0(t, s) = 1$ *and* $e_p(t, t) = 1$,
    (ii) $e_p(\sigma(t), s) = (1 + \mu(t) p(t)) e_p(t, s)$,
    (iii) $\frac{1}{e_p(t,s)} = e_{\ominus p}(t, s)$, *where* $\ominus p(t) = -\frac{p(t)}{1 + \mu(t) p(t)}$,
    (iv) $e_p(t, s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s, t)$,
    (v) $e_p(t, s) e_p(s, r) = e_p(t, r)$,
    (vi) $e_p^\triangle(., s) = p e_p(., s)$ *and* $\left(\frac{1}{e_p(.,s)}\right)^\triangle = -\frac{p(t)}{e_p^\sigma(.,s)}$.

Lemma 2.5 ([1])**.** *If* $p \in \mathcal{R}^+$, *then*

$$0 < e_p(t, s) \leq \exp\left(\int_s^t p(u) \triangle u\right), \ \forall t \in \mathbb{T}.$$

3.   Global asymptotic stability

In this section, we shall study the global asymptotic stability in C_{rd}^1 of the zero solution to (1.1). We introduce the following hypotheses.

    (H_1) $a \in C_{rd}([0, \infty) \cap \mathbb{T}, \mathbb{R})$, $g \in C_{rd}\left(([0, \infty) \cap \mathbb{T}) \times \mathbb{R}^N, \mathbb{R}\right)$,

    $f \in C_{rd}\left(([0, \infty) \cap \mathbb{T}) \times \mathbb{R}^{N+1}, \mathbb{R}\right)$, $\tau_j \in C_{rd}([0, \infty) \cap \mathbb{T}, (0, \infty) \cap \mathbb{T})$

    and $(id - \tau_j)([0, \infty) \cap \mathbb{T})$ is closed with $t - \tau_j(t) \to \infty$ as $t \to \infty$, $j = 1, ..., N$.

(H_2) For $t \in [0, \infty) \cap \mathbb{T}$, $g(t, 0, ..., 0) = f(t, 0, ..., 0) = 0$, and there exist $K_j, L, L_j \in C_{rd}([0, \infty) \cap \mathbb{T}, (0, \infty))$ such that

$$|g(t, x_1, ..., x_N) - g(t, y_1, ..., y_N)| \leq \sum_{j=1}^{N} K_j(t) |x_j - y_j|,$$

and

$$|f(t, x, x_1, ..., x_N) - f(t, y, y_1, ..., y_N)| \leq L(t) |x - y| + \sum_{j=1}^{N} L_j(t) |x_j - y_j|,$$

for any $x, x_i, y, y_i \in \mathbb{R}$, $j = 1, ..., N$.

(H_3) $a \in \mathcal{R}^+$ is bounded on $[0, \infty) \cap \mathbb{T}$ and

$$\liminf_{t \to \infty} \int_0^t \frac{1}{\mu(s)} \log(1 + \mu(s) a(s)) \Delta s > -\infty.$$

(H_4) There exists an $\alpha \in (0, 1)$ such that for $t \in [0, \infty) \cap \mathbb{T}$,

$$\int_0^t e_{\ominus a}(t, u) \left[\sum_{j=1}^{N} K_j(u) + L(u) + \sum_{j=1}^{N} L_j(u) \right] \Delta u \leq \alpha,$$

and

$$|a(t)| \int_0^{\sigma(t)} e_{\ominus a}(\sigma(t), u) \left[\sum_{j=1}^{N} K_j(u) + L(u) + \sum_{j=1}^{N} L_j(u) \right] \Delta u$$

$$+ \sum_{j=1}^{N} K_j(t) + L(t) + \sum_{j=1}^{N} L_j(t) \leq \alpha.$$

For each $t_0 \in [0, \infty) \cap \mathbb{T}$ denote $C_{rd}^1(t_0) = C_{rd}^1([d_{t_0}, t_0] \cap \mathbb{T}, \mathbb{R})$ with the norm defined by

$$|x|_{t_0} = \max_{t \in [d_{t_0}, t_0] \cap \mathbb{T}} \left\{ |x(t)|, |x^{\triangle}(t)| \right\}$$

for $x \in C_{rd}^1(t_0)$. In addition, denote

$$\Phi_{t_0} = \left\{ \varphi \in C_{rd}^1(t_0) : \varphi^{\triangle}(t_0) = -a(t_0) \varphi^{\sigma}(t_0) \right.$$
$$+ g\left(t_0, \varphi^{\widetilde{\triangle}}(t_0 - \tau_1(t_0)), ..., \varphi^{\widetilde{\triangle}}(t_0 - \tau_N(t_0)) \right)$$
$$\left. + f(t_0, \varphi(t_0), \varphi(t_0 - \tau_1(t_0)), ..., \varphi(t_0 - \tau_N(t_0))) \right\}.$$

For each $t_0 \in [0, \infty) \cap \mathbb{T}$, we always assume that the initial function for (1.1) is of the type $\varphi \in \Phi_{t_0}$. For convenience of stating our main result, we shall give the following definitions.

Definition 3.1. For each $(t_0, \varphi) \in [0, \infty) \cap \mathbb{T} \times \Phi_{t_0}$, x is said to be a solution of (1.1) through (t_0, φ) if $x \in C^1_{rd}([d_{t_0}, \infty) \cap \mathbb{T})$ satisfies (1.1) on $[t_0, \infty) \cap \mathbb{T}$ and $x(t) = \varphi(t)$ for $t \in [d_{t_0}, t_0] \cap \mathbb{T}$. We denote such a solution by $x(t) = x(t, t_0, \varphi)$.

Definition 3.2. (i) The zero solution of (1.1) is said to be stable in C^1_{rd} if, for any $t_0 \in [0, \infty) \cap \mathbb{T}$ and $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon, t_0)$ such that $\varphi \in \Phi_{t_0}$ and $|\varphi|_{t_0} < \delta$ implies

$$\max_{s \in [d_{t_0}, t] \cap \mathbb{T}} \left\{ |x(s)|, \left| x^\triangle(s) \right| \right\} < \varepsilon,$$

for $t \in [t_0, \infty) \cap \mathbb{T}$.

(ii) The zero solution of (1.1) is said to be globally asymptotically stable in C^1_{rd} if it is stable in C^1_{rd}, and for any $t_0 \in [0, \infty) \cap \mathbb{T}$, $\varphi \in \Phi_{t_0}$ implies

$$\lim_{t \to \infty} x(t, t_0, \varphi) = \lim_{t \to \infty} x^\triangle(t, t_0, \varphi) = 0.$$

In view of the definition of solution of (1.1), it is clear that the conditions imposed on the initial functions are very natural. From the above assumptions, it is easy to see that for each $(t_0, \varphi) \in [0, \infty) \cap \mathbb{T} \times \Phi_{t_0}$, there exists a unique solution $x(t) = x(t, t_0, \varphi)$ of (1.1) defined on $[d_{t_0}, \infty) \cap \mathbb{T}$. By (H_2), (1.1) has the zero solution.

Theorem 3.3. *Assume that* $(H_1) - (H_4)$ *hold. Then the zero solution of (1.1) is globally asymptotically stable in C^1_{rd} if and only if*

$$(3.1) \qquad \int_0^t \frac{1}{\mu(s)} \log(1 + \mu(s) a(s)) \Delta s \to \infty \ as \ t \to \infty.$$

Proof. (i) Suppose that (3.1) holds. For any $t_0 \in [0, \infty) \cap \mathbb{T}$, let

$$X = \left\{ x \in C^1_{rd}([d_{t_0}, \infty) \cap \mathbb{T}) : \lim_{t \to \infty} x(t) = \lim_{t \to \infty} x^\triangle(t) = 0 \right\},$$

with the norm defined by

$$\|x\|_{t_0} = \max_{t \in [d_{t_0}, \infty) \cap \mathbb{T}} \left\{ |x(t)|, \left| x^\triangle(t) \right| \right\},$$

for $x \in X$. Since X is a closed vector subspace of $C^1_{rd}([d_{t_0}, \infty) \cap \mathbb{T})$ and $C^1_{rd}([d_{t_0}, \infty) \cap \mathbb{T}, \|.\|_{t_0})$ is a Banach space, then $\left(X, \|.\|_{t_0}\right)$ is also a Banach space. For any $\varphi \in \Phi_{t_0}$, let

$$D = \{ x \in X : x(t) = \varphi(t) \ \text{for} \ t \in [d_{t_0}, t_0] \cap \mathbb{T} \}.$$

It is easy to see that D is a nonempty, closed subset of X. Define the operator $P : D \to C_{rd}([d_{t_0}, \infty) \cap \mathbb{T})$ by $(Px)(t) = \varphi(t)$ for $t \in [d_{t_0}, t_0] \cap \mathbb{T}$ and

$$(Px)(t) = \varphi(t_0) e_{\ominus a}(t, t_0)$$

$$+ \int_{t_0}^t e_{\ominus a}(t, u) \left[g\left(u, x^{\widetilde{\triangle}}(u - \tau_1(u)), ..., x^{\widetilde{\triangle}}(u - \tau_N(u)) \right) \right.$$

$$(3.2) \qquad + f(u, x(u), x(u - \tau_1(u)), ..., x(u - \tau_N(u))) \bigg] \Delta u,$$

for $t \in [t_0, \infty) \cap \mathbb{T}$.

Firstly, we prove $Px \in D$ for any $x \in D$. From (3.2), for $t > t_0$,

$$
\begin{aligned}
(Px)^{\triangle}(t) &= -\varphi(t_0)\, a(t)\, e_{\ominus a}(\sigma(t), t_0) \\
&\quad + e_{\ominus a}(\sigma(t), t)\left[g\left(t, x^{\widetilde{\triangle}}(t - \tau_1(t)), ..., x^{\widetilde{\triangle}}(t - \tau_N(t))\right) \right. \\
&\quad\left. + f(t, x(t), x(t - \tau_1(t)), ..., x(t - \tau_N(t)))\right] \\
&\quad - a(t) \int_{t_0}^{t} e_{\ominus a}(\sigma(t), u)\left[g\left(u, x^{\widetilde{\triangle}}(u - \tau_1(u)), ..., x^{\widetilde{\triangle}}(u - \tau_N(u))\right) \right. \\
&\quad\left. + f(u, x(u), x(u - \tau_1(u)), ..., x(u - \tau_N(u)))\right] \triangle u \\
&= -a(t)(Px)^{\sigma}(t) + g\left(t, x^{\widetilde{\triangle}}(t - \tau_1(t)), ..., x^{\widetilde{\triangle}}(t - \tau_N(t))\right) \\
&\quad + f(t, x(t), x(t - \tau_1(t)), ..., x(t - \tau_N(t))).
\end{aligned}
$$

(3.3)

By the definition of Φ_{t_0}, (3.3) yields

$$
\begin{aligned}
(Px)_{+}^{\triangle}(t_0) &= -a(t_0)\, \varphi^{\sigma}(t_0) + g\left(t_0, \varphi^{\widetilde{\triangle}}(t_0 - \tau_1(t_0)), ..., \varphi^{\widetilde{\triangle}}(t_0 - \tau_N(t_0))\right) \\
&\quad + f(t_0, \varphi(t_0), \varphi(t_0 - \tau_1(t_0)), ..., \varphi(t_0 - \tau_N(t_0))) \\
&= \varphi_{-}^{\triangle}(t_0).
\end{aligned}
$$

Hence, $Px \in C_{rd}^1([d_{t_0}, \infty) \cap \mathbb{T})$ for $x \in D$.

For $x \in D$, $\lim_{t \to \infty} x(t) = \lim_{t \to \infty} x^{\triangle}(t) = 0$. Note that $\lim_{t \to \infty} t - \tau_j(t) = \infty$, $j = 1, ..., N$. Therefore, for any $\varepsilon > 0$, there exists $T > 0$ such that for $t \geq T$,

(3.4)
$$
\max\left\{ |x(t)|, |x(t - \tau_j(t))|, \left|x^{\widetilde{\triangle}}(t - \tau_j(t))\right|, \ 1 \leq j \leq N \right\} < \varepsilon.
$$

It follows from (3.2), (3.4) and (H_2) and (H_4) that for $t > T$ and $x \in D$,

$$
\begin{aligned}
&|(Px)(t)| \\
&\leq |\varphi(t_0)|\, e_{\ominus a}(t, t_0) \\
&\quad + \int_{t_0}^{T} e_{\ominus a}(t, u)\left| g\left(u, x^{\widetilde{\triangle}}(u - \tau_1(u)), ..., x^{\widetilde{\triangle}}(u - \tau_N(u))\right)\right. \\
&\quad\left. + f(u, x(u), x(u - \tau_1(u)), ..., x(u - \tau_N(u)))\right| \triangle u \\
&\quad + \int_{T}^{t} e_{\ominus a}(t, u)\left| g\left(u, x^{\widetilde{\triangle}}(u - \tau_1(u)), ..., x^{\widetilde{\triangle}}(u - \tau_N(u))\right) - g(u, 0, ..., 0)\right. \\
&\quad\left. + f(u, x(u), x(u - \tau_1(u)), ..., x(u - \tau_N(u))) - f(u, 0, ..., 0)\right| \triangle u,
\end{aligned}
$$

thus

$$|(Px)(t)|$$

$$\leq e_{\ominus a}(t,t_0)\left[|\varphi(t_0)| + \int_{t_0}^{T} e_a(u,t_0)\left|g\left(u, x^{\widetilde{\triangle}}(u-\tau_1(u)),...,x^{\widetilde{\triangle}}(u-\tau_N(u))\right)\right.\right.$$

$$\left.+f(u,x(u),x(u-\tau_1(u)),...,x(u-\tau_N(u)))|\,\Delta u\right]$$

$$+\int_{T}^{t} e_{\ominus a}(t,u)\left[\sum_{j=1}^{N} K_j(u)\left|x^{\widetilde{\triangle}}(u-\tau_j(u))\right|\right.$$

$$\left.+L(u)|x(u)| + \sum_{j=1}^{N} L_j(u)|x(u-\tau_j(u))|\right]\Delta u$$

$$\leq e_{\ominus a}(t,t_0)\left[|\varphi(t_0)| + \int_{t_0}^{T} e_a(u,t_0)\left|g\left(u, x^{\widetilde{\triangle}}(u-\tau_1(u)),...,x^{\widetilde{\triangle}}(u-\tau_N(u))\right)\right.\right.$$

$$\left.+f(u,x(u),x(u-\tau_1(u)),...,x(u-\tau_N(u)))|\,\Delta u\right]$$

$$+\varepsilon\int_{T}^{t} e_{\ominus a}(t,u)\left[\sum_{j=1}^{N} K_j(u) + L(u) + \sum_{j=1}^{N} L_j(u)\right]\Delta u$$

$$\leq e_{\ominus a}(t,t_0)\left[|\varphi(t_0)| + \int_{t_0}^{T} e_a(u,t_0)\left|g\left(u, x^{\widetilde{\triangle}}(u-\tau_1(u)),...,x^{\widetilde{\triangle}}(u-\tau_N(u))\right)\right.\right.$$

$$\left.+f(u,x(u),x(u-\tau_1(u)),...,x(u-\tau_N(u)))|\,\Delta u\right]+\alpha\varepsilon.$$

From (3.1), there exists $T_1 > T$ such that for $t > T_1$,

$$e_{\ominus a}(t,t_0)\left[|\varphi(t_0)| + \int_{t_0}^{T} e_a(u,t_0)\left|g\left(u, x^{\widetilde{\triangle}}(u-\tau_1(u)),...,x^{\widetilde{\triangle}}(u-\tau_N(u))\right)\right.\right.$$

$$\left.+f(u,x(u),x(u-\tau_1(u)),...,x(u-\tau_N(u)))|\,\Delta u\right]<\varepsilon.$$

Hence, $\lim_{t\to\infty}(Px)(t) = 0$ for $x \in D$. In addition, it follows from (3.3) and (H_2) that

$$\left|(Px)^{\triangle}(t)\right|$$

$$\leq |a(t)(Px)^{\sigma}(t)| + \left|g\left(t, x^{\widetilde{\triangle}}(t-\tau_1(t)),...,x^{\widetilde{\triangle}}(t-\tau_N(t))\right) - g(t,0,...,0)\right|$$

$$+|f(t,x(t),x(t-\tau_1(t)),...,x(t-\tau_N(t))) - f(t,0,...,0)|$$

$$\leq |a(t)(Px)^{\sigma}(t)| + \sum_{j=1}^{N} K_j(t)\left|x^{\widetilde{\triangle}}(t-\tau_j(t))\right|$$

$$+L(t)|x(t)| + \sum_{j=1}^{N} L_j(t)|x(t-\tau_j(t))|.$$

This, together with (H_3) and (H_4), yields $\lim_{t\to\infty}(Px)^{\triangle}(t)=0$ for $x\in D$. Therefore, $Px\in D$ for $x\in D$, i.e. $P:D\to D$.

Secondly, we show that $P:D\to D$ is a contraction mapping. For any $x,y\in D$, it follows from (3.2), (H_2) and (H_4) that for $t\in[t_0,\infty)\cap\mathbb{T}$,

$$|(Px)(t)-(Py)(t)|$$

$$\leq\int_{t_0}^{t}e_{\ominus a}(t,u)\Big[\Big|g\Big(u,x^{\widetilde{\triangle}}(u-\tau_1(u)),...,x^{\widetilde{\triangle}}(u-\tau_N(u))\Big)$$

$$-g\Big(u,y^{\widetilde{\triangle}}(u-\tau_1(u)),...,y^{\widetilde{\triangle}}(u-\tau_N(u))\Big)\Big|$$

$$+|f(u,x(u),x(u-\tau_1(u)),...,x(u-\tau_N(u)))$$

$$-f(u,y(u),y(u-\tau_1(u)),...,y(u-\tau_N(u)))|\Big]\Delta u$$

$$\leq\|x-y\|_{t_0}\int_{t_0}^{t}e_{\ominus a}(t,u)\left[\sum_{j=1}^{N}K_j(u)+L(u)+\sum_{j=1}^{N}L_j(u)\right]\Delta u$$

$$(3.5)\qquad\leq\alpha\|x-y\|_{t_0}.$$

In addition, it follows from (3.3), (3.5), (H_2) and (H_4) that for $t\in[t_0,\infty)\cap\mathbb{T}$,

$$\left|(Px)^{\triangle}(t)-(Py)^{\triangle}(t)\right|$$

$$\leq|a(t)|\,|(Px)^{\sigma}(t)-(Py)^{\sigma}(t)|$$

$$+\left|g\Big(t,x^{\widetilde{\triangle}}(t-\tau_1(t)),...,x^{\widetilde{\triangle}}(t-\tau_N(t))\Big)\right.$$

$$-g\Big(t,y^{\widetilde{\triangle}}(t-\tau_1(t)),...,y^{\widetilde{\triangle}}(t-\tau_N(t))\Big)\Big|$$

$$+|f(t,x(t),x(t-\tau_1(t)),...,x(t-\tau_N(t)))$$

$$-f(t,y(t),y(t-\tau_1(t)),...,y(t-\tau_N(t)))|$$

$$\leq\|x-y\|_{t_0}\left\{|a(t)|\int_{t_0}^{\sigma(t)}e_{\ominus a}(\sigma(t),u)\left[\sum_{j=1}^{N}K_j(u)+L(u)+\sum_{j=1}^{N}L_j(u)\right]\Delta u\right.$$

$$\left.+\sum_{j=1}^{N}K_j(t)+L(t)+\sum_{j=1}^{N}L_j(t)\right\}$$

$$(3.6)$$
$$\leq\alpha\|x-y\|_{t_0}.$$

From (3.5) and (3.6), $P:D\to D$ is a contraction mapping. By the contraction mapping principle, P has a unique fixed point x in D, which is a unique solution of (1.1) through (t_0,φ) and satisfies

$$(3.7)\qquad\qquad\lim_{t\to\infty}x(t)=\lim_{t\to\infty}x^{\triangle}(t)=0.$$

Finally, we show that the zero solution of (1.1) is stable in C_{rd}^1. Let

$$K=\sup_{t\in[t_0,\infty)\cap\mathbb{T}}\{e_{\ominus a}(t,u)\}\text{ and }A=\sup_{t\in[t_0,\infty)\cap\mathbb{T}}\{|a(t)|\}.$$

From (3.1) and (H_3), $K, A \in (0, \infty) \cap \mathbb{T}$. For any $\varepsilon > 0$, let $\delta > 0$ such that

$$\delta < \varepsilon \min \left\{ 1, \frac{1-\alpha}{K}, \frac{1-\alpha}{KA} \right\}.$$

If $x(t) = x(t, t_0, \varphi)$ is a solution of (1.1) with $|\varphi|_{t_0} < \delta$, then $x(t) = (Px)(t)$ on $[t_0, \infty) \cap \mathbb{T}$. We claim that $\|x\|_{t_0} < \varepsilon$. Otherwise, there exists $t_1 > t_0$ such that

$$\max \left\{ |x(t_1)|, \left| x^\triangle(t_1) \right| \right\} = \varepsilon,$$

and

$$\max \left\{ |x(t)|, \left| x^\triangle(t) \right| \right\} < \varepsilon,$$

for $t \in [d_{t_0}, t_1) \cap \mathbb{T}$. If $|x(t_1)| = \varepsilon$, then it follows from (3.2), (H_2) and (H_4) that

$$
\begin{aligned}
|x(t_1)| &\leq |\varphi(t_0)| e_{\ominus a}(t_1, t_0) \\
&\quad + \int_{t_0}^{t_1} e_{\ominus a}(t_1, u) \left| g\left(u, x^{\tilde{\triangle}}(u - \tau_1(u)), ..., x^{\tilde{\triangle}}(u - \tau_N(u)) \right) \right. \\
&\quad + \left. f(u, x(u), x(u - \tau_1(u)), ..., x(u - \tau_N(u))) \right| \Delta u \\
&\leq K\delta + \varepsilon \int_{t_0}^{t_1} e_{\ominus a}(t_1, u) \left[\sum_{j=1}^N K_j(u) + L(u) + \sum_{j=1}^N L_j(u) \right] \Delta u \\
&\leq K\delta + \alpha\varepsilon \\
&< \varepsilon.
\end{aligned}
$$

This is a contradiction. If $\left| x^\triangle(t_1) \right| = \varepsilon$, then it follows from (3.3) and (H_2) and (H_4) that

$$
\begin{aligned}
\left| x^\triangle(t_1) \right| \\
\leq |\varphi(t_0) a(t_1)| e_{\ominus a}(\sigma(t_1), t_0) \\
+ \left| g\left(t_1, x^{\tilde{\triangle}}(t_1 - \tau_1(t_1)), ..., x^{\tilde{\triangle}}(t_1 - \tau_N(t_1)) \right) \right| \\
+ |f(t_1, x(t_1), x(t_1 - \tau_1(t_1)), ..., x(t_1 - \tau_N(t_1)))| \\
+ |a(t_1)| \int_{t_0}^{\sigma(t_1)} e_{\ominus a}(\sigma(t_1), u) \left| g\left(u, x^{\tilde{\triangle}}(u - \tau_1(u)), ..., x^{\tilde{\triangle}}(u - \tau_N(u)) \right) \right. \\
+ \left. f(u, x(u), x(u - \tau_1(u)), ..., x(u - \tau_N(u))) \right| \Delta u \\
\leq KA\delta \\
+ \varepsilon \left\{ |a(t_1)| \int_{t_0}^{\sigma(t_1)} e_{\ominus a}(\sigma(t_1), u) \left[\sum_{j=1}^N K_j(u) + L(u) + \sum_{j=1}^N L_j(u) \right] \Delta u \right. \\
+ \left. \sum_{j=1}^N K_j(t_1) + L(t_1) + \sum_{j=1}^N L_j(t_1) \right\} \\
\leq KA\delta + \alpha\varepsilon < \varepsilon.
\end{aligned}
$$

This is also a contradiction. Hence, the zero solution of (1.1) is stable in C_{rd}^1. This, together with (3.7), implies that the zero solution of (1.1) is globally asymptotically stable in C_{rd}^1.

(ii) Assume that the zero solution of (1.1) is globally asymptotically stable in C_{rd}^1. Now we prove that (3.1) holds. Otherwise, set

$$
\begin{aligned}
l &= \liminf_{t \mapsto \infty} \int_0^t \frac{1}{\mu(s)} \log\left(1 + \mu(s) a(s)\right) \Delta s, \quad K_0 = \sup_{t \in [0,\infty) \cap \mathbb{T}} \left\{ e_{\ominus a}(t, 0) \right\}, \\
A_0 &= \sup_{t \in [0,\infty) \cap \mathbb{T}} \left\{ |a(t)| \right\},
\end{aligned}
$$

thus it follows from (H_3) that $l \in (-\infty, \infty)$, $K_0 \in (0, \infty)$, $A_0 \in [0, \infty)$. Hence, there exists an increasing sequence $\{t_n\} \subset [0, \infty) \cap \mathbb{T}$ such that $\lim_{t \to \infty} t_n = \infty$ and

$$
(3.8) \qquad \lim_{n \mapsto \infty} \int_0^{t_n} \frac{1}{\mu(s)} \log\left(1 + \mu(s) a(s)\right) \Delta s = l.
$$

Denote

$$
I_n = \int_0^{t_n} e_a(u, 0) \left[\sum_{j=1}^N K_j(u) + L(u) + \sum_{j=1}^N L_j(u) \right] \Delta u, \quad n = 1, 2, \dots.
$$

From (H_4), it follows that

$$
\begin{aligned}
I_n &= e_a(t_n, 0) \int_0^{t_n} e_{\ominus a}(t_n, u) \left[\sum_{j=1}^N K_j(u) + L(u) + \sum_{j=1}^N L_j(u) \right] \Delta u \\
&\leq \alpha e_a(t_n, 0).
\end{aligned}
$$

This, together with (3.8), implies that the sequence $\{I_n\}$ is bounded. Further, there exists a convergent subsequence. For brevity of notation, we may assume that $\{I_n\}$ is convergent. Therefore, there exists a positive integer m such that for any integer $n > m$,

$$
(3.9) \qquad \int_{t_m}^{t_n} e_a(u, 0) \left[\sum_{j=1}^N K_j(u) + L(u) + \sum_{j=1}^N L_j(u) \right] \Delta u < \frac{1 - \alpha}{8B\left(e^{-l} + 1\right)},
$$

and

$$
(3.10) \qquad e_{\ominus a}(t_n, t_m) > \frac{1}{2}, \quad e_{\ominus a}(t_n, 0) < e^{-l} + 1, \quad e_a(t_m, 0) < e^l + 1,
$$

where $B = \max\left\{ K_0\left(e^l + 1\right), K_0 A_0\left(e^l + 1\right), 1 \right\}$.

For any $\delta > 0$, consider the solution $x(t) = x(t, t_m, \varphi)$ of (1.1) with $|\varphi|_{t_m} < \delta$ and $|\varphi(t_m)| > \delta/2$. It follows from (3.2), (3.3), (3.10), (H_2) and (H_4) that

for $t \in [t_m, \infty) \cap \mathbb{T}$,

$$
\begin{aligned}
&|x(t)| \\
&\leq |\varphi(t_m)| \, e_{\ominus a}(t, t_m) \\
&+ \int_{t_m}^{t} e_{\ominus a}(t, u) \left| g\left(u, x^{\widetilde{\triangle}}(u - \tau_1(u)), ..., x^{\widetilde{\triangle}}(u - \tau_N(u))\right) \right. \\
&+ f\left(u, x(u), x(u - \tau_1(u)), ..., x(u - \tau_N(u))\right) \Big| \, \Delta u \\
&\leq |\varphi(t_m)| \, e_{\ominus a}(t, 0) \, e_a(t_m, 0) \\
&+ \|x\|_{t_m} \int_{t_m}^{t} e_{\ominus a}(t, u) \left[\sum_{j=1}^{N} K_j(u) + L(u) + \sum_{j=1}^{N} L_j(u) \right] \Delta u \\
&\leq K_0 \left(e^l + 1 \right) \delta + \|x\|_{t_m} \int_{0}^{t} e_{\ominus a}(t, u) \left[\sum_{j=1}^{N} K_j(u) + L(u) + \sum_{j=1}^{N} L_j(u) \right] \Delta u \\
&\leq B\delta + \alpha \|x\|_{t_m},
\end{aligned}
$$

and

$$
\begin{aligned}
&\left| x^{\triangle}(t) \right| \\
&\leq |\varphi(t_m)| \, a(t) \, e_{\ominus a}(\sigma(t), t_m) + \left| g\left(t, x^{\widetilde{\triangle}}(t - \tau_1(t)), ..., x^{\widetilde{\triangle}}(t - \tau_N(t))\right) \right| \\
&+ |f(t, x(t), x(t - \tau_1(t)), ..., x(t - \tau_N(t)))| \\
&+ |a(t)| \int_{t_m}^{\sigma(t)} e_{\ominus a}(\sigma(t), u) \left| g\left(u, x^{\widetilde{\triangle}}(u - \tau_1(u)), ..., x^{\widetilde{\triangle}}(u - \tau_N(u))\right) \right. \\
&+ f\left(u, x(u), x(u - \tau_1(u)), ..., x(u - \tau_N(u))\right) \Big| \, \Delta u \\
&\leq K_0 A_0 \left(e^l + 1 \right) \delta \\
&+ \|x\|_{t_m} \left\{ |a(t)| \int_{t_m}^{\sigma(t)} e_{\ominus a}(\sigma(t), u) \left[\sum_{j=1}^{N} K_j(u) + L(u) + \sum_{j=1}^{N} L_j(u) \right] \Delta u \right. \\
&+ \sum_{j=1}^{N} K_j(t) + L(t) + \sum_{j=1}^{N} L_j(t) \Bigg\} \\
&\leq B\delta + \alpha \|x\|_{t_m}.
\end{aligned}
$$

Hence, $\|x\|_{t_m} \leq B\delta + \alpha \|x\|_{t_m}$, i.e.

$$
(3.11) \qquad\qquad \|x\|_{t_m} \leq \frac{B}{1-\alpha} \delta.
$$

It follows from (3.2),(3.9)-(3.11) and (H_2) that for any $n > m$,

$$|x(t_n)|$$
$$\geq |\varphi(t_m)| e_{\ominus a}(t_n, t_m)$$
$$- e_{\ominus a}(t_n, 0) \int_{t_m}^{t_n} e_a(u, 0) \left| g\left(u, x^{\tilde{\triangle}}(u - \tau_1(u)), ..., x^{\tilde{\triangle}}(u - \tau_N(u))\right)\right.$$
$$\left. + f(u, x(u), x(u - \tau_1(u)), ..., x(u - \tau_N(u)))\right| \Delta u$$
$$\geq |\varphi(t_m)| e_{\ominus a}(t_n, t_m)$$
$$- \|x\|_{t_m} e_{\ominus a}(t_n, 0) \int_{t_m}^{t_n} e_a(u, 0) \left[\sum_{j=1}^{N} K_j(u) + L(u) + \sum_{j=1}^{N} L_j(u)\right] \Delta u$$
$$> \frac{1}{4}\delta - \frac{B}{1-\alpha}\delta\left(e^{-l} + 1\right) \frac{1-\alpha}{8B\left(e^{-l}+1\right)} = \frac{1}{8}\delta.$$

This contradicts the fact that $\lim_{n\to\infty} t_n = \infty$ and the zero solution of (1.1) is globally asymptotically stable in C_{rd}^1. The proof is complete. $\qquad\square$

Acknowledgement

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