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CHARACTERIZATION OF $R\omega O(X)$ SETS BY USING $\delta\omega-$CLUSTER POINTS

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Abstract. The class of $R\omega-$open sets was defined by S. Murugesan. He showed that the collection of all $R\omega-$open sets forms a base of some topology on X denoted by $\tau_{\delta-\omega}$. The elements of $\tau_{\delta-\omega}$ are called $\delta_\omega-$open sets and the complement of a $\delta_\omega-$open set is called a $\delta_\omega-$closed set. In this paper we will introduce a new characterization of $\delta_\omega-$open and $\delta_\omega-$closed sets by using $\delta\omega-$cluster points. We show that all $\delta\omega-$open sets form a topology denoted by τ_{δ_ω} and equal to $\tau_{\delta-\omega}$. We discuss several properties of this topology and we give a characterization for the open sets in $\tau_{\delta-\omega}$. We investigate some of the relationship between the separation axioms of $(X, \tau_{\delta_\omega})$ and (X, τ). In the last section we study some of connectedness properties of $(X, \tau_{\delta_\omega})$ and some covering properties.

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1. Introduction and Preliminaries

Throughout this work a space will always mean a topological space in which no separation axioms are assumed unless explicitly stated. If A is a subset of the space (X, τ) then the closure of A, the interior of A and the relative topology on A in (X, τ) will be denoted by $cl(A)$, $Int(A)$ and τ_A, respectively.

Let A be a subset of the space (X, τ). A is called a regular open subset of (X, τ) if $A = Int(cl(A))$. The family of all regular open subsets of (X, τ) is denoted by $RO(X, \tau)$. The complement of a regular open set is called a regular closed set, this is equivalent to say that $A = cl(Int(A))$. A is called $\delta-$open [10] if and only if for each $x \in A$ there exists a regular open set G such that $x \in G \subseteq A$. It is well-know that the collection of all $\delta-$open sets in a topological space (X, τ) forms a topology τ_δ weaker than τ such that the regular

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open sets of (X,τ) form a base for τ_δ [10]. The space (X,τ_δ) is also called the semiregularization topology of (X,τ) [8]. The complement of a $\delta-$open set is called $\delta-$closed [10]. A point $x \in X$ is called a $\delta-$cluster point of A if and only if $Int(cl(V)) \cap A \neq \emptyset$, for each open set V containing x. The set of all $\delta-$cluster points of A is called the δ-closure of A [10], which is denoted by $cl_\delta(A)$. A space (X,τ) is said to be semi-regular [8] if $\tau_\delta = \tau$. Any regular space is semi-regular, but the converse is false.

In [6], the concept of ω-closed subsets was explored where a subset A of (X,τ) is $\omega-$closed if it contains all its condensation points. The complement of an $\omega-$closed set is called $\omega-$open or equivalently A is $\omega-$open [2] if for each $x \in A$, there exists an open set U containing x such that $U - A$ is countable. The family of all $\omega-$open subsets of a space (X,τ), denoted by τ_ω, forms a topology on X finer that τ. A space (X,τ) is called anti-locally countable [2] if each non-empty open subset of (X,τ) is uncountable.

A subset A of a space (X,τ) is called a $R\omega-$open set [9] if $A = Int(cl_\omega(A))$. The complement of a $R\omega-$open set is called $R\omega-$closed set. The collection of all $R\omega-$open sets is denoted by $R\omega O(X)$ and it forms a base for some topology on X denoted by $\tau_{\delta-\omega}$ [9]. Elements of $\tau_{\delta-\omega}$ are called $\delta_\omega-$open sets and the complement of a $\delta_\omega-$open set is $\delta_\omega-$closed. The closure of a subset A of X in $(X,\tau_{\delta-\omega})$ is denoted by $cl_{\delta_\omega}(A)$. Let A be a subset of a topological space (X,τ), then $Int(cl_\omega(A))$ is $R\omega-$open [9]. In this paper we will define the $\delta\omega-$cluster point of a set A, a $\delta\omega-$closed set and a $\delta\omega-$open set. We show that the set of all $\delta\omega-$open sets forms a topology denoted by τ_{δ_ω} and equal to $\tau_{\delta-\omega}$. We discuss several properties of this topology and we give a characterization of the open sets in τ_{δ_ω}. We investigate some of the relationship between the separation axioms of (X,τ_{δ_ω}) and (X,τ). In the last section we study some of the connectedness properties of (X,τ_{δ_ω}).

In this paper $\mathbb{R}, \mathbb{Q}$ and $\mathbb{N}$ denote, respectively, the set of real numbers, the set of rational numbers and the set of natural numbers.

Definition 1.1. [2] Let A be a subset of the space (X,τ). Then the intersection of all $\omega-$closed subsets of X containing A is called the $\omega-$closure of A in (X,τ) and it is denoted by $cl_\omega(A)$.

Note that $cl_\omega(A)$ is the closure of A in the space (X,τ_ω).

Lemma 1.2. *Let (X,τ) be a topological space and $A \subseteq X$. Then the following properties hold:*
(i) $cl_\delta(A) = cl(A)$ for every open set A [10].
(ii) If (X,τ) is an anti-locally countable space, then for all $A \in \tau_\omega$,
$cl_\omega(A) = cl(A)$ [2].
(iii) $(\tau_\omega)_\omega = \tau_\omega$ [2].
(iv) $(\tau_A)_\omega = (\tau_\omega)_A$ [2].

Lemma 1.3. *[7] Let (Y,σ) be a regular space. If $f : (X,\tau) \to (Y,\sigma)$ is continuous, then $f : (X,\tau_\delta) \to (Y,\sigma)$ is continuous.*

2. $\delta\omega-$cluster points

Definition 2.1. Let (X,τ) be a topological space and let $A \subseteq X$. A point $x \in X$ is said to be a $\delta\omega-$cluster point of A if for each open set U containing x, we have $Int(cl_\omega(U)) \cap A \neq \emptyset$. The set of all $\delta\omega-$cluster points of A is called the $\delta\omega-$closure of A, which is denoted by $cl_{\delta\omega}(A)$.

A subset $A \subseteq X$ is called $\delta\omega-$closed if $A = cl_{\delta\omega}(A)$. The complement of a $\delta\omega-$closed set is called $\delta\omega-$open. The family of all $\delta\omega-$open sets in (X,τ) will be denoted by τ_{δ_ω}.

It is clear that if (X,τ) is a countable space, then $\tau_{\delta_\omega} = \tau$.

Theorem 2.2. *Let (X,τ) be a topological space. Then:*
(i) $\emptyset$ and X are $\delta\omega-$closed sets.
(ii) Finite union of $\delta\omega-$closed sets is $\delta\omega-$closed.
(iii) Arbitrary intersection of $\delta\omega-$closed sets is $\delta\omega-$closed.

Proof. (i) It is obvious.
(ii) The proof is complete if we prove that $cl_{\delta\omega}(A \cup B) = cl_{\delta\omega}(A) \cup cl_{\delta\omega}(B)$. It is clear that $cl_{\delta\omega}(A) \cup cl_{\delta\omega}(B) \subseteq cl_{\delta\omega}(A \cup B)$. Let $x \notin cl_{\delta\omega}(A) \cup cl_{\delta\omega}(B)$, then there are $U, V \in \tau$ such that $x \in U \cap V$, $Int(cl_\omega(V)) \cap B = \emptyset$ and $Int(cl_\omega(U)) \cap A = \emptyset$. Thus we have $x \in U \cap V \in \tau$ and $Int(cl_\omega(U \cap V)) \cap (A \cup B) \subseteq (Int(cl_\omega(U)) \cap A) \cup (Int(cl_\omega(V)) \cap B) = \emptyset$.
(iii) Let $\{A_\alpha : \alpha \in \Delta\}$ be a family of $\delta\omega-$closed sets in (X,τ). Then for all $\alpha \in \Delta$, $cl_{\delta\omega}(A_\alpha) = A_\alpha$. We show that $cl_{\delta\omega}(\cap\{A_\alpha : \alpha \in \Delta\}) \subseteq \cap\{A_\alpha : \alpha \in \Delta\}$. Let $x \in cl_{\delta\omega}(\cap\{A_\alpha : \alpha \in \Delta\})$ and let $U \in \tau$ such that $x \in U$. Then $Int(cl_\omega(U)) \cap (\cap\{A_\alpha : \alpha \in \Delta\}) \neq \emptyset$. Therefore, $Int(cl_\omega(U)) \cap A_\alpha \neq \emptyset$ for all $\alpha \in \Delta$. It follows that $x \in \{cl_{\delta\omega}(A_\alpha) : \alpha \in \Delta\} = \cap\{A_\alpha : \alpha \in \Delta\}$. □

Theorem 2.3. *Let (X,τ) be a topological space. Then τ_{δ_ω} is a topology on X.*

Proof. It is follows directly from Theorem 2.2. □

Theorem 2.4. *Let (X,τ) be a topological space and $A \subseteq X$. Then A is $\delta\omega-$open if and only if for each $x \in A$, there exists $U \in \tau$ such that $x \in U \subseteq Int(cl_\omega(U)) \subseteq A$.*

The proof is obvious.

Proposition 2.5. *Let (X,τ) be a topological space. Then $\tau_{\delta-\omega} = \tau_{\delta_\omega}$*

Proof. If $A \in \tau_{\delta-\omega}$, then $A = \bigcup_{\alpha\in\Delta}\{O_\alpha : O_\alpha \in R\omega O(X),\ \alpha \in \Delta\}$. So if $x \in A$, then there exists $\alpha_\circ \in \Delta$ such that $x \in O_{\alpha_\circ} = Int(cl_\omega(O_\alpha)) \subseteq Int(cl_\omega(Int(cl_\omega(O_\alpha)))) \subseteq A$. Thus by Theorem 2.4, $A \in \tau_{\delta_\omega}$. Now if $A \in \tau_{\delta_\omega}$, then by Theorem 2.4, there exists $U \in \tau$ such that $x \in U \subseteq Int(cl_\omega(U)) \subseteq A$. But $Int(cl_\omega(U)) \in R\omega O$, thus $A \in \tau_{\delta-\omega}$. □

Therefore the open sets of $\tau_{\delta-\omega}$ coincide with the open sets of τ_{δ_ω}. So from now we shall use the notation $\delta_\omega-$open set instead of $\delta\omega-$open set.

It is clear that, in any space the singleton is $\delta_\omega-$open if and only if it is regular $\omega-$open.

Theorem 2.6. *Let (X,τ) be a topological space. Then:*
(i) $\tau_\delta \subseteq \tau_{\delta_\omega} \subseteq \tau$.
(ii) If (X,τ) is regular, then $\tau_\delta = \tau_{\delta_\omega} = \tau$.

Proof. (i) Follows from the definitions and the fact that τ_ω is a topology on X finer than τ.
(ii) It follows from the fact that if (X,τ) is regular then $\tau = \tau_\delta$. □

The equality in Theorem 2.6 part (i) does not hold in general as we show in the next example.

Example 2.7. Let $X = \mathbb{R}$ with the topology $\tau = \{U : 1 \in U\} \cup \{\emptyset\}$. Then $\tau_\delta = \tau_{ind}$ while $\tau_{\delta_\omega} = \tau$.

Example 2.8. Let $X = \mathbb{R}$ with the topology $\tau = \{\emptyset, X, \mathbb{R} - \mathbb{Q}\}$. Then $\tau_{\delta_\omega} = \tau_{ind} \neq \tau$.

Proposition 2.9. *Let (X,τ) be topological space and let $A \subseteq X$. Then*
(i) $cl(A) \subseteq cl_{\delta\omega}(A) \subseteq cl_\delta(A)$.
(ii) For each $A \in \tau_\omega$, $cl_{\delta\omega}(A) \subseteq cl(A)$.
(iii) For each $A \in \tau$, $cl_\delta(A) \subseteq cl_{\delta\omega}(A) \subseteq cl(A)$.

Proof. (i) Follows from Theorem 2.6 part (i).
(ii) Suppose, by the way of contradiction, that $x \in cl_{\delta\omega}(A) \cap (X - cl(A))$. Since $X - cl(A) \in \tau$, we have $Int(cl_\omega(X - cl(A))) \cap A \neq \emptyset$. Choose $y \in Int(cl_\omega(X - cl(A))) \cap A \subseteq cl_\omega(X - cl(A)) \cap A$. Since $A \in \tau_\omega$, then $(X - cl(A)) \cap A \neq \emptyset$. A contradiction.
(iii) Follows from part (ii) and Lemma 1.2. □

The following two examples show that the conditions in part (ii) and (iii) in Proposition 2.9 are essential.

Example 2.10. Let $X = \{a, b\}$ with the topology $\tau = \{\emptyset, X, \{a\}\}$ and let $A = \{b\}$. Then $a \in cl_\delta(A)$ but $a \notin cl_{\delta\omega}(A)$ since $\{a\} \in \tau$ and $\{b\} \cap Int(cl_\omega(\{a\})) = \emptyset$.

Example 2.11. Consider the space (X,τ) given in Example 2.8 and let $A = \mathbb{Q}$. Then $cl_{\delta\omega}(A) = \mathbb{R}$ while $cl(A) = \mathbb{Q}$.

Proposition 2.12. *Let (X,τ) be a space.*
(i) If (X,τ) is anti-locally countable, then $\tau_\delta = \tau_{\delta\omega}$.
(ii) $(\tau_\omega)_{\delta_\omega} = (\tau_\omega)_\delta$.

Proof. (i) The proof follows immediately from the definitions, Lemma 1.2 and Theorem 2.6 part (i).
(ii) From Theorem 2.6, we have $(\tau_\omega)_\delta \subseteq (\tau_\omega)_{\delta_\omega}$. To prove the reverse inclusion, let $A \in (\tau_\omega)_{\delta_\omega}$ and $x \in A$. Then there exists $W \in \tau_\omega$ such that $x \in W \subseteq Int_\omega(cl_{(\omega)_\omega}(W)) \subseteq A$. By Lemma 1.2 part (iii), $cl_{(\omega)_\omega}(W) = cl_\omega(W)$. Therefore, $Int_\omega(cl_{(\omega)_\omega}(W)) = Int_\omega(cl_\omega(W)) \in (\tau_\omega)_\delta$ and so $A \in (\tau_\omega)_\delta$. □

Note that Example 2.10 shows that the condition that X is anti-locally countable in Proposition 2.12 part(i) is essential.

Theorem 2.13. *Let (X, τ) be any space, then $(\tau_{\delta_\omega})_{\delta_\omega} = \tau_{\delta_\omega}$ if one of the following hold*
(i) (X, τ) is anti-locally countable.
(ii) (X, τ) is regular.
(iii) X is countable.

Proof. The proofs of (ii) and (iii) are clear, so we will prove (i). It is clear that $(\tau_{\delta_\omega})_{\delta_\omega} \subseteq \tau_{\delta_\omega}$. To prove the other subset, let $U \in \tau_{\delta_\omega}$ such that $x \in U$, so there exists an open set H such that $x \in H \subseteq Int(cl_\omega(H)) \subseteq U$. Take $G = Int(cl_\omega(H))$. Then $G \in \tau_{\delta_\omega}$ such that $x \in G \subseteq \mathrm{Int}_{\tau_{\delta_\omega}}(cl_{(\tau_{\delta_\omega})_\omega}(G))$ and by using Proposition 2.9 and Lemma 1.2 $Int_{\tau_{\delta_\omega}}(cl_{(\tau_{\delta_\omega})_\omega}(G)) \subseteq Int(cl_\omega(H)) \subseteq U$. $\square$

We can conclude from the proof of Theorem 2.13, that $cl_{\delta_\omega}(A) = cl_\omega(A)$, where A is an $\omega-$open subset of an anti-locally countable space (X, τ).

Proposition 2.14. *Let (X, τ) be a topological space. If $A \in \tau$ then $(\tau_{\delta_\omega})_A = (\tau_A)_{\delta_\omega}$.*

Proof. Let $B \in (\tau_{\delta_\omega})_A$. Then $B = W \cap A$, where $W \in \tau_{\delta_\omega}$. If $x \in B$ then $x \in W$, so by Theorem 2.4, there exists $V \in \tau$ such that $x \in V \subseteq Int(cl_\omega(V)) \subseteq W$. Now $Int_A(cl_{\omega_A}(V \cap A))$ = (Lemma 1.2, iv) $Int_A(cl_\omega(V \cap A) \cap A) = (Int_A (cl_\omega(V \cap A))) \cap A =$(A is open)$(Int\ (cl_\omega(V \cap A))) \cap A \subseteq Int\ (cl_\omega(V)) \cap A \subseteq W \cap A = B$. Therefore, $B \in (\tau_A)_{\delta_\omega}$. Now let $B \in (\tau_A)_{\delta_\omega}$. Then by Theorem 2.4, there exists $U_A = V \cap A \in \tau_A$ such that $x \in U_A = V \cap A \subseteq Int_A(cl_{\omega_A}(V \cap A)) \subseteq B$. As $Int\ (cl_\omega(V)) \cap A \subseteq Int\ (cl_\omega(V) \cap A) \subseteq Int(cl_\omega(V \cap A)) = Int\ (cl_{\omega_A}(V \cap A)) \subseteq Int_A(cl_{\omega_A}(V \cap A)) \subseteq B$, and $Int(cl_\omega(V)) \in (\tau_{\delta_\omega})_A$, so $B \in (\tau_{\delta_\omega})_A$. $\square$

In Proposition 2.14 it is essential that $A \subseteq X$ is open as we see in the next example.

Example 2.15. Again we consider the space (X, τ) given in Example 2.8. We take $A = \mathbb{Q} \cup \sqrt{2}$. Then $(\tau_{\delta_\omega})_A = \tau_{ind}$, while $\sqrt{2} \in (\tau_A)_{\delta_\omega}$.

If (X, τ) and (Y, σ) are two topological spaces, then $\delta \times \sigma$ will denote the product topology on $X \times Y$ [see[5]].

Lemma 2.16. *[1] Let (X, τ) and (Y, σ) be two topological spaces.*
(i) $(\tau \times \sigma)_\omega \subseteq \tau_\omega \times \sigma_\omega$.
(ii) If $A \subseteq X$ and $B \subseteq Y$, then $cl_\omega(A) \times cl_\omega(B) \subseteq cl_\omega(A \times B)$.

Theorem 2.17. *Let (X, τ) and (Y, σ) be two topological spaces. Then $(\tau \times \sigma)_{\delta_\omega} \subseteq \tau_{\delta_\omega} \times \sigma_{\delta_\omega}$.*

Proof. Let $W \in (\tau \times \sigma)_{\delta_\omega}$ and $(x, y) \in W$. There exist $U \in \tau$ and $V \in \sigma$ such that $(x, y) \in U \times V \subseteq Int(cl_\omega(U \times V)) \subseteq W$. By Theorem 2.4 and Lemma 2.16 we have $(x, y) \in U \times V \subseteq Int(cl_\omega(U)) \times Int(cl_\omega(V)) \subseteq Int(cl_\omega(U \times V)) \subseteq W$. $\square$

The following example shows that the reverse conclusion of the previous theorem is not true in general.

Example 2.18. Let $X = \mathbb{R}$ with topologies $\tau = \{\emptyset, \mathbb{R}, \mathbb{R} - \mathbb{Q}\}$ and $\sigma = \{\emptyset, \mathbb{R}, \mathbb{Q}\}$. Let $A = \mathbb{R}$ and $B = \mathbb{Q}$. Then $A \in \tau_{\delta_\omega}$, $B \in \sigma_{\delta_\omega}$. However, $A \times B \notin (\tau \times \sigma)_{\delta_\omega}$ since $cl_{(\tau\times\sigma)_\omega}(A \times B) = \mathbb{R} \times \mathbb{R}$.

Let $\{(X_\alpha, \tau_\alpha) : \alpha \in \Delta\}$ be a collection of topological spaces such that $X_\alpha \cap X_\beta = \emptyset$ for each $\alpha \neq \beta$. Let $X = \underset{\alpha\in\Delta}{\cup} X_\alpha$ be topologized by $\tau_s = \{G \subseteq X : G \cap X_\alpha \in \tau_\alpha$ for each $\alpha \in \Delta\}$. Then (X, τ_s) is called the sum of the spaces $\{(X_\alpha, \tau_\alpha) : \alpha \in \Delta\}$ and we write $X = \underset{\alpha\in\Delta}{\oplus} X_\alpha$ [see [5]].

Theorem 2.19. *For any collection of spaces* $\{(X_\alpha, \tau_\alpha) : \alpha \in \Delta\}$ *we have* $(\tau_s)_\omega = (\tau_{\alpha_\omega})_s$.

Proof. Let $W \in (\tau_s)_\omega$ and let $x \in W \cap X_\alpha$. Then $x \in W$ and so there exists $U \in \tau_s$ such that $x \in U$ and $U - W = C$ is a countable. Put $V = U \cap X_\alpha$. Then V is an open in X_α such that $x \in V$ and $V - (W \cap X_\alpha) = (U - W) \cap X_\alpha \subseteq U - W$ and so $V - (W \cap X_\alpha)$ is countable. Therefore, $W \cap X_\alpha \in \tau_{\alpha_\omega}$. Now, let $W \in (\tau_{\alpha_\omega})_s$ and let $x \in W$. Then there exists $\alpha_\circ \in \Delta$ such that $x \in X_{\alpha_\circ}$ and so $x \in W \cap X_{\alpha_\circ} \in \tau_{(\alpha_\circ)_\omega}$. So there exists an open set $V \subseteq X_{\alpha_\circ}$ such that $x \in V_{\alpha_\circ}$ and $V_{\alpha_\circ} - (W \cap X_\alpha)$ is countable. Since $V_{\alpha_\circ} = V_{\alpha_\circ} - (W \cap X_{\alpha_\circ})$ and $V_{\alpha_\circ} \in \tau_s$, thus $W \in (\tau_s)_\omega$. $\square$

Theorem 2.20. *Let* $\{(X_\alpha, \tau_\alpha) : \alpha \in \Delta\}$ *be a collection of spaces and* $A_{\alpha_\circ} \subseteq X_{\alpha_\circ}$, *then*
(i) $cl_{(\tau_{\alpha_\circ})_\omega}(A_{\alpha_\circ}) = cl_{(\tau_s)_\omega}(A_{\alpha_\circ})$.
(ii) $Int_{\tau_{\alpha_\circ}}(A_{\alpha_\circ}) = Int_{\tau_s}(A_{\alpha_\circ})$.

Proof. (i) Let $x \in cl_{(\tau_{\alpha_\circ})_\omega}(A_{\alpha_\circ})$ and let $W \in (\tau_s)_\omega$ such that $x \in W$. Then by Theorem 2.19, $W \cap X_{\alpha_\circ} \in (\tau_{\alpha_\circ})_\omega$ and so $\emptyset \neq W \cap X_{\alpha_\circ} \cap A_{\alpha_\circ} = W \cap A_{\alpha_\circ}$. Therefore, $x \in cl_{(\tau_s)_\omega}(A_{\alpha_\circ})$. Conversely, suppose that $x \in cl_{(\tau_s)_\omega}(A_{\alpha_\circ})$ and let $W \in (\tau_{\alpha_\circ})_\omega$ such that $x \in W$. So for each $\beta \neq \alpha_\circ$, $W \cap X_\beta = \emptyset$ and hence $W \in (\tau_{\tau_{\beta_\omega}})_s = (\tau_s)_\omega$. Therefore, $W \cap A_{\alpha_\circ} \neq \emptyset$. Thus $x \in cl_{(\tau_{\alpha_\circ})_\omega}(A_{\alpha_\circ})$.
(ii) Let $x \in Int_{\alpha_\circ}(A_{\alpha_\circ})$, so there exists $U_{\alpha_\circ} \in \tau_{\alpha_\circ}$ such that $x \in U_{\alpha_\circ} \subseteq A_{\alpha_\circ}$. Since $U_{\alpha_\circ} \in \tau_s$, then $x \in Int_s(A_{\alpha_\circ})$. Conversely, let $x \in Int_s(A_{\alpha_\circ})$, so there exists $U \in \tau_s$ such that $x \in U \subseteq A_{\alpha_\circ}$. Thus $x \in U \cap X_{\alpha_\circ} \subseteq A_{\alpha_\circ}$ and $U \cap X_{\alpha_\circ} \in \tau_{\alpha_\circ}$. Therefore, $x \in Int_{\alpha_\circ}(A_{\alpha_\circ})$. $\square$

Theorem 2.21. *For any collection of spaces* $\{(X_\alpha, \tau_\alpha) : \alpha \in \Delta\}$, *we have* $(\tau_s)_{\delta_\omega} = (\tau_{\alpha_{\delta_\omega}})_s$.

Proof. Let $A \in (\tau_s)_{\delta_\omega}$. Let $\alpha \in \Delta$ and $x \in A \cap X_\alpha$. So there exists $U \in \tau_s$ such that $x \in Int_s(cl_{(\tau_s)_\omega}(U)) \subseteq A$. Put $V = U \cap X_\alpha$. Then $V \in \tau_\alpha$ such that $x \in V$ and $Int_\alpha(cl_{(\tau_\alpha)_\omega}(V)) =$ (by Theorem 2.20, part (i)) $Int_\alpha(cl_{(\tau)_\omega}(U \cap X_\alpha)) =$ (by Theorem 2.20, part (ii)) $Int_s(cl_{(\tau_s)_\omega}(U \cap X_\alpha)) \subseteq Int_s((cl_{(\tau_s)_\omega}(U)) \cap (cl_{(\tau_s)_\omega}(X_\alpha))) = Int_s(cl_{(\tau_s)_\omega}(U)) \cap X_\alpha = Int_s(cl_{(\tau_s)_\omega}(U)) \cap Int_s(X_\alpha) = Int_s(cl_{(\tau_s)_\omega}(U)) \cap X_\alpha \subseteq A \cap X_\alpha$, i.e $A \cap X_\alpha \in \tau_{\alpha_{\delta_\omega}}$ and so $A \in (\tau_{\alpha_{\delta_\omega}})_s$. Now, let $A \in (\tau_{\alpha_{\delta_\omega}})_s$ and let $x \in A$. Then there exists $\alpha_\circ \in \Delta$ such that $x \in X_{\alpha_\circ}$. Now $x \in A \cap X_{\alpha_\circ} \in \tau_{(\alpha_\circ)_{\delta_\omega}}$ and so there exists $U \in \tau_{\alpha_\circ}$ such that $x \in Int_{\alpha_\circ}(cl_{(\alpha_\circ)_\omega}(U)) \subseteq A \cap X_{\alpha_\circ}$. Now $Int_s(cl_{(\tau_s)_\omega}(U)) =$

(by Theorem 2.20, part (i)) $Int_s(cl_{(\tau_{\alpha_\circ})_\omega}(U))$ = (by Theorem 2.20, part (ii)) $Int_{\alpha_\circ}(cl_{(\tau_{\alpha_\circ})_\omega}(U)) \subseteq A \cap X_{\alpha_\circ} \subseteq A$ and $U \in \tau_s$ and so $A \in (\tau_s)_{\delta_\omega}$. □

3. Separation Axioms

In topology and related fields of mathematics, there are several restrictions that one often makes on the kinds of topological spaces that one wishes to consider. Some of these restrictions are given by the separation axioms. In this section we investigate some of the relationship between the separation axioms of $(X, \tau_{\delta_\omega})$ and (X, τ).

Lemma 3.1. *Let (X, τ) be a topological space. If $U, V \in \tau$ such that $U \cap V = \emptyset$, then :*
(i) $Int\ (cl_\omega(U)) \cap Int\ (cl_\omega(V)) = \emptyset$.
(ii) $Int\ (cl_\omega(U)) \cap V = U \cap Int\ (cl_\omega(V)) = \emptyset$.

The proof is obvious

Corollary 3.2. *Let (X, τ) be a space. If $x \in U$, $y \in V$ such that U and V are two disjoint open sets, then there exist $U_1, V_1 \in \tau_{\delta_\omega}$ such that $x \in U_1, y \in V_1$ and $U_1 \cap V_1 = \emptyset$.*

Proposition 3.3. *If $(X, \tau_{\delta_\omega})$ is a $T_\circ-$space (resp., a T_1-space), then (X, τ) is a $T_\circ-$space (resp., a T_1-space).*

Proof. It follows from the fact that $\tau_{\delta_\omega} \subseteq \tau$. □

The next example shows that the converse of Proposition 3.3 is not true in general.

Example 3.4. Consider the space $(\mathbb{R}, \tau)$ consisting the set of real numbers and the cofinite topology τ on $\mathbb{R}$. Then $(\mathbb{R}, \tau)$ is T_1-space and so it is $T_\circ-$space. However, $(X, \tau_{\delta_\omega})$ is an indiscrete topology which is not $T_\circ-$space.

Proposition 3.5. $(X, \tau_{\delta_\omega})$ *is a T_2- space if and only if (X, τ) is T_2.*

Proof. Let $(X, \tau_{\delta_\omega})$ be a T_2-space. Since $\tau_{\delta_\omega} \subseteq \tau$, then (X, τ) is T_2. Conversely, if (X, τ) is a T_2-space and $x \neq y$, then there exist $U, V \in \tau$ such that $x \in U, y \in V$ and $U \cap V = \emptyset$. By Lemma 3.1, there exist $U_1, V_1 \in \tau_{\delta_\omega}$ such that $x \in U_1, y \in V_1$ and $U_1 \cap V_1 = \emptyset$. Thus $(X, \tau_{\delta_\omega})$ is T_2. □

Definition 3.6. A space (X, τ) is said to be τ $\omega-$Urysohn if for any pair (x, y) of distinct point in X there exist $U, V \in \tau$ such that $x \in U$, $y \in V$ and $cl_\omega(U) \cap cl_\omega(V) = \emptyset$.

Proposition 3.7. $(X, \tau_{\delta_\omega})$ *is a τ_{δ_ω} $\omega-$Urysohn space if and only if (X, τ) is a τ $\omega-$Urysohn space.*

Proof. Let $(X, \tau_{\delta_\omega})$ be a τ_{δ_ω} $\omega-$Urysohn space. Since $\tau_{\delta_\omega} \subseteq \tau$ and $(\tau_{\delta_\omega})_\omega \subseteq \tau_\omega$, then (X, τ) is τ $\omega-$Urysohn space. Conversely, suppose that (X, τ) is a τ $\omega-$Urysohn space. Then for any pair (x, y) of distinct points in X there exist $U, V \in \tau$ such that $x \in U$, $y \in V$ and $cl_\omega(U) \cap cl_\omega(V) = \emptyset$. Since $Int(cl_\omega(U))$, $Int\ (cl_w(V) \in \tau_{\delta_\omega}$ and $cl_\omega(Int(cl_\omega(U))) \cap cl_\omega(Int(cl_w(V))) \subseteq cl_\omega(cl_\omega(U)) \cap cl_\omega(cl_\omega(V)) = cl_\omega(U) \cap cl_\omega(V) = \emptyset$. Thus $(X, \tau_{\delta_\omega})$ is a τ_{δ_ω} $\omega-$Urysohn space. □

Continuing our study of the separation axioms we will study the relationship between the regularity of (X, τ) and $(X, \tau_{\delta_\omega})$.

Proposition 3.8. *Let (X, τ) be a topological space. Then the following are equivalent:*
(i) (X, τ) is a regular space.
(ii) For every closed set F and $x \notin F$ there exist $U \in \tau$ and $V \in \tau_{\delta_\omega}$ such that $F \subseteq U$, $x \in V$ and $U \cap V = \emptyset$.
(iii) For every closed set F and $x \notin F$ there exist $U \in \tau_{\delta_\omega}$ and $V \in \tau$ such that $F \subseteq U$, $x \in V$ and $U \cap V = \emptyset$.
(iv) For every closed set F and $x \notin F$ there exist $U, V \in \tau_{\delta_\omega}$ such that $F \subseteq U$, $x \in V$ and $U \cap V = \emptyset$.

Proof. It follows from Lemma 3.1. □

Proposition 3.9. *Let (X, τ) be a topological space. Then the following are equivalent:*
(i) $(X, \tau_{\delta_\omega})$ is a regular space.
(ii) For every $\delta_\omega-$closed set F and $x \notin F$ there exist $U \in \tau$ and $V \in \tau$ such that $F \subseteq U$, $x \in V$ and $U \cap V = \emptyset$.
(iii) For every $\delta_\omega-$closed set F and $x \notin F$ there exist $U \in \tau$ and $V \in \tau_{\delta_\omega}$ such that $F \subseteq U$, $x \in V$ and $U \cap V = \emptyset$.
(iv) For every $\delta_\omega-$closed set F and $x \notin F$ there exist $U \in \tau_{\delta_\omega}$ and $V \in \tau$ such that $F \subseteq U$, $x \in V$ and $U \cap V = \emptyset$.

Proof. It follows immediately form Lemma 3.1 and the fact that $\tau_{\delta_\omega} \subseteq \tau$. □

Proposition 3.10. *Let (X, τ) be a topological space. If (X, τ) is a regular space, then $(X, \tau_{\delta_\omega})$ is regular.*

It follows from Theorem 2.6.
The converse of the above proposition is not true as we see in the following example.

Example 3.11. Here we consider the space (X, τ) given in Example 2.8. Since $(X, \tau_{\delta_\omega})$ is the indiscrete space, it is regular. However, (X, τ) is not regular.

Proposition 3.12. *Let (X, τ) be a topological space. Then the following are equivalent:*
(i) $(X, \tau_{\delta_\omega})$ is a regular space.
(ii) For every $\delta_\omega-$open set U and $x \in U$ there exists $V \in \tau_{\delta_\omega}$ such that $x \in V \subseteq cl(V) \subseteq U$.
(iii) For every $\delta_\omega-$open set U and $x \in U$ there exists $V \in \tau_{\delta_\omega}$ such that $x \in V \subseteq cl_{\delta_\omega}(V) \subseteq U$.

It is follows from Proposition 3.9.

Definition 3.13. A space (X,τ) is said to be almost $\omega-$regular if for each $R\omega-$closed set A of X and $x\in X-A$, there exist U, $V\in\tau$ such that $A\subseteq U$, $x\in V$ and $U\cap V=\emptyset$.

Theorem 3.14. *A space (X,τ) almost $\omega-$regular if and only if (X,τ_{δ_ω}) is regular.*

Proof. Let (X,τ) be an almost $\omega-$regular space and F be a $\delta_\omega-$closed set in (X,τ) such that $x\notin F$. Since F is $\delta_\omega-$closed, then $F=\bigcap\limits_{\alpha\in\Delta} F_i$, where F_i is $R\omega-$ closed set in (X,τ). Thus there exists $\alpha_\circ\in\Delta$ such that $x\in X-F_{\alpha_\circ}$. Since (X,τ) is almost $\omega-$regular, there exist $U,V\in\tau$ such that $F\subseteq F_{\alpha_\circ}\subset U, x\in V$ and $U\cap V=\emptyset$. By Lemma 3.1, $Int\ (cl_\omega(U))\cap Int\ (cl_\omega(V))=\emptyset$ such that $F\subseteq F_{\alpha_\circ}=Int\ (cl_\omega(F_{\alpha_\circ}))\subseteq Int\ (cl_\omega(U))$ and $x\in Int\ (cl_\omega(V))\in\tau_{\delta_\omega}$. Therefore, (X,τ_{δ_ω}) is regular. Conversely, suppose that A is $R\omega-$closed such that $x\in X-A$. By the regularity of (X,τ_{δ_ω}) there exist $U,V\in\tau_{\delta_\omega}\subseteq\tau$ such that $A\subseteq U$, $x\in V$ and $U\cap V=\emptyset$. Thus (X,τ) is an almost $\omega-$regular space. □

Corollary 3.15. *A space (X,τ) is regular if and only if it is semi regular and almost $\omega-$regular space.*

Definition 3.16. A space (X,τ) is said to be almost completely $\omega-$regular if for each $R\omega-$ closed set A of X and $x\in X-A$, there exists a continuous function $f:(X,\tau)\longrightarrow[0,1]$ such that $f(x)=1$ and $f(A)=0$.

Theorem 3.17. *A space (X,τ) is almost completely $\omega-$regular if and only if (X,τ_{δ_ω}) is completely regular.*

Proof. Let F be closed in τ_{δ_ω} and $x\notin F$. Since F is $\delta_\omega-$closed in (X,τ) then $F=\bigcap\limits_{\alpha\in\Delta} F_i$, where F_i is a $R\omega-$ closed set. Thus there exists $\alpha_\circ\in\Delta$ such that $x\in X-F_{\alpha_\circ}$ where $F_{\alpha_\circ}$ is $R\omega-$closed. Then there exists a continuous function $f:(X,\tau)\longrightarrow[0,1]$ such that $f(x)=1$ and $f(F_{\alpha_\circ})=0$. As $[0,1]$ is regular, so by Lemma 1.3, $f:(X,\tau_{\delta_\omega})\longrightarrow[0,1]$ is continuous such that $f(x)=1$ and $f(F)=0$. Conversely, suppose A is $R\omega-$closed set and $x\in X-A$. Since (X,τ_{δ_ω}) is completely regular there is a continuous function $f:(X,\tau_{\delta_\omega})\longrightarrow[0,1]$ such that $f(x)=1$ and $f(A)=0$. Since $\tau_{\delta_\omega}\subseteq\tau$ so $f:(X,\tau)\longrightarrow[0,1]$ is continuous such that $f(x)=1$ and $f(A)=0$. □

4. Connectedness of the space (X,τ_{δ_ω})

In this section we shall study some of connectedness properties of (X,τ_{δ_ω}) and some covering properties.

The following proposition gives the relationship between the connectedness properties of (X,τ_{δ_ω}) and (X,τ).

Proposition 4.1. *A topological space (X, τ) is connected if and only if $(X, \tau_{\delta_\omega})$ is connected.*

Proof. Let (X, τ) be connected. To show that $(X, \tau_{\delta_\omega})$ is connected we need to show that the only subsets of $(X, \tau_{\delta_\omega})$ which are both open and closed (clopen sets) are X and the empty set. Since $\tau_{\delta_\omega} \subseteq \tau$, so if A is clopen in $(X, \tau_{\delta_\omega})$ then A is clopen in (X, τ). But as (X, τ) is connected, so A must be either X or the empty set. Thus $(X, \tau_{\delta_\omega})$ is connected. Conversely, let $(X, \tau_{\delta_\omega})$ be connected. If A is clopen in (X, τ), then $X - A$ and A are open in $\tau_\delta \subseteq \tau_{\delta_\omega}$, so $X - A, A \in \tau_{\delta_\omega}$. But as $(X, \tau_{\delta_\omega})$ is connected, so A must be either X or the empty set. Thus (X, τ) is connected. $\square$

We can conclude from the proof of Proposition 4.1 that the collection of clopen sets of (X, τ) coincides with the collection of clopen sets of $(X, \tau_{\delta_\omega})$.

A subset A of a topological space (X, τ) is said to be connected if (A, τ_A) is connected and A is called a connected set in (X, τ) if A can not be written as a union of two disjoint open sets in (X, τ). Now we will present the definition of $\delta_\omega-$connected relative to X.

Definition 4.2. A subset A of a space (X, τ) is called a $\delta_\omega-$connected set in (X, τ) if A is a connected set in $(X, \tau_{\delta_\omega})$.

Proposition 4.3. *Let (X, τ) be a topological space and $A \subseteq X$. If A is a connected set in (X, τ), then A is a $\delta_\omega-$connected set in (X, τ).*

Proof. As A is a connected set in (X, τ) and $\tau_{\delta_\omega} \subseteq \tau$ so A is a $\delta_\omega-$connected set in (X, τ). $\square$

The converse of Proposition 4.3, is not true as we will see in the following example.

Example 4.4. Consider the space $(\mathbb{N}, \tau)$ consisting the set of the natural numbers and the cofinite topology τ on $\mathbb{N}$. Let $A = \{1, 2\}$. Then $(X, \tau_{\delta_\omega})$ is the indiscrete topology, so A is a disconnected set in $(\mathbb{N}, \tau)$ but it is a $\delta_\omega-$connected set in $(\mathbb{N}, \tau)$.

So what are the additional conditions that make the reversal of previous relationships true? This is what will be shown in the following proposition.

Proposition 4.5. *Let (X, τ) be a topological space and A be an open subset of (X, τ). Then the following are equivalent:*
(i) (A, τ_A) is connected.
(ii) A is a connected set in (X, τ).
(iii) A is a $\delta_\omega-$connected set in (X, τ).

Proof. (i)$\rightarrow$(ii) Follows from the definitions.
(ii)$\rightarrow$(iii) It is follows from Proposition 4.3.
(iii)$\rightarrow$(i) Suppose, by the way of contradiction, that (A, τ_A) is disconnected.

Then there exist two disjoint non empty sets $U, V \in \tau_A$ such that $A = U \cup V$. Since A is open, by Lemma 3.1, it follows that $A = (Int(cl_\omega(U)) \cap A) \cup (Int(cl_\omega(V)) \cap A)$, which contradicts the assumption. $\square$

Definition 4.6. [5] A space (X, τ) is called:
(i) locally connected at x if for every open set V containing x there exists a connected open set U in (X, τ) with $x \in U \subset V$. The space X is said to be locally connected if it is locally connected at x for all x in X.
(ii) path-connected if for any two points $x \neq y$ in X there is a continuous function $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$.

Since $\tau_{\delta_\omega} \subseteq \tau$, we can get the following proposition.

Proposition 4.7. *If (X, τ) is path connected then $(X, \tau_{\delta_\omega})$ is path connected.*

The converse of the above proposition is not true. Again, we consider the space $(\mathbb{N}, \tau)$ given in Example 4.4. Then $(\mathbb{N}, \tau)$ is not path connected, but $(X, \tau_{\delta_\omega})$ is the indiscrete topology so it is path connected.

Definition 4.8. A topological space (X, τ) is said to be almost $\omega-$locally connected at a point $x \in X$ if whenever U is a $R\omega-$open set containing x, then there exists an open connected set V such that $x \in V \subseteq U$. A space (X, τ) is almost $\omega-$locally connected if it is almost locally connected at each of it is points.

Theorem 4.9. *A topological space (X, τ) is almost $\omega-$locally connected if and only if $(X, \tau_{\delta_\omega})$ is locally connected.*

Proof. Let (X, τ) be almost $\omega-$locally connected. Let $x \in X$ and $A \in \tau_{\delta_\omega}$ such that $x \in A$. Then there exists a $R\omega-$ open set U in (X, τ) such that $x \in U \subseteq A$. As (X, τ) is almost $\omega-$locally connected there is a connected open set V in (X, τ) such that $x \in V \subseteq U$. So $x \in V \subseteq Int(cl_\omega(V)) \subseteq Int(cl_\omega(U)) = U \subseteq A$. Since V is a connected set in (X, τ), then $Int\ (cl_\omega(V))$ is connected in (X, τ), so $Int(cl_\omega(V))$ is an open set in $(X, \tau_{\delta_\omega})$ and it's $\delta_\omega-$connected set in (X, τ). Thus $(X, \tau_{\delta_\omega})$ is locally connected. Conversely, let $x \in X$ and U be a $R\omega-$ open set containing x. Since $(X, \tau_{\delta_\omega})$ is locally connected then there exists an open set V in $(X, \tau_{\delta_\omega})$ containing x which is $\delta_\omega-$connected in (X, τ) such that $V \subseteq U$. Also as V is open in (X, τ), hence V is connected in (X, τ). Thus (X, τ) is almost $\omega-$ locally connected. $\square$

From Theorem 4.9, we can get the following two corollaries.

Corollary 4.10. *If (X, τ) is locally connected then $(X, \tau_{\delta_\omega})$ is locally connected.*

Corollary 4.11. *If (X, τ) is semi-regular then it is locally connected if and only if it is almost $\omega-$ locally connected.*

At the end of this section we have the following discussion of some covering properties.

Using the fact that $\tau_{\delta_\omega} \subseteq \tau$ we easily obtain the next result.

Proposition 4.12. *If (X,τ) is a Lindelöf space, then (X,τ_{δ_ω}) is Lindelöf.*

The following example shows that the converse of Proposition 4.12 is not true in general.

Example 4.13. Let $X = A \cup B$ with the topology $\tau = \{U : U \subseteq A\} \cup \{U : A \subseteq U\}$ where A and B are uncountable sets such that $A \cap B = \emptyset$. To prove that $\tau_{\delta_\omega} = \{U : U \subseteq A\}$, we consider the following two cases:

(i) If $U \in \tau$ such that $U \subseteq A$ then $U \in \tau_{\delta_\omega}$, since if U is countable, then $cl_\omega(U) = U$ and if U is uncountable, then $cl_\omega(U) = U \cup B$. To show that, let $x \in B$ and suppose there exists $W \in \tau_\omega$ such that $x \in W$ and $W \cap U = \emptyset$. There exists an open set V such that $x \in V$ and $V - W$ is a countable set. Now $A \cup \{x\} \subseteq V$ and $A \cup \{x\} = c' \cup W$, where c' is a countable set. Thus $U = U \cap (A \cup \{x\}) \subseteq (c' \cup W) \cap U \subseteq c' \cap U$, a contradiction.

(ii) If $U \in \tau$ and $A \subseteq U$, then $U = A \cup U'$ where $U' \subseteq B$ and $cl_\omega(U) = cl_\omega(U') \cup cl_\omega(A)$. We show that, $cl_\omega(A) = X$. Let $x \in B$ and suppose there exists $W \in \tau_\omega$ such that $x \in W$ and $W \cap A = \emptyset$, so $W \subseteq B$. Hence, there exists an open set V in (X,τ) such that $x \in V$ and $V - W$ is countable. Then $V = A \cup V'$ where $V' \subseteq B$. Thus $A \subseteq A \cup V' - W$ countable, which is a contradiction.

Note that if $x_\circ \in B$, then the only open set containing x is X and so (X,τ_{δ_ω}) is Lindelöf. But (X,τ) is not Lindelöf, since $\{A \cup \{x\} : x \in B\}$ is an open cover which has no countable subcover.

Recall that a space (X,τ) is called almost Lindelöf [11] (resp. weakly Lindelöf [4], nearly Lindelöf [3]) if whenever $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ is an open cover of (X,τ) there exists a countable subset $\Delta_\circ$ of Δ such that $X = \underset{\alpha\in\Delta_\circ}{\cup} cl(U_\alpha)$ (resp. $X = cl(\underset{\alpha\in\Delta_\circ}{\cup} U_\alpha), X = Int(cl(\underset{\alpha\in\Delta_\circ}{\cup} U_\alpha)))$.

Proposition 4.14. *Let (X,τ) be any space.*
(i) (X,τ) is weakly Lindelöf if and only if (X,τ_{δ_ω}) is weakly Lindelöf.
(ii)(X,τ) is almost Lindelöf if and only if (X,τ_{δ_ω}) is almost Lindelöf.
(iii) (X,τ) is nearly Lindelöf if and only if (X,τ_{δ_ω}) is nearly Lindelöf.

Proof. (i) Necessity follows directly from the fact that $\tau_{\delta_\omega} \subseteq \tau$. To prove sufficiency, let $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ be an open cover of (X,τ). Then $\mathcal{W} = \{Int(cl_\omega(U_\alpha)) : \alpha \in \Delta\}$ is a $\tau_{\delta_\omega}-$open cover of the weakly Lindelöf space (X,τ_{δ_ω}). Therefore, there exists a countable subset $\Delta_\circ$ of Δ such that $X = cl_{\delta_\omega}(\underset{\alpha\in\Delta_\circ}{\cup}(Int(cl_\omega(U_\alpha))))$. By Proposition 2.9, we get

$$\begin{aligned} X &= cl(\underset{\alpha\in\Delta_\circ}{\cup} Int(cl_\omega(U_\alpha))) \\ &\subseteq cl(\underset{\alpha\in\Delta_\circ}{\cup} cl_\omega(U_\alpha)) \quad (\tau_\omega \text{ is a topology on } X) \\ &\subseteq cl(cl_\omega(\underset{\alpha\in\Delta_\circ}{\cup}(U_\alpha))) \\ &\subseteq cl(\underset{\alpha\in\Delta_\circ}{\cup} U_\alpha). \end{aligned}$$

Thus (X,τ) is weakly Lindelöf.
(ii) The proof is similar to part (i).
(iii) Necessity. Let $\mathcal{U}$ be a $\tau_{\delta_\omega}-$open cover of (X,τ_{δ_ω}). For every $x\in X$, choose $U_x\in\mathcal{U}$ and $V_x\in\tau$ such that $x\in Int(cl_\omega(V_x))\subseteq U_x$. Then, the collection $\{Int(cl_\omega(V_x)): x\in X\}$ is a $\tau-$open cover of the nearly Lindelöf space (X,τ) and so there exist a countable subset X' of X such that

$$\begin{aligned} X &= \bigcup_{x\in X'} Int(cl(Int(cl_\omega(V_x)))) \\ &= \bigcup_{x\in X'} Int(cl_{\delta_\omega}(Int(cl_\omega(V_x)))) \text{ (by Proposition 2.9)} \\ &= \bigcup_{x\in X'} Int_{\delta_\omega}(cl_{\delta_\omega}(Int(cl_\omega(V_x)))) \\ &\subseteq \bigcup_{x\in X'} Int_{\delta_\omega}(cl_{\delta_\omega}((U_x)). \end{aligned}$$

Therefore, (X,τ_{δ_ω}) is nearly Lindelöf. To prove sufficiency, let $\mathcal{U}$ be a $\tau-$open cover of (X,τ). For every $x\in X$, choose $U_x\in\mathcal{U}$ such that $x\in U_x$. Then, the collection $\{Int(cl_\omega(U_x)): x\in X\}$ is a $\tau_{\delta_\omega}-$open cover of (X,τ_{δ_ω}) and so there exists a countable subset X' of X such that

$$\begin{aligned} X &= \bigcup_{x\in X'} Int_{\delta_\omega}(cl_{\delta_\omega}(Int(cl_\omega(U_x)))) \\ &= \bigcup_{x\in X'} Int_{\delta_\omega}(cl(Int(cl_\omega(U_x))) \\ &= \bigcup_{x\in X'} Int(cl(Int(cl_\omega(U_x))) \subseteq \bigcup_{x\in X'} Int(cl(cl_\omega(U_x)) \\ &\subseteq \bigcup_{x\in X'} Int(cl(U_x)). \end{aligned}$$

Therefore, (X,τ) is nearly Lindelöf. □

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