NOVI SAD J. MATH. VOL. 50, NO. 1, 2020, 17-36
https://doi.org/10.30755/NSJOM.06942

# A $k$-dimensional system of Langevin Hadamard-type fractional differential inclusions with $2k$ different fractional orders

**Ali Mansoori,[1] Hamidreza Rahimi[2] and Shahram Rezapour[3] [4][5]**

**Abstract.** We investigate the existence of solution for a $k$-dimensional system of Langevin Hadamard-type fractional differential inclusions with $2k$ different fractional orders. We provide an example to illustrate our main result.

*AMS Mathematics Subject Classification* (2010): 34A60; 34B15

*Key words and phrases:* Fixed point; Hadamard-type fractional differential Langevin inclusion; $k$-dimensional system

## 1. Introduction

Langevin equation, introduced by Langevin in 1908, is an effective tool to describe the evolution of physical phenomena in fluctuating environments ([13] and [24]). Later, some works appeared on fractional Langevin equations and inclusions (see for example, [3], [4], [5], [6], [8] and [22]). Some researchers have studied fractional differential inclusions from different views (see for example, [1], [19], [21], [23]).

The Hadamard fractional integral of order $\alpha > 0$ for a function $f \in L^p[y, x]$ $(0 \leq y \leq t \leq x \leq \infty)$ is defined by $I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_1^t (\ln \frac{t}{s})^{\alpha-1} \frac{f(s)}{s} ds$ ([9]). Let $[y, x] \subseteq \mathbb{R}$, $\delta = t\frac{d}{dt}$ and $AC_\delta^n[y, x] = \{g : [y, x] \to \mathbb{R} : \delta^{n-1}[g(t)] \in AC[y, x]\}$. The Hadamard derivative of fractional order $\alpha$ for a function $f \in AC_\delta^n[y, x]$ is defined by $D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} (t\frac{d}{dt})^n \int_1^t (\ln \frac{t}{s})^{n-\alpha-1} \frac{f(s)}{s} ds$, where $n = [\alpha] + 1$ ([9]). In fact, the function $f$ should have some properties such that the integral exists. It was proved that the general solution of the Hadamard fractional differential equation $D^\alpha x(t) = 0$ is given by

$$x(t) = c_1(\ln t)^{\alpha-1} + c_2(\ln t)^{\alpha-2} + ... + c_n(\ln t)^{\alpha-n},$$

[1]Department of Mathematics, South Tehran Branch, Islamic Azad University, Tehran, Iran, e-mail: s_t_mansouri@azad.ac.ir

[2]Department of Mathematics, Faculty of Science, Central Tehran Branch, Islamic Azad University,P.O. Box 13185/768, Tehran, Iran, e-mail: rahimi@iauctb.ac.ir

[3]Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran, e-mail: sh.rezapour@azaruniv.ac.ir

[4]Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan, e-mail: rezapourshahram@yahoo.ca

[5]Corresponding author

where $c_1, ..., c_n$ are real constants and $n = [\alpha] + 1$ ([7] and [18]).

The Hadamard-type differential inclusion problem $D^\alpha x(t) \in F(t, x(t))$ with boundary conditions $x(1) = 0$, $x(e) = I^\beta x(\eta)$ was investigated in 2013, where $1 < t < e$, $1 < \alpha \leq 2$, $\beta > 0$, $1 < \eta < e$, $D^\alpha$ is the Hadamard fractional differential, $I^\beta$ is the Hadamard fractional integral and $F : [1, e] \times \mathbb{R} \to P(\mathbb{R})$ is a multifunction ([7]). For more details on Hadamard type fractional differential equations and inclusions, we refer the reader to a recent text ([2]). In 2016, Hedayati and Rezapour investigated the $k$-dimensional systems of fractional differential inclusions

$$\begin{cases} {}^cD^{\alpha_1} u_1(t) \in F_1(t, u_1(t), \cdots, u_k(t), {}^cD^{\gamma_{11}^1} u_1(t), \cdots, {}^cD^{\gamma_{1k}^1} u_k(t), {}^cD^{\gamma_{11}^2} u_1(t), \\ \cdots, {}^cD^{\gamma_{1k}^2} u_k(t)), \\ {}^cD^{\alpha_2} u_2(t) \in F_2(t, u_1(t), \cdots, u_k(t), {}^cD^{\gamma_{21}^1} u_1(t), \cdots, {}^cD^{\gamma_{2k}^1} u_k(t), {}^cD^{\gamma_{21}^2} u_1(t), \\ \cdots, {}^cD^{\gamma_{2k}^2} u_k(t)), \\ \vdots \\ {}^cD^{\alpha_k} u_k(t) \in F_k(t, u_1(t), \cdots, u_k(t), {}^cD^{\gamma_{k1}^1} u_1(t), \cdots, {}^cD^{\gamma_{kk}^1} u_k(t), {}^cD^{\gamma_{k1}^2} u_1(t), \\ \cdots, {}^cD^{\gamma_{kk}^2} u_k(t)), \end{cases}$$

with the anti-periodic boundary value conditions $u_i(0) = -u_i(1)$,

$$t^{\beta_{1i}-1c} D^{\beta_{1i}} u_i(t)|_{t \to 0} = -t^{\beta_{1i}-1c} D^{\beta_{1i}} u_i(t)|_{t=1}$$

and $t^{\beta_{2i}-2c} D^{\beta_{2i}} u_i(t)|_{t \to 0} = -t^{\beta_{2i}-2c} D^{\beta_{2i}} u_i(t)|_{t=1}$ for $i = 1, \cdots, k$, where $k \geq 2$, $t \in J$, $2 < \alpha_i \leq 3$, $0 < \gamma_{ij}^1 \leq 1$, $1 < \gamma_{ij}^2 \leq 2$, $0 < \beta_{1i} < 1 < \beta_{2i} < 2$ and $F_i : J \times \mathbb{R}^{3k} \to 2^{\mathbb{R}}$ is a multifunction for all $1 \leq i, j \leq k$ ([17]). Using the idea of [7] and [17], we investigate the existence of solutions for a $k$-dimensional system of Langevin Hadamard-type fractional differential inclusions

(1)
$$\begin{cases} D^{\beta_1}(D^{\alpha_1} + \lambda_1) x_1(t) \in F_1(t, x_1(t), \cdots, x_k(t), I^{\nu_1} x_1(t), \cdots, I^{\nu_k} x_k(t)) \\ +G_1(t, x_1(t), \cdots, x_k(t)), \\ D^{\beta_2}(D^{\alpha_2} + \lambda_2) x_2(t) \in F_2(t, x_1(t), \cdots, x_k(t), I^{\nu_1} x_1(t), \cdots, I^{\nu_k} x_k(t)) \\ +G_2(t, x_1(t), \cdots, x_k(t)), \\ \vdots \\ D^{\beta_k}(D^{\alpha_k} + \lambda_k) x_k(t) \in F_k(t, x_1(t), \cdots, x_k(t), I^{\nu_1} x_1(t), \cdots, I^{\nu_k} x_k(t)) \\ +G_k(t, x_1(t), \cdots, x_k(t)), \end{cases}$$

with the boundary conditions

$$(2) \qquad \begin{cases} x_i(1) = 0, \\ I^{\gamma_i} x_i(\eta) + D^{\gamma_i} x_i(\eta) = 0, \\ I^{\gamma_i} x_i(e) + D^{\gamma_i} x_i(e) = 0 \end{cases}$$

for $i = 1, \cdots, k$, where $1 < \beta_i \leq 2$, $0 < \gamma_i < \alpha_i < 1$, $\nu_i > 0$ for all $i$, $1 < \eta < e$, $t \in [1, e]$, $D^{(\cdot)}$ is the fractional Hadamard derivative, $I^{(\cdot)}$ is the Hadamard integral, $F_1, \ldots, F_k : [1, e] \times \mathbb{R}^{2k} \to 2^{\mathbb{R}}$ and $G_1, \ldots, G_k : [1, e] \times \mathbb{R}^k \to 2^{\mathbb{R}}$ are some multifunctions. We say that $G : [1, e] \times \mathbb{R}^k \to 2^{\mathbb{R}}$ is a Caratheodory multifunction whenever $t \mapsto G(t, x_1, \cdots, x_k)$ is measurable for all $x_1, \cdots, x_k \in \mathbb{R}$ and $(x_1, \cdots, x_k) \mapsto G(t, x_1, \cdots, x_k)$ is an upper semi-continuous map for almost all $t \in [1, e]$ (see [10], [15] and [19]). Also, a Caratheodory multifunction $G : [1, e] \times \mathbb{R}^k \to 2^{\mathbb{R}}$ is $L^1$-Caratheodory whenever for each $\rho > 0$ there exists $\phi_\rho \in L^1([1, e], \mathbb{R}^+)$ such that

$$\| G(t, x_1, \cdots, x_k) \| = \sup\{|v| : v \in G(t, x_1, \cdots, x_k)\} \leq \phi_\rho(t)$$

for all $|x_1|, \cdots, |x_k| \leq \rho$ and for almost all $t \in [1, e]$ (see [10], [15] and [19]). Define the space $X = C([1, e], \mathbb{R})$ endowed with the norm $\|x\| = \sup_{t\in[1,e]} |x(t)|$. Then, $(X, \|.\|)$ and the product space $(X^k = \underbrace{X \times X \times \cdots \times X}_{k}, \|.\|_*)$ endowed with the norm $\|(x_1, x_2, \ldots, x_k)\|_* = \|x_1\| + \|x_2\| + \cdots + \|x_k\|$ are Banach spaces. As argued in [8], [12], [21] and [23], we define the set of the selections of the multifunctions $F_i$ and $G_i$ at $(x_1, ..., x_k)$ by

$$\begin{aligned} &S_{F_i,(x_1,\ldots,x_k)} \\ &= \{v \in L^1[1, e] : v(t) \in F_i(t, x_1(t), \cdots, x_k(t), I^{\nu_1} x_1(t), \cdots, I^{\nu_k} x_k(t)) \\ &\quad \text{for a.e. } t \in [1, e]\} \end{aligned}$$

and

$$\begin{aligned} &S_{G_i,(x_1,\ldots,x_k)} \\ &= \{v \in L^1[1, e] : v(t) \in G_i(t, x_1(t), \cdots, x_k(t)) \text{ for almost all } t \in [1, e]\} \end{aligned}$$

for all $1 \leq i \leq k$.

Let $(X, d)$ be a metric space. Denote by $P(X)$ and $2^X$ the class of all subsets and the class of all nonempty subsets of $X$, respectively. Also, we denote by $P_{cl}(X)$, $P_{bd}(X)$, $P_{cv}(X)$ and $P_{cp}(X)$ the class of all closed, bounded, convex and compact subsets of $X$, respectively. A mapping $Q : X \to 2^X$ is called a multifunction on $X$ and $x \in X$ is called a fixed point of $Q$ whenever $x \in Qx$ ([14]). A multifunction $Q : X \to P_{cl}(X)$ is lower semi-continuous (briefly $l.s.c$) whenever the set $Q^{-1}(A) := \{x \in X : Qx \cap A \neq \emptyset\}$ is open for any open subset $A$ of $X$. Also, $Q$ is upper semi-continuous (briefly $u.s.c$) whenever $\{x \in X : Qx \subset A\}$ is open for all open set $A$ of $X$. A multifunction $Q : X \to P_{cp}(X)$ is compact whenever $\overline{Q(S)}$ is a compact subset of $X$ for each bounded subsets $S$ of $X$. A multifunction $G : [a, b] \to P_{cl}(\mathbb{R})$ is said to be measurable whenever the function $t \mapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$ is measurable for all $y \in \mathbb{R}$. Define the Pompeiu-Hausdorff metric $H : 2^X \times 2^X \to [0, \infty)$ by $H(A, B) = \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\}$, where $d(A, b) = \inf_{a\in A} d(a; b)$ ([11]). Then $(P_{b,cl}(X), H)$ is a metric space and $(P_{cl}(X), H)$ is a generalized metric space ((([11]) and [19]). A multifunction $N : X \to P_{cl}(X)$ is contractive whenever there exists $\gamma \in (0, 1)$ such that $H(N(x), N(y)) \leq \gamma d(x, y)$ for all $x, y \in X$. In 1970, Covitz and Nadler proved that each closed valued contractive multifunction on a complete metric space has a fixed point ([14]).

We need the following theorems to prove our main results.

**Lemma 1.1.** *([15]) If $G : X \to P_{cl}(Y)$ is upper semi-continuous, then $Gr(G)$ is a closed subset of $X \times Y$. If $G$ is completely continuous and has a closed graph, then it is upper semi-continuous.*

**Lemma 1.2.** *([20]) Let $X$ be a Banach space, $F : J \times X \to P_{cp,cv}(X)$ an $L^1$-Caratheodory multifunction and $\Theta$ a linear continuous mapping from $L^1(J, X)$ to $C(J, X)$. Then the operator $\Theta o S_F : C(J, X) \to P_{cp,cv}(C(J), X)$ defined by $(\Theta o S_F)(x) = \Theta(S_{F,x})$ is a closed graph operator in $C(J, X) \times C(J, X)$.*

**Lemma 1.3.** *[16] Suppose that $B(0, r)$ and $B[0, r]$ denote the open and closed balls centered at the origin and of radius $r$ in a Banach space $X$. Also, assume that $\Phi_1 : X \to P_{bd,cl,cv}(X)$ and $\Phi_2 : B[0, r] \to P_{cp,cv}(X)$ are two multivalued operators such that $\Phi_1$ is contraction and $\Phi_2$ is upper semi-continuous and completely continuous. Then either the operator inclusion $x \in \Phi_1(x) + \Phi_2(x)$ has a solution in $B[0, r]$ or there exists $u \in X$ with $\|u\| = r$ such that $\lambda u \in \Phi_1(u) + \Phi_2(u)$.*

## 2. Main results

**Lemma 2.1.** *Let $v \in AC([1, e], \mathbb{R})$, $\lambda \in \mathbb{R}$, $\beta \in (1, 2]$, $\alpha \in (0, 1]$, $\eta \in (1, e)$, $D^{(.)}$ be the Hadamard fractional derivative and $I^{(.)}$ the Hadamard fractional integral. Then the unique solution of the fractional problem $D^\beta(D^\alpha + \lambda)x(t) = v(t)$ with boundary conditions*

$$\begin{cases} x(1) = 0, \\ I^\gamma x(\eta) + D^\gamma x(\eta) = 0, \\ I^\gamma x(e) + D^\gamma x(e) = 0 \end{cases} \tag{3}$$

*is given by*

$$\begin{aligned}
&x(t) = \\
&\frac{1}{\Gamma(\alpha)} \int_1^t (\ln \frac{t}{s})^{\alpha-1} \frac{1}{s} \left( \frac{1}{\Gamma(\beta)} \int_1^s (\ln \frac{s}{u})^{\beta-1} \frac{v(u)}{u} du - \lambda x(s) \right) ds \\
&- \Delta_1 (\ln t)^{\alpha+\beta-1} \left[ N_1 \int_1^\eta \left( \frac{1}{\Gamma(\alpha+\gamma)} (\ln \frac{\eta}{s})^{\alpha+\gamma-1} + \frac{1}{\Gamma(\alpha-\gamma)} (\ln \frac{\eta}{s})^{\alpha-\gamma-1} \right) \frac{1}{s} \right. \\
&\times \left( \frac{1}{\Gamma(\beta)} \int_1^s (\ln \frac{s}{u})^{\beta-1} \frac{v(u)}{u} du - \lambda x(s) \right) ds \Bigg] - N_2 \int_1^e \left( \frac{1}{\Gamma(\alpha+\gamma)} (\ln \frac{e}{s})^{\alpha+\gamma-1} \right. \\
&+ \left. \frac{1}{\Gamma(\alpha-\gamma)} (\ln \frac{e}{s})^{\alpha-\gamma-1} \right) \frac{1}{s} \left( \frac{1}{\Gamma(\beta)} \int_1^s (\ln \frac{s}{u})^{\beta-1} \frac{v(u)}{u} du - \lambda x(s) \right) ds \Bigg] \\
&- \Delta_2 (\ln t)^{\alpha+\beta-1} \int_1^\eta \left( \frac{1}{\Gamma(\alpha+\gamma)} (\ln \frac{\eta}{s})^{\alpha+\gamma-1} + \frac{1}{\Gamma(\alpha-\gamma)} (\ln \frac{\eta}{s})^{\alpha-\gamma-1} \right) \\
&\times \frac{1}{s} \left( \frac{1}{\Gamma(\beta)} \int_1^s (\ln \frac{s}{u})^{\beta-1} \frac{v(u)}{u} du - \lambda x(s) \right) ds + \Delta_3 (\ln t)^{\alpha+\beta-2} \\
&\times \left[ N_1 \int_1^\eta \left( \frac{1}{\Gamma(\alpha+\gamma)} (\ln \frac{\eta}{s})^{\alpha+\gamma-1} + \frac{1}{\Gamma(\alpha-\gamma)} (\ln \frac{\eta}{s})^{\alpha-\gamma-1} \right) \frac{1}{s} \right.
\end{aligned}$$

$$\times\left(\frac{1}{\Gamma(\beta)}\int_1^s(\ln\frac{s}{u})^{\beta-1}\frac{v(u)}{u}du-\lambda x(s)\right)ds$$
$$-N_2\int_1^e\left(\frac{1}{\Gamma(\alpha+\gamma)}(\ln\frac{e}{s})^{\alpha+\gamma-1}+\frac{1}{\Gamma(\alpha-\gamma)}(\ln\frac{e}{s})^{\alpha-\gamma-1}\right)$$
(4)
$$\times\frac{1}{s}\left(\frac{1}{\Gamma(\beta)}\int_1^s(\ln\frac{s}{u})^{\beta-1}\frac{v(u)}{u}du-\lambda x(s)\right)ds\Big]$$

*where* $N_1=\Big((\alpha+\beta+\gamma)+\Gamma(\alpha+\beta-\gamma)\Big)$,

$N_2=-\Big(\Gamma(\alpha+\beta+\gamma)(\ln\eta)^{\alpha+\beta-\gamma-1}+\Gamma(\alpha+\beta-\gamma)(\ln\eta)^{\alpha+\beta+\gamma-1}\Big)$,

$\Delta_1=-\frac{\Big(\Gamma(\alpha+\beta+\gamma-1)(\ln\eta)^{\alpha+\beta-\gamma-2}+\Gamma(\alpha+\beta-\gamma-1)(\ln\eta)^{\alpha+\beta+\gamma-2}\Big)\Gamma(\alpha+\beta-\gamma)\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha+\beta)\Big(\Gamma(\alpha+\beta+\gamma)(\ln\eta)^{\alpha+\beta-\gamma-1}+\Gamma(\alpha+\beta-\gamma)(\ln\eta)^{\alpha+\beta+\gamma-1}\Big)\Delta}$,

$\Delta_2=-\frac{\Gamma(\alpha+\beta-\gamma)\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha+\beta)\Big(\Gamma(\alpha+\beta+\gamma)(\ln\eta)^{\alpha+\beta-\gamma-1}+\Gamma(\alpha+\beta-\gamma)(\ln\eta)^{\alpha+\beta+\gamma-1}\Big)}$,

$\Delta_3=\frac{\Gamma(\alpha+\beta-\gamma-1)\Gamma(\alpha+\beta+\gamma-1)}{\Gamma(\alpha+\beta-1)\Delta}$ *and*

$$\begin{aligned}\Delta=&\Big(\Gamma(\alpha+\beta-\gamma-1)+\Gamma(\alpha+\beta+\gamma-1)\Big)\\&\cdot\Big(\Gamma(\alpha+\beta+\gamma)(\ln\eta)^{\alpha+\beta-\gamma-1}+\Gamma(\alpha+\beta-\gamma)(\ln\eta)^{\alpha+\beta+\gamma-1}\Big)\\&-\Big(\Gamma(\alpha+\beta-\gamma)+\Gamma(\alpha+\beta+\gamma)\Big)\\&\cdot\Big(\Gamma(\alpha+\beta+\gamma-1)(\ln\eta)^{\alpha+\beta-\gamma-2}+\Gamma(\alpha+\beta-\gamma-1)(\ln\eta)^{\alpha+\beta+\gamma-2}\Big)\end{aligned}$$

*Proof.* It is known that the general solution of the equation $D^\beta(D^\alpha+\lambda)x(t)=v(t)$ can be written as

$$\begin{aligned}x(t)\;=\;&\frac{1}{\Gamma(\alpha)}\int_1^t(\ln\frac{t}{s})^{\alpha-1}\frac{1}{s}\left(\frac{1}{\Gamma(\beta)}\int_1^s(\ln\frac{s}{u})^{\beta-1}\frac{v(u)}{u}du-\lambda x(s)\right)ds\\&+c_1\frac{\Gamma(\beta)(\ln t)^{\beta+\alpha-1}}{\Gamma(\alpha+\beta)}+c_2\frac{\Gamma(\beta-1)(\ln t)^{\beta+\alpha-1}}{\Gamma(\alpha+\beta-1)}+c_3(\ln t)^{\alpha-1}\end{aligned}\tag{5}$$

where $c_1,c_2,c_3$ are arbitrary constants and $t\in[1,e]$. From (5), we have

$$\begin{aligned}&I^\gamma x(t)\\&=\frac{1}{\Gamma(\gamma+\alpha)}\int_1^t(\ln\frac{t}{s})^{\alpha+\gamma-1}\frac{1}{s}\left(\frac{1}{\Gamma(\beta)}\int_1^s(\ln\frac{s}{u})^{\beta-1}\frac{v(u)}{u}du-\lambda x(s)\right)ds\\&+c_1\frac{\Gamma(\beta)(\ln t)^{\beta+\gamma+\alpha-1}}{\Gamma(\alpha+\beta+\gamma)}+c_2\frac{\Gamma(\beta-1)(\ln t)^{\beta+\gamma+\alpha-2}}{\Gamma(\alpha+\gamma+\beta-1)}+c_3\frac{\Gamma(\alpha)(\ln t)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)}\end{aligned}$$

and

$$
\begin{aligned}
&D^{\gamma}x(t)\\
&= \frac{1}{\Gamma(\alpha-\gamma)}\int_1^t (\ln\frac{t}{s})^{\alpha-\gamma-1}\frac{1}{s}\left(\frac{1}{\Gamma(\beta)}\int_1^s (\ln\frac{s}{u})^{\beta-1}\frac{v(u)}{u}du-\lambda x(s)\right)ds\\
&\quad +c_1\frac{\Gamma(\beta)(\ln t)^{\alpha+\beta-\gamma-1}}{\Gamma(\alpha+\beta-\gamma)}+c_2\frac{\Gamma(\beta-1)(\ln t)^{\alpha+\beta-\gamma-2}}{\Gamma(\alpha+\gamma-\beta-1)}+c_3\frac{\Gamma(\alpha)(\ln t)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}.
\end{aligned}
$$

Using the boundary we obtain $c_3=0$,

$$
\begin{aligned}
&c_1\\
&= -\frac{\Big(\Gamma(\alpha+\beta+\gamma-1)(\ln\eta)^{\alpha+\beta-\gamma-2}+\Gamma(\alpha+\beta-\gamma-1)(\ln\eta)^{\alpha+\beta+\gamma-2}\Big)}{\Gamma(\beta)\Big(\Gamma(\alpha+\beta+\gamma)(\ln\eta)^{\alpha+\beta-\gamma-1}+\Gamma(\alpha+\beta-\gamma)(\ln\eta)^{\alpha+\beta+\gamma-1}\Big)\Delta}\\
&\quad\times\Gamma(\alpha+\beta-\gamma)\Gamma(\alpha+\beta+\gamma)\Big(\Gamma(\alpha+\beta+\gamma)+\Gamma(\alpha+\beta-\gamma)\Big)\\
&\quad\times\int_1^{\eta}\left(\frac{1}{\Gamma(\alpha+\gamma)}(\ln\frac{\eta}{s})^{\alpha+\gamma-1}+\frac{1}{\Gamma(\alpha-\gamma)}(\ln\frac{\eta}{s})^{\alpha-\gamma-1}\right)\\
&\quad\times\frac{1}{s}\left(\frac{1}{\Gamma(\beta)}\int_1^s(\ln\frac{s}{u})^{\beta-1}\frac{v(u)}{u}du-\lambda x(s)\right)ds\\
&\quad-\Big(\Gamma(\alpha+\beta+\gamma)(\ln\eta)^{\alpha+\beta-\gamma-1}+\Gamma(\alpha+\beta-\gamma)(\ln\eta)^{\alpha+\beta+\gamma-1}\Big)\\
&\quad\times\int_1^{e}\left(\frac{1}{\Gamma(\alpha+\gamma)}(\ln\frac{e}{s})^{\alpha+\gamma-1}+\frac{1}{\Gamma(\alpha-\gamma)}(\ln\frac{e}{s})^{\alpha-\gamma-1}\right)\\
&\quad\times\frac{1}{s}\left(\frac{1}{\Gamma(\beta)}\int_1^s(\ln\frac{s}{u})^{\beta-1}\frac{v(u)}{u}du-\lambda x(s)\right)ds\Big]\\
&\quad-\frac{\Gamma(\alpha+\beta-\gamma)\Gamma(\alpha+\beta+\gamma)}{\Gamma(\beta)\Big(\Gamma(\alpha+\beta+\gamma)(\ln\eta)^{\alpha+\beta-\gamma-1}+\Gamma(\alpha+\beta-\gamma)(\ln\eta)^{\alpha+\beta+\gamma-1}\Big)}\\
&\quad\times\int_1^{\eta}\left(\frac{1}{\Gamma(\alpha+\gamma)}(\ln\frac{\eta}{s})^{\alpha+\gamma-1}+\frac{1}{\Gamma(\alpha-\gamma)}(\ln\frac{\eta}{s})^{\alpha-\gamma-1}\right)\\
&\quad\times\frac{1}{s}\left(\frac{1}{\Gamma(\beta)}\int_1^s(\ln\frac{s}{u})^{\beta-1}\frac{v(u)}{u}du-\lambda x(s)\right)ds
\end{aligned}
$$

and

$$
\begin{aligned}
&c_2\\
&= \frac{\Gamma(\alpha+\beta-\gamma-1)\Gamma(\alpha+\beta+\gamma-1)}{\Gamma(\beta-1)\Delta}\Big[\Big(\Gamma(\alpha+\beta+\gamma)+\Gamma(\alpha+\beta-\gamma)\Big)\\
&\quad\times\int_1^{\eta}\left(\frac{1}{\Gamma(\alpha+\gamma)}(\ln\frac{\eta}{s})^{\alpha+\gamma-1}+\frac{1}{\Gamma(\alpha-\gamma)}(\ln\frac{\eta}{s})^{\alpha-\gamma-1}\right)\\
&\quad\times\frac{1}{s}\left(\frac{1}{\Gamma(\beta)}\int_1^s(\ln\frac{s}{u})^{\beta-1}\frac{v(u)}{u}du-\lambda x(s)\right)ds
\end{aligned}
$$

$$-\Big(\Gamma(\alpha+\beta+\gamma)(\ln\eta)^{\alpha+\beta-\gamma-1}+\Gamma(\alpha+\beta-\gamma)(\ln\eta)^{\alpha+\beta+\gamma-1}\Big)$$
$$\int_1^e\Big(\frac{1}{\Gamma(\alpha+\gamma)}(\ln\frac{e}{s})^{\alpha+\gamma-1}+\frac{1}{\Gamma(\alpha-\gamma)}(\ln\frac{e}{s})^{\alpha-\gamma-1}\Big)$$
$$\times\frac{1}{s}\Big(\frac{1}{\Gamma(\beta)}\int_1^s(\ln\frac{s}{u})^{\beta-1}\frac{v(u)}{u}du-\lambda x(s)\Big)ds\Big].$$

Inserting the values of $c_1$, $c_2$ and $c_3$ in (5), we obtain the solution (4). The converse follows by direct computation. This completes the proof. □

We say that an element $(x_1,x_2,...,x_k)\in X^k$ is a solution for the $k$-dimensional inclusions problem (1)-(2) whenever there exist functions

$$(v_1,v_2,...,v_k),(v'_1,v'_2,...,v'_k)\in\underbrace{L^1[1,e]\times L^1[1,e]\times...\times L^1[1,e]}_{k}$$

such that

$$v_i(t)\in F_i(t,x_1(t),\cdots,x_k(t),I^{\nu_1}x_1(t),\cdots,I^{\nu_k}x_k(t))v'_i(t)\in G_i(t,x_1(t),\cdots,x_k(t))$$

and $D^{\beta_i}(D^{\alpha_i}+\lambda_i)x_i(t)=v_i(t)+v'_i(t)$ for almost all $t\in[1,e]$, $x_i(1)=0$, $I^{\gamma_i}x_i(\eta)+D^{\gamma_i}x_i(\eta)=0$ and $I^{\gamma_i}x_i(e)+D^{\gamma_i}x_i(e)=0$ for $i=1,\cdots,k$. Now, put

$$\begin{aligned}\Lambda_1^i\\ =\;&\frac{1}{\Gamma(\alpha_i+\beta_i+1)}+\Big(\frac{d_i\Gamma(\alpha_i+\beta_i-\gamma_i)\Gamma(\alpha_i+\beta_i+\gamma_i)}{\Gamma(\alpha_i+\beta_i)e_i|\Delta_i|}\\
&+\frac{\Gamma(\alpha_i+\beta_i-\gamma_i-1)\Gamma(\alpha_i+\beta_i+\gamma_i-1)}{\Gamma(\alpha_i+\beta_i-1)|\Delta_i|}\Big)\\
&\Big((\Gamma(\alpha_i+\beta_i+\gamma_i)+\Gamma(\alpha_i+\beta_i-\gamma_i))(\frac{(\ln\eta)^{\alpha_i+\beta_i+\gamma_i}}{\Gamma(\alpha_i+\beta_i+\gamma_i+1)}\\
&+\frac{(\ln\eta)^{\alpha_i+\beta_i-\gamma_i}}{\Gamma(\alpha_i+\beta_i-\gamma_i+1)})+e_i(\frac{1}{\Gamma(\alpha_i+\beta_i+\gamma_i+1)}+\frac{1}{\Gamma(\alpha_i+\beta_i-\gamma_i+1)})\Big)\\
&+\frac{\Gamma(\alpha_i+\beta_i-\gamma_i)\Gamma(\alpha_i+\beta_i+\gamma_i)}{\Gamma(\alpha_i+\beta_i)e_i}\\
(6)\quad&\Big(\frac{(\ln\eta)^{\alpha_i+\beta_i+\gamma_i}}{\Gamma(\alpha_i+\beta_i+\gamma_i+1)}+\frac{(\ln\eta)^{\alpha_i+\beta_i-\gamma_i}}{\Gamma(\alpha_i+\beta_i-\gamma_i+1)}\Big)\end{aligned}$$

and

$$\begin{aligned}\Lambda_2^i=&\frac{1}{\Gamma(\alpha_i+1)}+\\
&\Big(\frac{d_i\Gamma(\alpha_i+\beta_i-\gamma_i)\Gamma(\alpha_i+\beta_i+\gamma_i)}{\Gamma(\alpha_i+\beta_i)e_i|\Delta_i|}+\frac{\Gamma(\alpha_i+\beta_i-\gamma_i-1)\Gamma(\alpha_i+\beta_i+\gamma_i-1)}{\Gamma(\alpha_i+\beta_i-1)|\Delta_i|}\Big)\\
&\Big((\Gamma(\alpha_i+\beta_i+\gamma_i)+\Gamma(\alpha_i+\beta_i-\gamma_i))(\frac{(\ln\eta)^{\alpha_i+\gamma_i}}{\Gamma(\alpha_i+\gamma_i+1)}+\frac{(\ln\eta)^{\alpha_i-\gamma_i}}{\Gamma(\alpha_i-\gamma_i+1)})\end{aligned}$$

$$+ e_i(\frac{1}{\Gamma(\alpha_i+\gamma_i+1)} + \frac{1}{\Gamma(\alpha_i-\gamma_i+1)})\Big)$$

(7)

$$+ \frac{\Gamma(\alpha_i+\beta_i-\gamma_i)\Gamma(\alpha_i+\beta_i+\gamma_i)}{\Gamma(\alpha_i+\beta_i)e_i}\Big(\frac{(\ln\eta)^{\alpha_i+\gamma_i}}{\Gamma(\alpha_i+\gamma_i+1)} + \frac{(\ln\eta)^{\alpha_i-\gamma_i}}{\Gamma(\alpha_i-\gamma_i+1)}\Big)\Big]$$

for all $1 \le i \le k$, where

$$d_i = \Big(\Gamma(\alpha_i+\beta_i+\gamma_i-1)(\ln\eta)^{\alpha_i+\beta_i-\gamma_i-2} + \Gamma(\alpha_i+\beta_i-\gamma_i-1)(\ln\eta)^{\alpha_i+\beta_i+\gamma_i-2}\Big)$$

and $e_i = \Big(\Gamma(\alpha_i+\beta_i+\gamma_i)(\ln\eta)^{\alpha_i+\beta_i-\gamma_i-1} + \Gamma(\alpha_i+\beta_i-\gamma_i)(\ln\eta)^{\alpha_i+\beta_i+\gamma_i-1}\Big).$

**Theorem 2.2.** *Suppose that $G_1,\dots,G_k : [1,e]\times\mathbb{R}^k \to P_{cp,cv}(\mathbb{R})$ are some Caratheodory multifunctions and $F_1,\dots,F_k : [1,e]\times\mathbb{R}^{2k} \to P_{cp,cv}(\mathbb{R})$ are multifunctions such that $F_i(.,x_1,...,x_{2k}) : [1,e] \to P_{cp,cv}(\mathbb{R})$ are measurable and there exist continuous functions $p_i, m_i : [1,e] \to (0,\infty)$ and continuous nondecreasing functions $\psi_i : [0,\infty) \to [0,\infty)$ such that*

$$\|G_i(t,x_1,\cdots,x_k)\| = \sup\{|v| : v \in G_i(t,x_1,\cdots,x_k)\} \le p_i(t)\psi_i(\sum_{i=1}^{k}|x_i|),$$

$$\|F_i(t,x_1,\cdots,x_{2k})\| = \sup\{|v| : v \in F_i(t,x_1,\cdots,x_{2k})\} \le m_i(t)$$

*and*

$$H(F_i(t,x_1,...,x_{2k}),F_i(t,y_1,...,y_{2k})) \le m_i(t)\sum_{i=1}^{2k}|x_i-y_i|$$

*for all $x_1,...,x_{2k},y_1,...,y_{2k} \in \mathbb{R}$, $t \in [1,e]$ and $1 \le i \le k$. If*

$$\sum_{i=1}^{k}\Big(\|m_i\|\Lambda_1^i\sum_{j=1}^{k}(1+\frac{1}{\Gamma(\nu_j+1)}) + |\lambda_i|\Lambda_2^i\Big) < 1,$$

*then the $k$-dimensional system of fractional differential inclusions* (1) *has at least one solution in* $[1,e]$.

*Proof.* Let $r$ be a real number such that

$$\frac{\sum_{i=1}^{k}\|p_i\|\psi_i(r)\Lambda_1^i}{1-\sum_{i=1}^{k}\Big(\|m_i\|\Lambda_1^i\sum_{j=1}^{k}(1+\frac{1}{\Gamma(\nu_j+1)}) + 2|\lambda_i|\Lambda_2^i\Big)} < r$$

and $B(0,r)$ the related open ball in $X^k$. Consider the multi-valued operators

$A,B : X^k \to P(X^k)$ by $A(x_1,\cdots,x_k) = \begin{pmatrix} A_1(x_1,\cdots,x_k) \\ A_2(x_1,\cdots,x_k) \\ \vdots \\ A_k(x_1,\cdots,x_k)\end{pmatrix}$, $B(x_1,\cdots,x_k) =$

$\begin{pmatrix} B_1(x_1,\cdots,x_k) \\ B_2(x_1,\cdots,x_k) \\ \vdots \\ B_k(x_1,\cdots,x_k) \end{pmatrix}$, where $\Lambda_1^i$ and $\Lambda_2^i$ are defined by (6) and (7) for all $1 \le i \le k$,

$$A_i(x_1,\cdots,x_k) := \{u \in X : \exists v \in S_{F_i,(x_1,\cdots,x_k)} \text{ such that } u(t) = w_i(v,t) \text{ for all } t \in [1,e]\},$$

$$B_i(x_1,\cdots,x_k) := \{u \in X : \exists v \in S_{G_i,(x_1,\cdots,x_k)} \text{ such that } u(t) = w_i(v,t) \text{ for all } t \in [1,e]\},$$

$$w_i(v,t) = \frac{1}{\Gamma(\alpha_i)}\int_1^t (\ln\frac{t}{s})^{\alpha_i-1}\frac{1}{s}\Big(\frac{1}{\Gamma(\beta_i)}\int_1^s(\ln\frac{s}{u})^{\beta_i-1}\frac{v(u)}{u}du - \lambda_i x_i(s)\Big)ds$$
$$-\frac{d_i\Gamma(\alpha_i+\beta_i-\gamma_i)\Gamma(\alpha_i+\beta_i+\gamma_i)(\ln t)^{\alpha_i+\beta_i-1}}{\Gamma(\alpha_i+\beta_i)e_i\Delta_i}\times$$
$$\Big[\Big(\Gamma(\alpha_i+\beta_i+\gamma_i)+\Gamma(\alpha_i+\beta_i-\gamma_i)\Big)\int_1^\eta\Big(\frac{1}{\Gamma(\alpha_i+\gamma_i)}(\ln\frac{\eta}{s})^{\alpha_i+\gamma_i-1}+\frac{1}{\Gamma(\alpha_i-\gamma_i)}(\ln\frac{\eta}{s})^{\alpha_i-\gamma_i-1}\Big)$$
$$\frac{1}{s}\Big(\frac{1}{\Gamma(\beta_i)}\int_1^s(\ln\frac{s}{u})^{\beta_i-1}\frac{v(u)}{u}du - \lambda_i x_i(s)\Big)ds$$
$$-e_i\int_1^e\Big(\frac{1}{\Gamma(\alpha_i+\gamma_i)}(\ln\frac{e}{s})^{\alpha_i+\gamma_i-1}+\frac{1}{\Gamma(\alpha_i-\gamma_i)}(\ln\frac{e}{s})^{\alpha_i-\gamma_i-1}\Big)\frac{1}{s}\Big(\frac{1}{\Gamma(\beta_i)}\int_1^s(\ln\frac{s}{u})^{\beta_i-1}\frac{v(u)}{u}du-\lambda_i x_i(s)\Big)ds\Big]$$
$$-\frac{\Gamma(\alpha_i+\beta_i-\gamma_i)\Gamma(\alpha_i+\beta_i+\gamma_i)(\ln t)^{\alpha_i+\beta_i-1}}{\Gamma(\alpha_i+\beta_i)e_i}\times\int_1^\eta\Big(\frac{1}{\Gamma(\alpha_i+\gamma_i)}(\ln\frac{\eta}{s})^{\alpha_i+\gamma_i-1}+\frac{1}{\Gamma(\alpha_i-\gamma_i)}(\ln\frac{\eta}{s})^{\alpha_i-\gamma_i-1}\Big)$$
$$\frac{1}{s}\Big(\frac{1}{\Gamma(\beta_i)}\int_1^s(\ln\frac{s}{u})^{\beta_i-1}\frac{v(u)}{u}du - \lambda_i x_i(s)\Big)ds + \frac{\Gamma(\alpha_i+\beta_i-\gamma_i-1)\Gamma(\alpha_i+\beta_i+\gamma_i-1)(\ln t)^{\alpha_i+\beta_i-2}}{\Gamma(\alpha_i+\beta_i-1)\Delta_i}$$
$$\Big[\Big(\Gamma(\alpha_i+\beta_i+\gamma_i)+\Gamma(\alpha_i+\beta_i-\gamma_i)\Big)\int_1^\eta\Big(\frac{1}{\Gamma(\alpha_i+\gamma_i)}(\ln\frac{\eta}{s})^{\alpha_i+\gamma_i-1}+\frac{1}{\Gamma(\alpha_i-\gamma_i)}(\ln\frac{\eta}{s})^{\alpha_i-\gamma_i-1}\Big)$$
$$\frac{1}{s}\Big(\frac{1}{\Gamma(\beta_i)}\int_1^s(\ln\frac{s}{u})^{\beta_i-1}\frac{v(u)}{u}du - \lambda_i x_i(s)\Big)ds$$
$$-e_i\times\int_1^e\Big(\frac{1}{\Gamma(\alpha_i+\gamma_i)}(\ln\frac{e}{s})^{\alpha_i+\gamma_i-1}+\frac{1}{\Gamma(\alpha_i-\gamma_i)}(\ln\frac{e}{s})^{\alpha_i-\gamma_i-1}\Big)$$
$$\frac{1}{s}\Big(\frac{1}{\Gamma(\beta_i)}\int_1^s(\ln\frac{s}{u})^{\beta_i-1}\frac{v(u)}{u}du - \lambda_i x_i(s)\Big)ds\Big]$$

and $\Delta_i = \Big(\Gamma(\alpha_i+\beta_i-\gamma_i-1)+\Gamma(\alpha_i+\beta_i+\gamma_i-1)\Big)\times e_i - \Big(\Gamma(\alpha_i+\beta_i-\gamma_i)+\Gamma(\alpha_i+\beta_i+\gamma_i)\Big)\times d_i$ for all $1 \le i \le k$. Thus, the $k$-dimensional system of fractional differential inclusions (1)-(2) is equivalent to the inclusion problem $(x_1,...,x_k) \in A(x_1,...,x_k)+B(x_1,...,x_k)$. We show that the multifunctions $A$ and $B$ satisfy the conditions of Lemma 1.3. First, we show that $B(x_1,...,x_k) \in P_{cl}(X^k)$ for all $(x_1,...,x_k) \in X^k$. Let $\{(u_1^n,\cdots,u_k^n)\}_{n\ge1}$ be a sequence in $B(x_1,\cdots,x_k)$ such that $(u_1^n,\cdots,u_k^n) \to (u_1^0,\cdots,u_k^0)$. Choose

$$(v_1^n,\cdots,v_k^n) \in S_{G_1,(x_1,\cdots,x_k)} \times S_{G_2,(x_1,\cdots,x_k)} \times \cdots \times S_{G_k,(x_1,\cdots,x_k)}$$

such that $u_i^n(t) = w_i(v_i^n,t)$ for all $t \in [1,e]$ and $i = 1,\cdots,k$. Since $G_i$ is compact valued for all $i$, $\{v_i^n\}_{n\ge1}$ has a convergent subsequence to some $v_i^0 \in L^1([1,e],\mathbb{R})$. Denote the subsequence again by $\{v_i^n\}_{n\ge1}$. It is easy to check that $v_i^0 \in S_{G_i(x_1,\cdots,x_k)}$ and $u_i^0(t) = w_i(v_i^0,t)$ for all $t \in [1,e]$. This implies that $u_i^0 \in B_i(x_1,\cdots,x_k)$ for all $i$ and so $(u_1^0,\cdots,u_k^0) \in B(x_1,\cdots,x_k)$. Now, we show that $B(x_1,\cdots,x_k)$ is convex for all $(x_1,\cdots,x_k)\in X^k$. Let $(h_1,\cdots,h_k)$ and $(h'_1,\cdots,h'_k)$ be elements of $B(x_1,\cdots,x_k)$ and $0\le\mu\le 1$. Choose $v_i,v'_i\in S_{G_i,(x_1,\cdots,x_k)}$ such that $h_i(t)=w_i(v_i,t)$ and $h'_i(t)=w_i(v'_i,t)$ for almost all $t\in[1,e]$ and $1\le i\le k$. Then, we have

$$[\mu h_i+(1-\mu)h'_i](t)=\frac{1}{\Gamma(\alpha_i)}\int_1^t(\ln\frac{t}{s})^{\alpha_i-1}\frac{1}{s}\Big(\frac{1}{\Gamma(\beta_i)}\int_1^s(\ln\frac{s}{u})^{\beta_i-1}\frac{[\mu v_i(u)+(1-\mu)v'_i(u)]}{u}du-\lambda_i x_i(s)\Big)ds$$

$$-\frac{d_i\Gamma(\alpha_i+\beta_i-\gamma_i)\Gamma(\alpha_i+\beta_i+\gamma_i)(\ln t)^{\alpha_i+\beta_i-1}}{\Gamma(\alpha_i+\beta_i)e_i\Delta_i}\times$$

$$\Big[\Big(\Gamma(\alpha_i+\beta_i+\gamma_i)+\Gamma(\alpha_i+\beta_i-\gamma_i)\Big)\int_1^\eta\Big(\frac{1}{\Gamma(\alpha_i+\gamma_i)}(\ln\frac{\eta}{s})^{\alpha_i+\gamma_i-1}+\frac{1}{\Gamma(\alpha_i-\gamma_i)}(\ln\frac{\eta}{s})^{\alpha_i-\gamma_i-1}\Big)$$

$$\frac{1}{s}\Big(\frac{1}{\Gamma(\beta_i)}\int_1^s(\ln\frac{s}{u})^{\beta_i-1}\frac{[\mu v_i(u)+(1-\mu)v'_i(u)]}{u}du-\lambda_i x_i(s)\Big)ds$$

$$-e_i\times\int_1^e\Big(\frac{1}{\Gamma(\alpha_i+\gamma_i)}(\ln\frac{e}{s})^{\alpha_i+\gamma_i-1}+\frac{1}{\Gamma(\alpha_i-\gamma_i)}(\ln\frac{e}{s})^{\alpha_i-\gamma_i-1}\Big)$$

$$\frac{1}{s}\Big(\frac{1}{\Gamma(\beta_i)}\int_1^s(\ln\frac{s}{u})^{\beta_i-1}\frac{[\mu v_i(u)+(1-\mu)v'_i(u)]}{u}du-\lambda_i x_i(s)\Big)ds\Big]$$

$$-\frac{\Gamma(\alpha_i+\beta_i-\gamma_i)\Gamma(\alpha_i+\beta_i+\gamma_i)(\ln t)^{\alpha_i+\beta_i-1}}{\Gamma(\alpha_i+\beta_i)e_i}\times\int_1^\eta\Big(\frac{1}{\Gamma(\alpha_i+\gamma_i)}(\ln\frac{\eta}{s})^{\alpha_i+\gamma_i-1}+\frac{1}{\Gamma(\alpha_i-\gamma_i)}(\ln\frac{\eta}{s})^{\alpha_i-\gamma_i-1}\Big)$$

$$\frac{1}{s}\Big(\frac{1}{\Gamma(\beta_i)}\int_1^s(\ln\frac{s}{u})^{\beta_i-1}\frac{[\mu v_i(u)+(1-\mu)v'_i(u)]}{u}du-\lambda_i x_i(s)\Big)ds$$

$$+\frac{\Gamma(\alpha_i+\beta_i-\gamma_i-1)\Gamma(\alpha_i+\beta_i+\gamma_i-1)(\ln t)^{\alpha_i+\beta_i-2}}{\Gamma(\alpha_i+\beta_i-1)\Delta_i}$$

$$\Big[\Big(\Gamma(\alpha_i+\beta_i+\gamma_i)+\Gamma(\alpha_i+\beta_i-\gamma_i)\Big)\int_1^\eta\Big(\frac{1}{\Gamma(\alpha_i+\gamma_i)}(\ln\frac{\eta}{s})^{\alpha_i+\gamma_i-1}+\frac{1}{\Gamma(\alpha_i-\gamma_i)}(\ln\frac{\eta}{s})^{\alpha_i-\gamma_i-1}\Big)\times$$

$$\frac{1}{s}\Big(\frac{1}{\Gamma(\beta_i)}\int_1^s(\ln\frac{s}{u})^{\beta_i-1}\frac{[\mu v_i(u)+(1-\mu)v'_i(u)]}{u}du-\lambda_i x_i(s)\Big)ds$$

$$-e_i\int_1^e\Big(\frac{1}{\Gamma(\alpha_i+\gamma_i)}(\ln\frac{e}{s})^{\alpha_i+\gamma_i-1}+\frac{1}{\Gamma(\alpha_i-\gamma_i)}(\ln\frac{e}{s})^{\alpha_i-\gamma_i-1}\Big)$$

$$\frac{1}{s}\Big(\frac{1}{\Gamma(\beta_i)}\int_1^s(\ln\frac{s}{u})^{\beta_i-1}\frac{[\mu v_i(u)+(1-\mu)v'_i(u)]}{u}du-\lambda_i x_i(s)\Big)ds\Big]=w_i(\mu v_i+(1-\mu)v'_i,t).$$

Since $G_i$ is convex-valued, $S_{G_i,(x_1,\ldots,x_k)}$ is convex for all $1\le i\le k$ and so $[\mu h_i+(1-\mu)h'_i]$ belongs to $B_i(x_1,x_2,\ldots,x_k)$. Hence, $\mu(h_1,\cdots,h_k)+(1-\mu)(h'_1,\cdots,h'_k)\in B(x_1,\cdots,x_k)$. Now, we show that $B$ maps bounded sets of $X^k$ into bounded sets. Let $\rho>0$ and

$$B_\rho=\{(x_1,\cdots,x_k)\in X^k:\|(x_1,\cdots,x_k)\|_*\le\rho\}.$$

For $(x_1,\cdots,x_k)\in B_\rho$ and $(h_1,\cdots,h_k)\in B(x_1,\cdots,x_k)$ choose $(v_1,\cdots,v_k)$ in $S_{G_1,(x_1,\cdots,x_k)}\times\cdots\times S_{G_k,(x_1,\cdots,x_k)}$ such that $h_i(t)=w_i(v_i,t)$ for almost all $t\in[1,e]$ and $1\le i\le k$. Thus,

$$|h_i(t)|\le\frac{1}{\Gamma(\alpha_i)}\int_1^t(\ln\frac{t}{s})^{\alpha_i-1}\frac{1}{s}\Big(\frac{1}{\Gamma(\beta_i)}\int_1^s(\ln\frac{s}{u})^{\beta_i-1}\frac{|v_i(u)|}{u}du+|\lambda_i||x_i(s)|\Big)ds$$

$$+\frac{d_i\times\Gamma(\alpha_i+\beta_i-\gamma_i)\Gamma(\alpha_i+\beta_i+\gamma_i)(\ln t)^{\alpha_i+\beta_i-1}}{\Gamma(\alpha_i+\beta_i)e_i|\Delta_i|}\times$$

$$\Big[\Big(\Gamma(\alpha_i+\beta_i+\gamma_i)+\Gamma(\alpha_i+\beta_i-\gamma_i)\Big)\int_1^\eta\Big(\frac{1}{\Gamma(\alpha_i+\gamma_i)}(\ln\frac{\eta}{s})^{\alpha_i+\gamma_i-1}+\frac{1}{\Gamma(\alpha_i-\gamma_i)}(\ln\frac{\eta}{s})^{\alpha_i-\gamma_i-1}\Big)$$

$$\frac{1}{s}\Big(\frac{1}{\Gamma(\beta_i)}\int_1^s(\ln\frac{s}{u})^{\beta_i-1}\frac{|v_i(u)|}{u}du+|\lambda_i||x_i(s)|\Big)ds$$

$$+e_i \times \int_1^e \Big(\frac{1}{\Gamma(\alpha_i+\gamma_i)}(\ln\frac{e}{s})^{\alpha_i+\gamma_i-1} + \frac{1}{\Gamma(\alpha_i-\gamma_i)}(\ln\frac{e}{s})^{\alpha_i-\gamma_i-1}\Big)$$

$$\frac{1}{s}\Big(\frac{1}{\Gamma(\beta_i)}\int_1^s (\ln\frac{s}{u})^{\beta_i-1}\frac{|v_i(u)|}{u}du + |\lambda_i||x_i(s)|\Big)ds\Big]$$

$$+\frac{\Gamma(\alpha_i+\beta_i-\gamma_i)\Gamma(\alpha_i+\beta_i+\gamma_i)(\ln t)^{\alpha_i+\beta_i-1}}{\Gamma(\alpha_i+\beta_i)\times e_i}\times\int_1^\eta \Big(\frac{1}{\Gamma(\alpha_i+\gamma_i)}(\ln\frac{\eta}{s})^{\alpha_i+\gamma_i-1}+\frac{1}{\Gamma(\alpha_i-\gamma_i)}(\ln\frac{\eta}{s})^{\alpha_i-\gamma_i-1}\Big)$$

$$\frac{1}{s}\Big(\frac{1}{\Gamma(\beta_i)}\int_1^s (\ln\frac{s}{u})^{\beta_i-1}\frac{|v_i(u)|}{u}du + |\lambda_i||x_i(s)|\Big)ds + \frac{\Gamma(\alpha_i+\beta_i-\gamma_i-1)\Gamma(\alpha_i+\beta_i+\gamma_i-1)(\ln t)^{\alpha_i+\beta_i-2}}{\Gamma(\alpha_i+\beta_i-1)|\Delta_i|}$$

$$\times\Big[\Big(\Gamma(\alpha_i+\beta_i+\gamma_i)+\Gamma(\alpha_i+\beta_i-\gamma_i)\Big)\int_1^\eta \Big(\frac{1}{\Gamma(\alpha_i+\gamma_i)}(\ln\frac{\eta}{s})^{\alpha_i+\gamma_i-1}+\frac{1}{\Gamma(\alpha_i-\gamma_i)}(\ln\frac{\eta}{s})^{\alpha_i-\gamma_i-1}\Big)$$

$$\frac{1}{s}\Big(\frac{1}{\Gamma(\beta_i)}\int_1^s (\ln\frac{s}{u})^{\beta_i-1}\frac{|v_i(u)|}{u}du + |\lambda_i||x_i(s)|\Big)ds$$

$$+e_i \times \int_1^e \Big(\frac{1}{\Gamma(\alpha_i+\gamma_i)}(\ln\frac{e}{s})^{\alpha_i+\gamma_i-1} + \frac{1}{\Gamma(\alpha_i-\gamma_i)}(\ln\frac{e}{s})^{\alpha_i-\gamma_i-1}\Big)$$

$$\frac{1}{s}\Big(\frac{1}{\Gamma(\beta_i)}\int_1^s (\ln\frac{s}{u})^{\beta_i-1}\frac{|v_i(u)|}{u}du + |\lambda_i||x_i(s)|\Big)ds\Big]$$

$$\le \frac{\|p_i\|\psi(\|(x_1,\dots,x_k)\|_*)}{\Gamma(\alpha_i+\beta_i+1)} + \frac{|\lambda_i|\|(x_1,\dots,x_k)\|_*}{\Gamma(\alpha_i+1)} + \frac{d_i\Gamma(\alpha_i+\beta_i-\gamma_i)\Gamma(\alpha_i+\beta_i+\gamma_i)}{\Gamma(\alpha_i+\beta_i)e_i|\Delta_i|}\times$$

$$\Big[\Big(\Gamma(\alpha_i+\beta_i+\gamma_i)+\Gamma(\alpha_i+\beta_i-\gamma_i)\Big)\Big(\frac{\|p_i\|\psi(\|(x_1,\dots,x_k)\|_*)(\ln\eta)^{\alpha_i+\beta_i+\gamma_i}}{\Gamma(\alpha_i+\beta_i+\gamma_i+1)} + \frac{|\lambda_i|\|(x_1,\dots,x_k)\|_*(\ln\eta)^{\alpha_i+\gamma_i}}{\Gamma(\alpha_i+\gamma_i+1)}$$

$$+\frac{\|p_i\|\psi(\|(x_1,\dots,x_k)\|_*)(\ln\eta)^{\alpha_i+\beta_i-\gamma_i}}{\Gamma(\alpha_i+\beta_i-\gamma_i+1)} + \frac{|\lambda_i|\|(x_1,\dots,x_k)\|_*(\ln\eta)^{\alpha_i-\gamma_i}}{\Gamma(\alpha_i-\gamma_i+1)}\Big)+$$

$$\Big(\Gamma(\alpha_i+\beta_i+\gamma_i)(\ln\eta)^{\alpha_i+\beta_i-\gamma_i-1}+\Gamma(\alpha_i+\beta_i-\gamma_i)(\ln\eta)^{\alpha_i+\beta_i+\gamma_i-1}\Big)$$

$$\Big(\frac{\|p_i\|\psi(\|(x_1,\dots,x_k)\|_*)}{\Gamma(\alpha_i+\beta_i+\gamma_i+1)} + \frac{|\lambda_i|\|(x_1,\dots,x_k)\|_*}{\Gamma(\alpha_i+\gamma_i+1)} + \frac{\|p_i\|\psi(\|(x_1,\dots,x_k)\|_*)}{\Gamma(\alpha_i+\beta_i-\gamma_i+1)} + \frac{|\lambda_i|\|(x_1,\dots,x_k)\|_*}{\Gamma(\alpha_i-\gamma_i+1)}\Big)\Big]$$

$$+\frac{\Gamma(\alpha_i+\beta_i-\gamma_i)\Gamma(\alpha_i+\beta_i+\gamma_i)}{\Gamma(\alpha_i+\beta_i)e_i}\times\Big(\frac{\|p_i\|\psi(\|(x_1,\dots,x_k)\|_*)(\ln\eta)^{\alpha_i+\beta_i+\gamma_i}}{\Gamma(\alpha_i+\beta_i+\gamma_i+1)} + \frac{|\lambda_i|\|(x_1,\dots,x_k)\|_*(\ln\eta)^{\alpha_i+\gamma_i}}{\Gamma(\alpha_i+\gamma_i+1)}$$

$$+\frac{\|p_i\|\psi(\|(x_1,\dots,x_k)\|_*)(\ln\eta)^{\alpha_i+\beta_i-\gamma_i}}{\Gamma(\alpha_i+\beta_i-\gamma_i+1)} + \frac{|\lambda_i|\|(x_1,\dots,x_k)\|_*(\ln\eta)^{\alpha_i-\gamma_i}}{\Gamma(\alpha_i-\gamma_i+1)}\Big)$$

$$+\frac{\Gamma(\alpha_i+\beta_i-\gamma_i-1)\Gamma(\alpha_i+\beta_i+\gamma_i-1)}{\Gamma(\alpha_i+\beta_i-1)|\Delta_i|}\Big[\Big(\Gamma(\alpha_i+\beta_i+\gamma_i)+\Gamma(\alpha_i+\beta_i-\gamma_i)\Big)$$

$$\Big(\frac{\|p_i\|\psi(\|(x_1,\dots,x_k)\|_*)(\ln\eta)^{\alpha_i+\beta_i+\gamma_i}}{\Gamma(\alpha_i+\beta_i+\gamma_i+1)} + \frac{|\lambda_i|\|(x_1,\dots,x_k)\|_*(\ln\eta)^{\alpha_i+\gamma_i}}{\Gamma(\alpha_i+\gamma_i+1)}$$

$$+\frac{\|p_i\|\psi(\|(x_1,\dots,x_k)\|_*)(\ln\eta)^{\alpha_i+\beta_i-\gamma_i}}{\Gamma(\alpha_i+\beta_i-\gamma_i+1)} + \frac{|\lambda_i|\|(x_1,\dots,x_k)\|_*(\ln\eta)^{\alpha_i-\gamma_i}}{\Gamma(\alpha_i-\gamma_i+1)}\Big)$$

$$+e_i\times\Big(\frac{\|p_i\|\psi(\|(x_1,\dots,x_k)\|_*)}{\Gamma(\alpha_i+\beta_i+\gamma_i+1)} + \frac{|\lambda_i|\|(x_1,\dots,x_k)\|_*}{\Gamma(\alpha_i+\gamma_i+1)} + \frac{\|p_i\|\psi(\|(x_1,\dots,x_k)\|_*)}{\Gamma(\alpha_i+\beta_i-\gamma_i+1)} + \frac{|\lambda_i|\|(x_1,\dots,x_k)\|_*}{\Gamma(\alpha_i-\gamma_i+1)}\Big)\Big]$$

$$= \|p_i\|\psi_i(\rho)\Big[\frac{1}{\Gamma(\alpha_i+\beta_i+1)}+$$

$$\Big(\frac{d_i\Gamma(\alpha_i+\beta_i-\gamma_i)\Gamma(\alpha_i+\beta_i+\gamma_i)}{\Gamma(\alpha_i+\beta_i)e_i|\Delta_i|} + \frac{\Gamma(\alpha_i+\beta_i-\gamma_i-1)\Gamma(\alpha_i+\beta_i+\gamma_i-1)}{\Gamma(\alpha_i+\beta_i-1)|\Delta_i|}\Big)\times$$

$$\Big((\Gamma(\alpha_i+\beta_i+\gamma_i)+\Gamma(\alpha_i+\beta_i-\gamma_i))(\frac{(\ln\eta)^{\alpha_i+\beta_i+\gamma_i}}{\Gamma(\alpha_i+\beta_i+\gamma_i+1)} + \frac{(\ln\eta)^{\alpha_i+\beta_i-\gamma_i}}{\Gamma(\alpha_i+\beta_i-\gamma_i+1)})+$$

$$(\Gamma(\alpha_i+\beta_i+\gamma_i)(\ln\eta)^{\alpha_i+\beta_i-\gamma_i-1}+\Gamma(\alpha_i+\beta_i-\gamma_i)(\ln\eta)^{\alpha_i+\beta_i+\gamma_i-1})(\frac{1}{\Gamma(\alpha_i+\beta_i+\gamma_i+1)}+\frac{1}{\Gamma(\alpha_i+\beta_i-\gamma_i+1)})\Big)$$

$$+\frac{\Gamma(\alpha_i+\beta_i-\gamma_i)\Gamma(\alpha_i+\beta_i+\gamma_i)}{\Gamma(\alpha_i+\beta_i)e_i}\Big(\frac{(\ln\eta)^{\alpha_i+\beta_i+\gamma_i}}{\Gamma(\alpha_i+\beta_i+\gamma_i+1)} + \frac{(\ln\eta)^{\alpha_i+\beta_i-\gamma_i}}{\Gamma(\alpha_i+\beta_i-\gamma_i+1)}\Big)\Big]$$

$$+\rho|\lambda_i|\Big[\frac{1}{\Gamma(\alpha_i+1)} + \Big(\frac{d_i\Gamma(\alpha_i+\beta_i-\gamma_i)\Gamma(\alpha_i+\beta_i+\gamma_i)}{\Gamma(\alpha_i+\beta_i)e_i|\Delta_i|} + \frac{\Gamma(\alpha_i+\beta_i-\gamma_i-1)\Gamma(\alpha_i+\beta_i+\gamma_i-1)}{\Gamma(\alpha_i+\beta_i-1)|\Delta_i|}\Big)$$

$$\Big((\Gamma(\alpha_i+\beta_i+\gamma_i)+\Gamma(\alpha_i+\beta_i-\gamma_i))(\frac{(\ln\eta)^{\alpha_i+\gamma_i}}{\Gamma(\alpha_i+\gamma_i+1)}+\frac{(\ln\eta)^{\alpha_i-\gamma_i}}{\Gamma(\alpha_i-\gamma_i+1)})+$$

$$(\Gamma(\alpha_i+\beta_i+\gamma_i)(\ln\eta)^{\alpha_i+\beta_i-\gamma_i-1}+\Gamma(\alpha_i+\beta_i-\gamma_i)(\ln\eta)^{\alpha_i+\beta_i+\gamma_i-1})(\frac{1}{\Gamma(\alpha_i+\gamma_i+1)}+\frac{1}{\Gamma(\alpha_i-\gamma_i+1)})\Big)$$

$$+\frac{\Gamma(\alpha_i+\beta_i-\gamma_i)\Gamma(\alpha_i+\beta_i+\gamma_i)}{\Gamma(\alpha_i+\beta_i)e_i}\Big(\frac{(\ln\eta)^{\alpha_i+\gamma_i}}{\Gamma(\alpha_i+\gamma_i+1)}+\frac{(\ln\eta)^{\alpha_i-\gamma_i}}{\Gamma(\alpha_i-\gamma_i+1)}\Big)\Big]$$

$$=\|p_i\|\psi_i(\rho)\Lambda_1^i+\rho|\lambda_i|\Lambda_2^i$$

for all $t\in[1,e]$ and $1\le i\le k$. Hence, $\|h_i\|\le\|p_i\|\psi_i(\rho)\Lambda_1^i+\rho|\lambda_i|\Lambda_2^i$ and so

$$\|(h_1,\cdots,h_k)\|=\sum_{i=1}^k\|h_i\|\le\sum_{i=1}^k\Big(\|p_i\|\psi_i(\rho)\Lambda_1^i+\rho|\lambda_i|\Lambda_2^i\Big).$$

Now, we show that $B$ maps bounded sets to equi-continuous subsets of $X^k$. Let $t_1,t_2\in[1,e]$ with $t_1<t_2$, $(x_1,\cdots,x_k)\in B_\rho$ and $(h_1,\cdots,h_k)\in B(x_1,\cdots,x_k)$. Then, we have

$$|h_i(t_2)-h_i(t_1)|=\Big|\frac{1}{\Gamma(\alpha_i)}\int_1^{t_2}(\ln\frac{t_2}{s})^{\alpha_i-1}\frac{1}{s}\Big(\frac{1}{\Gamma(\beta_i)}\int_1^s(\ln\frac{s}{u})^{\beta_i-1}\frac{v(u)}{u}du-\lambda_ix_i(s)\Big)ds$$

$$-\frac{1}{\Gamma(\alpha_i)}\int_1^{t_1}(\ln\frac{t_1}{s})^{\alpha_i-1}\frac{1}{s}\Big(\frac{1}{\Gamma(\beta_i)}\int_1^s(\ln\frac{s}{u})^{\beta_i-1}\frac{v(u)}{u}du-\lambda_ix_i(s)\Big)ds$$

$$-\frac{d_i\Gamma(\alpha_i+\beta_i-\gamma_i)\Gamma(\alpha_i+\beta_i+\gamma_i)(\ln t_2)^{\alpha_i+\beta_i-1}}{\Gamma(\alpha_i+\beta_i)e_i\Delta_i}\times$$

$$\Big[\Big(\Gamma(\alpha_i+\beta_i+\gamma_i)+\Gamma(\alpha_i+\beta_i-\gamma_i)\Big)\int_1^\eta\Big(\frac{1}{\Gamma(\alpha_i+\gamma_i)}(\ln\frac{\eta}{s})^{\alpha_i+\gamma_i-1}+\frac{1}{\Gamma(\alpha_i-\gamma_i)}(\ln\frac{\eta}{s})^{\alpha_i-\gamma_i-1}\Big)$$

$$\frac{1}{s}\Big(\frac{1}{\Gamma(\beta_i)}\int_1^s(\ln\frac{s}{u})^{\beta_i-1}\frac{v(u)}{u}du-\lambda_ix_i(s)\Big)ds$$

$$-e_i\times\int_1^e\Big(\frac{1}{\Gamma(\alpha_i+\gamma_i)}(\ln\frac{e}{s})^{\alpha_i+\gamma_i-1}+\frac{1}{\Gamma(\alpha_i-\gamma_i)}(\ln\frac{e}{s})^{\alpha_i-\gamma_i-1}\Big)$$

$$\frac{1}{s}\Big(\frac{1}{\Gamma(\beta_i)}\int_1^s(\ln\frac{s}{u})^{\beta_i-1}\frac{v(u)}{u}du-\lambda_ix_i(s)\Big)ds\Big]$$

$$+\frac{d_i\Gamma(\alpha_i+\beta_i-\gamma_i)\Gamma(\alpha_i+\beta_i+\gamma_i)(\ln t_1)^{\alpha_i+\beta_i-1}}{\Gamma(\alpha_i+\beta_i)e_i\Delta_i}\times$$

$$\Big[\Big(\Gamma(\alpha_i+\beta_i+\gamma_i)+\Gamma(\alpha_i+\beta_i-\gamma_i)\Big)\int_1^\eta\Big(\frac{1}{\Gamma(\alpha_i+\gamma_i)}(\ln\frac{\eta}{s})^{\alpha_i+\gamma_i-1}+\frac{1}{\Gamma(\alpha_i-\gamma_i)}(\ln\frac{\eta}{s})^{\alpha_i-\gamma_i-1}\Big)$$

$$\frac{1}{s}\Big(\frac{1}{\Gamma(\beta_i)}\int_1^s(\ln\frac{s}{u})^{\beta_i-1}\frac{v(u)}{u}du-\lambda_ix_i(s)\Big)ds$$

$$-e_i\times\int_1^e\Big(\frac{1}{\Gamma(\alpha_i+\gamma_i)}(\ln\frac{e}{s})^{\alpha_i+\gamma_i-1}+\frac{1}{\Gamma(\alpha_i-\gamma_i)}(\ln\frac{e}{s})^{\alpha_i-\gamma_i-1}\Big)$$

$$\frac{1}{s}\Big(\frac{1}{\Gamma(\beta_i)}\int_1^s(\ln\frac{s}{u})^{\beta_i-1}\frac{v(u)}{u}du-\lambda_ix_i(s)\Big)ds\Big]$$

$$-\frac{\Gamma(\alpha_i+\beta_i-\gamma_i)\Gamma(\alpha_i+\beta_i+\gamma_i)(\ln t_2)^{\alpha_i+\beta_i-1}}{\Gamma(\alpha_i+\beta_i)e_i}\times$$

$$\int_1^\eta\Big(\frac{1}{\Gamma(\alpha_i+\gamma_i)}(\ln\frac{\eta}{s})^{\alpha_i+\gamma_i-1}+\frac{1}{\Gamma(\alpha_i-\gamma_i)}(\ln\frac{\eta}{s})^{\alpha_i-\gamma_i-1}\Big)\frac{1}{s}\Big(\frac{1}{\Gamma(\beta_i)}\int_1^s(\ln\frac{s}{u})^{\beta_i-1}\frac{v(u)}{u}du-\lambda_ix_i(s)\Big)ds$$

$$+\frac{\Gamma(\alpha_i+\beta_i-\gamma_i)\Gamma(\alpha_i+\beta_i+\gamma_i)(\ln t_1)^{\alpha_i+\beta_i-1}}{\Gamma(\alpha_i+\beta_i)e_i}\times$$

$$\int_1^\eta\Big(\frac{1}{\Gamma(\alpha_i+\gamma_i)}(\ln\frac{\eta}{s})^{\alpha_i+\gamma_i-1}+\frac{1}{\Gamma(\alpha_i-\gamma_i)}(\ln\frac{\eta}{s})^{\alpha_i-\gamma_i-1}\Big)\frac{1}{s}\Big(\frac{1}{\Gamma(\beta_i)}\int_1^s(\ln\frac{s}{u})^{\beta_i-1}\frac{v(u)}{u}du-\lambda_ix_i(s)\Big)ds$$

$$+\frac{\Gamma(\alpha_i+\beta_i-\gamma_i-1)\Gamma(\alpha_i+\beta_i+\gamma_i-1)(\ln t_2)^{\alpha_i+\beta_i-2}}{\Gamma(\alpha_i+\beta_i-1)\Delta_i}$$

$$\Big[\Big(\Gamma(\alpha_i+\beta_i+\gamma_i)+\Gamma(\alpha_i+\beta_i-\gamma_i)\Big)\int_1^\eta\Big(\frac{1}{\Gamma(\alpha_i+\gamma_i)}(\ln\frac{\eta}{s})^{\alpha_i+\gamma_i-1}+\frac{1}{\Gamma(\alpha_i-\gamma_i)}(\ln\frac{\eta}{s})^{\alpha_i-\gamma_i-1}\Big)$$
$$\frac{1}{s}\Big(\frac{1}{\Gamma(\beta_i)}\int_1^s(\ln\frac{s}{u})^{\beta_i-1}\frac{v(u)}{u}du-\lambda_i x_i(s)\Big)ds$$
$$-e_i\times\int_1^e\Big(\frac{1}{\Gamma(\alpha_i+\gamma_i)}(\ln\frac{e}{s})^{\alpha_i+\gamma_i-1}+\frac{1}{\Gamma(\alpha_i-\gamma_i)}(\ln\frac{e}{s})^{\alpha_i-\gamma_i-1}\Big)$$
$$\frac{1}{s}\Big(\frac{1}{\Gamma(\beta_i)}\int_1^s(\ln\frac{s}{u})^{\beta_i-1}\frac{v(u)}{u}du-\lambda_i x_i(s)\Big)ds\Big]$$
$$-\frac{\Gamma(\alpha_i+\beta_i-\gamma_i-1)\Gamma(\alpha_i+\beta_i+\gamma_i-1)(\ln t_1)^{\alpha_i+\beta_i-2}}{\Gamma(\alpha_i+\beta_i-1)\Delta_i}\times$$
$$\Big[\Big(\Gamma(\alpha_i+\beta_i+\gamma_i)+\Gamma(\alpha_i+\beta_i-\gamma_i)\Big)\int_1^\eta\Big(\frac{1}{\Gamma(\alpha_i+\gamma_i)}(\ln\frac{\eta}{s})^{\alpha_i+\gamma_i-1}+\frac{1}{\Gamma(\alpha_i-\gamma_i)}(\ln\frac{\eta}{s})^{\alpha_i-\gamma_i-1}\Big)$$
$$\frac{1}{s}\Big(\frac{1}{\Gamma(\beta_i)}\int_1^s(\ln\frac{s}{u})^{\beta_i-1}\frac{v(u)}{u}du-\lambda_i x_i(s)\Big)ds$$
$$-e_i\times\int_1^e\Big(\frac{1}{\Gamma(\alpha_i+\gamma_i)}(\ln\frac{e}{s})^{\alpha_i+\gamma_i-1}+\frac{1}{\Gamma(\alpha_i-\gamma_i)}(\ln\frac{e}{s})^{\alpha_i-\gamma_i-1}\Big)$$
$$\frac{1}{s}\Big(\frac{1}{\Gamma(\beta_i)}\int_1^s(\ln\frac{s}{u})^{\beta_i-1}\frac{v(u)}{u}du-\lambda_i x_i(s)\Big)ds\Big]\Big|$$
$$\leq\frac{1}{\Gamma(\alpha_i)}\int_1^{t_1}\Big((\ln\frac{t_2}{s})^{\alpha_i-1}-(\ln\frac{t_1}{s})^{\alpha_i-1}\Big)\frac{1}{s}\Big(\frac{\|p_i\|\psi_i(\rho)}{\Gamma(\beta_i+1)}(\ln(s))^{\beta_i}+\rho|\lambda_i|\Big)ds$$
$$+\frac{1}{\Gamma(\alpha_i)}\int_{t_1}^{t_2}(\ln\frac{t_2}{s})^{\alpha_i-1}\frac{1}{s}\Big(\frac{\|p_i\|\psi_i(\rho)}{\Gamma(\beta_i+1)}(\ln(s))^{\beta_i}+\rho|\lambda_i|\Big)ds$$
$$+\frac{d_i\Gamma(\alpha_i+\beta_i-\gamma_i)\Gamma(\alpha_i+\beta_i+\gamma_i)\Big((\ln t_2)^{\alpha_i+\beta_i-1}-(\ln t_1)^{\alpha_i+\beta_i-1}\Big)}{\Gamma(\alpha_i+\beta_i)e_i|\Delta_i|}\times$$
$$\Big[\Big(\Gamma(\alpha_i+\beta_i+\gamma_i)+\Gamma(\alpha_i+\beta_i-\gamma_i)\Big)\int_1^\eta\Big(\frac{1}{\Gamma(\alpha_i+\gamma_i)}(\ln\frac{\eta}{s})^{\alpha_i+\gamma_i-1}+\frac{1}{\Gamma(\alpha_i-\gamma_i)}(\ln\frac{\eta}{s})^{\alpha_i-\gamma_i-1}\Big)$$
$$\frac{1}{s}\Big(\frac{1}{\Gamma(\beta_i)}\int_1^s(\ln\frac{s}{u})^{\beta_i-1}\frac{v(u)}{u}du-\lambda_i x_i(s)\Big)ds$$
$$-e_i\times\int_1^e\Big(\frac{1}{\Gamma(\alpha_i+\gamma_i)}(\ln\frac{e}{s})^{\alpha_i+\gamma_i-1}+\frac{1}{\Gamma(\alpha_i-\gamma_i)}(\ln\frac{e}{s})^{\alpha_i-\gamma_i-1}\Big)$$
$$\frac{1}{s}\Big(\frac{1}{\Gamma(\beta_i)}\int_1^s(\ln\frac{s}{u})^{\beta_i-1}\frac{v(u)}{u}du-\lambda_i x_i(s)\Big)ds\Big]$$
$$+\frac{\Gamma(\alpha_i+\beta_i-\gamma_i)\Gamma(\alpha_i+\beta_i+\gamma_i)\Big((\ln t_2)^{\alpha_i+\beta_i-1}-(\ln t_1)^{\alpha_i+\beta_i-1}\Big)}{\Gamma(\alpha_i+\beta_i)e_i}\times$$
$$\int_1^\eta\Big(\frac{1}{\Gamma(\alpha_i+\gamma_i)}(\ln\frac{\eta}{s})^{\alpha_i+\gamma_i-1}+\frac{1}{\Gamma(\alpha_i-\gamma_i)}(\ln\frac{\eta}{s})^{\alpha_i-\gamma_i-1}\Big)\frac{1}{s}\Big(\frac{1}{\Gamma(\beta_i)}\int_1^s(\ln\frac{s}{u})^{\beta_i-1}\frac{v(u)}{u}du-\lambda_i x_i(s)\Big)ds$$
$$+\frac{\Gamma(\alpha_i+\beta_i-\gamma_i-1)\Gamma(\alpha_i+\beta_i+\gamma_i-1)\Big((\ln t_2)^{\alpha_i+\beta_i-1}-(\ln t_1)^{\alpha_i+\beta_i-1}\Big)}{\Gamma(\alpha_i+\beta_i-1)|\Delta_i|}$$
$$\Big[\Big(\Gamma(\alpha_i+\beta_i+\gamma_i)+\Gamma(\alpha_i+\beta_i-\gamma_i)\Big)\int_1^\eta\Big(\frac{1}{\Gamma(\alpha_i+\gamma_i)}(\ln\frac{\eta}{s})^{\alpha_i+\gamma_i-1}+\frac{1}{\Gamma(\alpha_i-\gamma_i)}(\ln\frac{\eta}{s})^{\alpha_i-\gamma_i-1}\Big)$$
$$\frac{1}{s}\Big(\frac{1}{\Gamma(\beta_i)}\int_1^s(\ln\frac{s}{u})^{\beta_i-1}\frac{v(u)}{u}du-\lambda_i x_i(s)\Big)ds$$
$$-e_i\times\int_1^e\Big(\frac{1}{\Gamma(\alpha_i+\gamma_i)}(\ln\frac{e}{s})^{\alpha_i+\gamma_i-1}+\frac{1}{\Gamma(\alpha_i-\gamma_i)}(\ln\frac{e}{s})^{\alpha_i-\gamma_i-1}\Big)$$
$$\frac{1}{s}\Big(\frac{1}{\Gamma(\beta_i)}\int_1^s(\ln\frac{s}{u})^{\beta_i-1}\frac{v(u)}{u}du-\lambda_i x_i(s)\Big)ds\Big]$$

for each $1\leq i\leq k$. Note that the right-hand side of the above inequality tends to zero independently of $(x_1,...,x_k)\in B_\rho$ as $t_2\to t_1$. This implies that

$$\lim_{t_2\to t_1}|(h_1(t_2)-h_1(t_1),\cdots,h_k(t_2)-h_k(t_1))|=0.$$

Hence by using the Arzela-Ascoli theorem, $B$ is completely continuous and since $B(x_1,...,x_k)$ is closed-valued, $B(x_1,...,x_k) \in P_{cp,cv}(X^k)$. By using a similar proof, we can show that $A(x_1,...,x_k) \in P_{cl,bd,cv}(X^k)$. Now, we show that $B$ has a closed graph. For each $n$, suppose that $(u_1^n,\cdots,u_k^n) \in B(x_1^n,\cdots,x_k^n)$ such that $(x_1^n,\cdots,x_k^n) \to (x_1^0,\cdots,x_k^0)$ and $(u_1^n,\cdots,u_k^n) \to (u_1^0,\cdots,u_k^0)$. We show that $(u_1^0,\cdots,u_k^0) \in B(x_1^0,\cdots,x_k^0)$. For each natural number $n$, choose $(v_1^n,\cdots,v_k^n) \in S_{G_1,(x_1^n,\cdots,x_k^n)} \times \cdots \times S_{G_k,(x_1^n,\cdots,x_k^n)}$ such that $u_i^n(t) = w_i(v_i^n,t)$ for all $t \in [1,e]$ and $1 \le i \le k$. Consider the continuous linear operator $\theta_i : L^1([1,e],\mathbb{R}) \to X$ by $\theta_i(v)(t) = w_i(v,t)$. By using Lemma 1.2, $\theta_i o S_{G_i}$ is a closed graph operator. Since $u_i^n \in \theta_i(S_{G_i,(x_1^n,\cdots,x_k^n)})$ for all $n$, $1 \le i \le k$ and $(x_1^n,\cdots,x_k^n) \to (x_1^0,\cdots,x_k^0)$, there exists $v_i^0 \in S_{G_i,(x_1^0,\cdots,x_k^0)}$ such that $u_i^0(t) = w_i(v_i^0,t)$. Hence, $u_i^0 \in B_i(x_1^0,\cdots,x_k^0)$ for all $1 \le i \le k$. This implies that $B_i$ has a closed graph for all $1 \le i \le k$ and so $B$ has a closed graph and this shows that the operator $B$ is upper semi-continuous. Now, we show that $A$ is a contractive multifunction. Let $(x_1,\cdots,x_k),(y_1,\cdots,y_k) \in X^k$ and $(h_1,\cdots,h_k) \in A(y_1,\cdots,y_k)$ be given. Then, we can choose $(v_1,\cdots,v_k) \in S_{F_1,(y_1,\cdots,y_k)} \times S_{F_2,(y_1,\cdots,y_k)} \times \cdots \times S_{F_k,(y_1,\cdots,y_k)}$ such that $h_i(t) = w_i(v_i,t)$ for all $t \in [1,e]$ and $i = 1,\cdots,k$. Since

$$H\Big(F_i(t,x_1(t),\cdots,x_k(t),I^{\nu_1}x_1(t),\cdots,I^{\nu_k}x_k(t)),F_i(t,y_1(t),\cdots,y_k(t),I^{\nu_1}y_1(t),\cdots,I^{\nu_k}y_k(t)\Big)$$

$$\le m_i(t)\sum_{i=1}^{k}(|x_i(t)-y_i(t)|+|I^{\nu_i}x_i(t)-I^{\nu_i}y_i(t)|)$$

for almost all $t \in [1,e]$ and $i = 1,\cdots,k$, there exists

$$u_i \in F_i(t,x_1(t),\cdots,x_k(t),I^{\nu_1}x_1(t),...,I^{\nu_k}x_k(t))$$

such that $|v_i(t) - u_i| \le m_i(t)\sum_{i=1}^{k}(|x_i(t)-y_i(t)|+|I^{\nu_i}x_i(t)-I^{\nu_i}y_i(t)|)$ for almost all $t \in [1,e]$ and $i = 1,\cdots,k$. Consider the multifunction $U_i : [1,e] \to 2^{\mathbb{R}}$ defined by

$$U_i(t) = \{w \in \mathbb{R} : |v_i(t)-w| \le m_i(t)\sum_{i=1}^{k}(|x_i(t)-y_i(t)|+|I^{\nu_i}x_i(t)-I^{\nu_i}y_i(t)|) \text{ for a.e. } t \in [1,e]\}.$$

Since $U_i(t) \cap F_i(t,x_1(t),\cdots,x_k(t),I^{\nu_1}x_1(t),...,I^{\nu_k}x_k(t))$ is a measurable multifunction, we can choose $v_i'(t) \in F_i(t,x_1(t),\cdots,x_k(t),I^{\nu_1}x_1(t),...,I^{\nu_k}x_k(t))$ such that

$$|v_i(t) - v_i'(t)| \le m_i(t)\sum_{i=1}^{k}(|x_i(t)-y_i(t)|+|I^{\nu_i}x_i(t)-I^{\nu_i}y_i(t)|).$$

For each $t \in [1.e]$ and $i = 1,\cdots,k$, consider the map $h_i'(t) = w_i(v_i',t)$. Since

$$|h_i(t)-h_i'(t)| \le \frac{1}{\Gamma(\alpha_i)}\int_1^t(\ln\frac{t}{s})^{\alpha_i-1}\frac{1}{s}\Big(\frac{1}{\Gamma(\beta_i)}\int_1^s(\ln\frac{s}{u})^{\beta_i-1}\frac{|v_i(u)-v_i'(u)|}{u}du+|\lambda_i||x_i(s)-y_i(s)|\Big)ds$$

$$+\frac{d_i\Gamma(\alpha_i+\beta_i-\gamma_i)\Gamma(\alpha_i+\beta_i+\gamma_i)(\ln t)^{\alpha_i+\beta_i-1}}{\Gamma(\alpha_i+\beta_i)e_i|\Delta_i|}\times$$

$$\Big[\Big(\Gamma(\alpha_i+\beta_i+\gamma_i)+\Gamma(\alpha_i+\beta_i-\gamma_i)\Big)\int_1^\eta\Big(\frac{1}{\Gamma(\alpha_i+\gamma_i)}(\ln\frac{\eta}{s})^{\alpha_i+\gamma_i-1}+\frac{1}{\Gamma(\alpha_i-\gamma_i)}(\ln\frac{\eta}{s})^{\alpha_i-\gamma_i-1}\Big)$$

$$\frac{1}{s}\Big(\frac{1}{\Gamma(\beta_i)}\int_1^s(\ln\frac{s}{u})^{\beta_i-1}\frac{|v_i(u)-v_i'(u)|}{u}du+|\lambda_i||x_i(s)-y_i(s)|\Big)ds+e_i\times$$

$$\int_1^e\Big(\frac{1}{\Gamma(\alpha_i+\gamma_i)}(\ln\frac{e}{s})^{\alpha_i+\gamma_i-1}+\frac{1}{\Gamma(\alpha_i-\gamma_i)}(\ln\frac{e}{s})^{\alpha_i-\gamma_i-1}\Big)$$

$$\frac{1}{s}\Big(\frac{1}{\Gamma(\beta_i)}\int_1^s(\ln\frac{s}{u})^{\beta_i-1}\frac{|v_i(u)-v_i'(u)|}{u}du+|\lambda_i||x_i(s)-y_i(s)|\Big)ds\Big]$$

$$+\frac{\Gamma(\alpha_i+\beta_i-\gamma_i)\Gamma(\alpha_i+\beta_i+\gamma_i)(\ln t)^{\alpha_i+\beta_i-1}}{\Gamma(\alpha_i+\beta_i)e_i}$$

$$\times\int_1^\eta\Big(\frac{1}{\Gamma(\alpha_i+\gamma_i)}(\ln\frac{\eta}{s})^{\alpha_i+\gamma_i-1}+\frac{1}{\Gamma(\alpha_i-\gamma_i)}(\ln\frac{\eta}{s})^{\alpha_i-\gamma_i-1}\Big)\times$$

$$\frac{1}{s}\Big(\frac{1}{\Gamma(\beta_i)}\int_1^s(\ln\frac{s}{u})^{\beta_i-1}\frac{|v_i(u)-v_i'(u)|}{u}du+|\lambda_i||x_i(s)-y_i(s)|\Big)ds$$

$$+\frac{\Gamma(\alpha_i+\beta_i-\gamma_i-1)\Gamma(\alpha_i+\beta_i+\gamma_i-1)(\ln t)^{\alpha_i+\beta_i-2}}{\Gamma(\alpha_i+\beta_i-1)|\Delta_i|}$$

$$\Big[\Big(\Gamma(\alpha_i+\beta_i+\gamma_i)+\Gamma(\alpha_i+\beta_i-\gamma_i)\Big)\int_1^\eta\Big(\frac{1}{\Gamma(\alpha_i+\gamma_i)}(\ln\frac{\eta}{s})^{\alpha_i+\gamma_i-1}+\frac{1}{\Gamma(\alpha_i-\gamma_i)}(\ln\frac{\eta}{s})^{\alpha_i-\gamma_i-1}\Big)$$

$$\frac{1}{s}\Big(\frac{1}{\Gamma(\beta_i)}\int_1^s(\ln\frac{s}{u})^{\beta_i-1}\frac{|v_i(u)-v_i'(u)|}{u}du+|\lambda_i||x_i(s)-y_i(s)|\Big)ds+e_i\times$$

$$\int_1^e\Big(\frac{1}{\Gamma(\alpha_i+\gamma_i)}(\ln\frac{e}{s})^{\alpha_i+\gamma_i-1}+\frac{1}{\Gamma(\alpha_i-\gamma_i)}(\ln\frac{e}{s})^{\alpha_i-\gamma_i-1}\Big)$$

$$\frac{1}{s}\Big(\frac{1}{\Gamma(\beta_i)}\int_1^s(\ln\frac{s}{u})^{\beta_i-1}\frac{|v_i(u)-v_i'(u)|}{u}du+|\lambda_i||x_i(s)-y_i(s)|\Big)ds\Big]$$

$$\leq\frac{\|m_i\|\|(x_1-y_1,\dots,x_k-y_k)\|_*\sum_{j=1}^k(1+\frac{1}{\Gamma(\nu_j+1)})}{\Gamma(\alpha_i+\beta_i+1)}+\frac{|\lambda_i|\|(x_1-y_1,\dots,x_k-y_k)\|_*}{\Gamma(\alpha_i+1)}$$

$$+\frac{d_i\Gamma(\alpha_i+\beta_i-\gamma_i)\Gamma(\alpha_i+\beta_i+\gamma_i)}{\Gamma(\alpha_i+\beta_i)e_i|\Delta_i|}\times\Big[\Big(\Gamma(\alpha_i+\beta_i+\gamma_i)+\Gamma(\alpha_i+\beta_i-\gamma_i)\Big)$$

$$\Big(\frac{\|m_i\|\|(x_1-y_1,\dots,x_k-y_k)\|_*\sum_{i=1}^k(1+\frac{1}{\Gamma(\nu_i+1)})(\ln\eta)^{\alpha_i+\beta_i+\gamma_i}}{\Gamma(\alpha_i+\beta_i+\gamma_i+1)}\times$$

$$+\frac{|\lambda_i|\|(x_1-y_1,\dots,x_k-y_k)\|_*(\ln\eta)^{\alpha_i+\gamma_i}}{\Gamma(\alpha_i+\gamma_i+1)}$$

$$+\frac{\|m_i\|\|(x_1-y_1,\dots,x_k-y_k)\|_*\sum_{j=1}^k(1+\frac{1}{\Gamma(\nu_j+1)}))(\ln\eta)^{\alpha_i+\beta_i-\gamma_i}}{\Gamma(\alpha_i+\beta_i-\gamma_i+1)}$$

$$+\frac{|\lambda_i|\|(x_1-y_1,\dots,x_k-y_k)\|_*(\ln\eta)^{\alpha_i-\gamma_i}}{\Gamma(\alpha_i-\gamma_i+1)}\Big)+e_i\times$$

$$\Big(\frac{\|m_i\|\|(x_1-y_1,\dots,x_k-y_k)\|_*\sum_{j=1}^k(1+\frac{1}{\Gamma(\nu_j+1)}}{\Gamma(\alpha_i+\beta_i+\gamma_i+1)}$$

$$+\frac{|\lambda_i|\|(x_1,\dots,x_k)\|_*}{\Gamma(\alpha_i+\gamma_i+1)}+\frac{\|m_i\|\|(x_1-y_1,\dots,x_k-y_k)\|_*\sum_{j=1}^k(1+\frac{1}{\Gamma(\nu_j+1)})}{\Gamma(\alpha_i+\beta_i-\gamma_i+1)}$$

$$+\frac{|\lambda_i|\|(x_1-y_1,\dots,x_k-y_k)\|_*}{\Gamma(\alpha_i-\gamma_i+1)}\Big)\Big]+\frac{\Gamma(\alpha_i+\beta_i-\gamma_i)\Gamma(\alpha_i+\beta_i+\gamma_i)}{\Gamma(\alpha_i+\beta_i)e_i}\times$$

$$\Big(\frac{\|m_i\|\|(x_1-y_1,\dots,x_k-y_k)\|_*\sum_{j=1}^k(1+\frac{1}{\Gamma(\nu_j+1)})(\ln\eta)^{\alpha_i+\beta_i+\gamma_i}}{\Gamma(\alpha_i+\beta_i+\gamma_i+1)}$$

$$+\frac{|\lambda_i|\|(x_1-y_1,\dots,x_k-y_k)\|_*(\ln\eta)^{\alpha_i+\gamma_i}}{\Gamma(\alpha_i+\gamma_i+1)}$$

$$+\frac{\|m_i\|\|(x_1-y_1,\dots,x_k-y_k)\|_*\sum_{j=1}^k(1+\frac{1}{\Gamma(\nu_j+1)})(\ln\eta)^{\alpha_i+\beta_i-\gamma_i}}{\Gamma(\alpha_i+\beta_i-\gamma_i+1)}+\frac{|\lambda_i|\|(x_1,\dots,x_k)\|_*(\ln\eta)^{\alpha_i-\gamma_i}}{\Gamma(\alpha_i-\gamma_i+1)}\Big)$$

$$+\frac{\Gamma(\alpha_i+\beta_i-\gamma_i-1)\Gamma(\alpha_i+\beta_i+\gamma_i-1)}{\Gamma(\alpha_i+\beta_i-1)|\Delta_i|}\Big[\Big(\Gamma(\alpha_i+\beta_i+\gamma_i)+\Gamma(\alpha_i+\beta_i-\gamma_i)\Big)$$

$$\Big(\frac{\|m_i\|\|(x_1-y_1,\dots,x_k-y_k)\|_*\sum_{j=1}^k(1+\frac{1}{\Gamma(\nu_j+1)})(\ln\eta)^{\alpha_i+\beta_i+\gamma_i}}{\Gamma(\alpha_i+\beta_i+\gamma_i+1)}$$

$$+\frac{|\lambda_i|\|(x_1-y_1,\dots,x_k-y_k)\|_*(\ln\eta)^{\alpha_i+\gamma_i}}{\Gamma(\alpha_i+\gamma_i+1)}$$

$$+\frac{\|m_i\|\|(x_1-y_1,\dots,x_k-y_k)\|_*\sum_{j=1}^k(1+\frac{1}{\Gamma(\nu_j+1)})(\ln\eta)^{\alpha_i+\beta_i-\gamma_i}}{\Gamma(\alpha_i+\beta_i-\gamma_i+1)}$$

$$+\frac{|\lambda_i|\|(x_1-y_1,\dots,x_k-y_k)\|_*(\ln\eta)^{\alpha_i-\gamma_i}}{\Gamma(\alpha_i-\gamma_i+1)}\Big)+e_i\times$$

$$\Big(\frac{\|m_i\|\|(x_1-y_1,\dots,x_k-y_k)\|_*\sum_{j=1}^k(1+\frac{1}{\Gamma(\nu_j+1)})}{\Gamma(\alpha_i+\beta_i+\gamma_i+1)}+\frac{|\lambda_i|\|(x_1,\dots,x_k)\|_*}{\Gamma(\alpha_i+\gamma_i+1)}$$

$$+\frac{\|m_i\|\|(x_1-y_1,\dots,x_k-y_k)\|_*\sum_{j=1}^k(1+\frac{1}{\Gamma(\nu_j+1)})}{\Gamma(\alpha_i+\beta_i-\gamma_i+1)}+\frac{|\lambda_i|\|(x_1,\dots,x_k)\|_*}{\Gamma(\alpha_i-\gamma_i+1)}\Big)\Big]$$

$$=\|(x_1-y_1,\dots,x_k-y_k)\|_*\|m_i\|\Lambda_1^i\sum_{j=1}^k(1+\frac{1}{\Gamma(\nu_j+1)})+\|(x_1-y_1,\dots,x_k-y_k)\|_*|\lambda_i|\Lambda_2^i$$

$$=\Big(\|m_i\|\Lambda_1^i\sum_{j=1}^k(1+\frac{1}{\Gamma(\nu_j+1)})+|\lambda_i|\Lambda_2^i\Big)\|(x_1-y_1,\dots,x_k-y_k)\|_*,$$

we get $\|h_i-h_i'\|\leq\Big(\|m_i\|\Lambda_1^i\sum_{j=1}^k(1+\frac{1}{\Gamma(\nu_j+1)})+|\lambda_i|\Lambda_2^i\Big)\|(x_1-y_1,...,x_k-y_k)\|_*$ for all $i=1,\cdots,k$. Hence,

$$\|(h_1,\cdots,h_k)-(h_1',\cdots,h_k')\|_*=\sum_{i=1}^k\|h_i-h_i'\|$$

$$\leq\sum_{i=1}^k\Big(\|m_i\|\Lambda_1^i\sum_{j=1}^k(1+\frac{1}{\Gamma(\nu_j+1)})+|\lambda_i|\Lambda_2^i\Big)\|(x_1-y_1,...,x_k-y_k)\|_*.$$

This implies that

$$H(A(x_1,\cdots,x_k),A(y_1,\cdots,y_k))$$

$$\leq\sum_{i=1}^k\Big(\|m_i\|\Lambda_1^i\sum_{j=1}^k(1+\frac{1}{\Gamma(\nu_j+1)})+|\lambda_i|\Lambda_2^i\Big)\|(x_1-y_1,...,x_k-y_k)\|_*.$$

Since $\sum_{i=1}^k\Big(\|m_i\|\Lambda_1^i\sum_{j=1}^k(1+\frac{1}{\Gamma(\nu_j+1)})+|\lambda_i|\Lambda_2^i\Big)<1$, $A$ is contractive multifunction. Let $(x_1,...,x_k)$ be a possible solution for the problem $\lambda(x_1,...,x_k)\in A(x_1,...,x_k)+B(x_1,...,x_k)$ for some real number $\lambda>1$ with $\|(x_1,...,x_k)\|_*=1$. Choose

$$(v_1,\cdots,v_k)\in S_{F_1,(x_1,\cdots,x_k)}\times S_{F_2,(x_1,\cdots,x_k)}\times\cdots\times S_{F_k,(x_1,\cdots,x_k)}$$

and $(v_1',\cdots,v_k')\in S_{G_1,(x_1,\cdots,x_k)}\times S_{G_2,(x_1,\cdots,x_k)}\times\cdots\times S_{G_k,(x_1,\cdots,x_k)}$ such that

$$x_i(t)=\lambda^{-1}(w_i(v_i,t)+w_i(v_i',t))$$

for all $t \in [1, e]$ and $1 \leq i \leq k$. By using a simple calculation, we can see that

$$\|x_i\| \leq \left( \|m_i\| \Lambda_1^i \sum_{j=1}^{k} (1 + \frac{1}{\Gamma(\nu_j + 1)}) + |\lambda_i| \Lambda_2^i \right) \|(x_1, ..., x_k)\|_*$$

$$+ \|p_i\| \psi_i(\|(x_1, ..., x_k)\|_*) \Lambda_1^i + |\lambda_i| \Lambda_2^i \|(x_1, ..., x_k)\|_*$$

and so $\|(x_1, ..., x_k)\|_* \leq \frac{\sum_{i=1}^{k} \|p_i\| \psi_i(\|(x_1,...,x_k)\|_*) \Lambda_1^i}{1 - \sum_{i=1}^{k} \left( \|m_i\| \Lambda_1^i \sum_{j=1}^{k} (1 + \frac{1}{\Gamma(\nu_j+1)}) + 2|\lambda_i| \Lambda_2^i \right)}$. If we have $\|(x_1, ..., x_k)\|_* \to r$, then we obtain $r \leq \frac{\sum_{i=1}^{k} \|p_i\| \psi_i(r) \Lambda_1^i}{1 - \sum_{i=1}^{k} \left( \|m_i\| \Lambda_1^i \sum_{j=1}^{k} (1 + \frac{1}{\Gamma(\nu_j+1)}) + 2|\lambda_i| \Lambda_2^i \right)}$ which is a contradiction. Using Lemma 1.3, there exists $(x_1, ..., x_k) \in B[0, r]$ such that $(x_1, ..., x_k) \in A(x_1, ..., x_k) + B(x_1, ..., x_k)$ which is a solution for the $k-$dimensional system of fractional differential inclusions (1). □

Here, we give an example to illustrate our main result.

**Example 2.3.** Consider the 2-dimensional system of fractional differential inclusions

(8)

$$\begin{cases} D^{\frac{3}{2}}(D^{\frac{1}{2}} + \pi^{-4}) u(t) \in F_1(t, u(t), v(t), I^{\frac{1}{4}} u(t), I^{\frac{1}{3}} v(t)) + G_1(t, u(t), v(t)), \\ D^{\frac{5}{4}}(D^{\frac{3}{4}} + \frac{1}{75}) v(t) \in F_1(t, u(t), v(t), I^{\frac{1}{4}} u(t), I^{\frac{1}{3}} v(t)) + G_1(t, u(t), v(t)), \end{cases}$$

with boundary conditions

$$\begin{cases} u(1) = v(1) = 0, \ I^{\frac{1}{3}} u(2) + D^{\frac{1}{3}} u(2) = 0, \ I^{\frac{1}{2}} v(2) + D^{\frac{1}{2}} v(2) = 0, \\ I^{\frac{1}{3}} u(e) + D^{\frac{1}{3}} u(e) = 0 \ \ \textit{and} \ \ I^{\frac{1}{2}} u(e) + D^{\frac{1}{2}} u(e) = 0. \end{cases} \quad (9)$$

Define the set-valued maps $F_1, F_2 : [1, e] \times \mathbb{R}^4 \to P(\mathbb{R})$ and $G_1, G_2 : [1, e] \times \mathbb{R}^2 \to P(\mathbb{R})$ by

$$F_1(t, x_1, x_2, x_3, x_4) = \left[ -1, \frac{e^{t-e} \sin x_1}{150\pi} + \frac{t}{e^8} \cos x_2 + \frac{|x_3|}{\cosh 7(1 + |x_3|)} + \frac{2x_4^2}{10^3(1 + x_4^2)} \right],$$

$$F_2(t, x_1, x_2, x_3, x_4) = \left[ 0, \frac{e^t}{25\pi^5} (\frac{|x_1| + |x_2| + |x_3| + |x_4|}{1 + |x_1| + |x_2| + |x_3| + |x_4|}) \right],$$

$$G_1(t, x_1, x_2, x_3, x_4) = \left[ e^{-|x_1|} - \frac{x_2^2}{1 + x_2^2} + \cos t, 2t + \frac{|x_1|}{1 + |x_1|} + \sin y + t^2 \right],$$

and $G_2(t, x_1, x_2, x_3, x_4) = \left[ \frac{x_1}{4(1+x_1)} + \frac{x_2}{1+x_2} + 2 + t, \sin x_1 + \cos x_2 + 4t \right]$. Put $k = 2$, $\alpha_1 = \frac{1}{2}$, $\alpha_2 = \frac{3}{4}$, $\beta_1 = \frac{3}{2}$, $\beta_2 = \frac{5}{4}$, $\gamma_1 = \frac{1}{3}$, $\gamma_2 = \frac{1}{2}$, $\nu_1 = \frac{1}{4}$, $\nu_2 = \frac{1}{3}$, $\eta = 2$, $\lambda_1 = \pi^{-4}$ and $\lambda_2 = \frac{1}{75}$. Note that, $\|G_1(t, x_1, x_2)\| = \sup\{|v| : v \in G_1(t, x_1, x_2)\} \leq 17$,

$$\|G_2(t, x_1, x_2)\| = \sup\{|v| : v \in G_2(t, x_1, x_2)\} \leq 14,$$

$$\|F_2(t, x_1, x_2, x_3, x_4)\| = \sup\{|v| : v \in F_2(t, x_1, x_2, x_3, x_4)\} \leq \frac{e^t}{25\pi^5},$$

and $\|F_1(t, x_1, x_2, x_3, x_4)\| = \sup\{|v| : v \in F_1(t, x_1, x_2, x_3, x_4)\} \leq \frac{e^{t-e}}{150\pi} + \frac{t}{e^8} + \frac{1}{\cosh 7} + \frac{2}{10^3}$. Define $p_1(t) = 1$, $p_2(t) = 1$, $\psi_1(t) = 17$, $\psi_2(t) = 14$, $m_1(t) = \frac{e^{t-e}}{150\pi} + \frac{t}{e^8} + \frac{1}{\cosh 7} + \frac{2}{10^3}$ and $m_2(t) = \frac{e^t}{25\pi^5}$. Then by using some calculations, we get $\Lambda_1^1 \approx 6.799$, $\Lambda_1^2 \approx 6.93$, $\Lambda_2^1 \approx 17.93$, $\Lambda_2^2 \approx 15.8$, $\Delta_1 \approx 1.54$, $\Delta_2 \approx 2.05$, $\sum_{i=1}^{2}\left(\|m_i\|\Lambda_1^i \sum_{j=1}^{2}(1 + \frac{1}{\Gamma(\nu_j+1)}) + |\lambda_i|\Lambda_2^i\right) = \frac{5}{1000} \times 6.799 \times 4.22 + \pi^{-4} \times 17.93 + \frac{1}{1000} \times 6.93 \times 4.22 + \frac{1}{75} \times 15.8 = 0.66 < 1$ and

$$\frac{\sum_{i=1}^{2} \|p_i\|\psi_i(r)\Lambda_1^i}{1 - \sum_{i=1}^{2}\left(\|m_i\|\Lambda_1^i \sum_{j=1}^{2}(1 + \frac{1}{\Gamma(\nu_j+1)}) + 2|\lambda_i|\Lambda_2^i\right)} = \frac{17 \times 6.97 + 14 \times 6.93}{1 - 0.66} = 633.852.$$

By using Theorem 2.2, the 2-dimensional system of fractional differential inclusions (8) – (9) has a solution.

## 3. Conclusions

As we know, fractional differential inclusions could be applied, for modeling of any phenomena including chaos, such as some economical behaviors. Thus, study of integro-differential inclusions could increase our ability for modeling of different natural phenomena. In this work, we reviewed the existence of solution for a $k$-dimensional system of Langevin Hadamard-type fractional differential inclusions with $2k$ different fractional orders. Also, we provided an example to illustrate our main result.

## 4. Acknowledgement

The authors would like to thank dear unknown referees, specially the third dear referee, for their very important comments which improved final version of this manuscript.

## References

[1] Agarwal, R. P., O'Regan, D., and Stanek, S. Positive solutions for mixed problems of singular fractional differential equations. *Math. Naschr. 285*, 1 (2012), 24–71.

[2] Ahmad, B., Alsaedi, A., Ntouyas, S., and Tariboon, J. Hadamard-type fractional differential equations, inclusions and inequalities. vol. xiii of *Cham.* Springer-Verlag, 2017, pp. 173–208.

[3] Ahmad, B., and Nieto, J. J. Solvability of nonlinear Langevin equation involving two fractional orders with Dirichlet boundary conditions. *Int. J. Diff. Eq. 2010*, Article ID 649486 (2010), 10 pages.

[4] Ahmad, B., Nieto, J. J., and Alsaedi, A. A nonlocal three-point inclusion problem of Langevin equation with two different fractional orders. *Adv. Diff. Eq. 2012* (2012), 2012:54.

[5] Ahmad, B., Nieto, J. J., Alsaedi, A., and El-Shahed, M. A study of nonlinear Langevin equation involving two fractional orders in different intervals. *Nonlinear Anal. 13* (2010), 599–606.

[6] Ahmad, B., Ntouyas, S., and Alsaedi, A. Existence results for Langevin fractional differential inclusions involving two fractional orders with four-point multiterm fractional integral boundary conditions. *Abst. Appl. Anal. 2013*, Article ID 869837 (2013), 17 pages.

[7] Ahmad, B., Ntouyas, S., and Alsaedi, A. New results for boundary value problems of Hadamard-type fractional differential inclusions and integral boundary conditions. *Bound. Value Probl. 2013* (2013), 2013:275.

[8] Ahmad, B., Ntouyas, S., and Alsaedi, A. On fractional differential inclusions with with anti-periodic type integral boundary conditions. *Bound. Value Probl. 2013* (2013), 2013:82.

[9] Aljoudi, S., Ahmad, B., Nieto, J. J., and Alsaedi, A. A coupled system of Hadamard type sequential fractional differential equations with coupled strip conditions. *Chaos Solit. Fract. 91* (2016), 39–46.

[10] Aubin, J., and Ceuina, A. Differential inclusions: set-valued maps and viability theory. vol. 264 of *Fundamental Principles of Mathematical Sciences.* Springer-Verlag, 1984, pp. 37–92.

[11] Berinde, V., and Pacurar, M. The role of the Pompeiu-Hausdorff metric in fixed point theory. *Creat. Math. Inform. 22*, 2 (2013), 35–42.

[12] Bragdi, M., Debbouche, A., and Baleanu, D. Existence of solutions for fractional differential inclusions with separated boundary conditions in Banach spaces. *Adv. Math. Physics 2013*, Article ID 426061 (2013), 5 pages.

[13] Coffey, W., Kalmykov, Y., and Wadorn, J. The Langevin equation. vol. xxiv of *Contemporary Chemical Physics.* World Scientific Publishing Co., 2004, pp. 173–208.

[14] Covitz, H., and Nadler, S. Multivalued contraction mappings in generalized metric spaces. *Israel J. Math. 8* (1970), 5–11.

[15] Deimling, K. Multi-valued differential equations. vol. xii of *Nonlinear Analysis and Applications.* Walter de Gruyter Co., Berlin, 1992.

[16] Dhage, B. Multi-valued mappings and fixed points. *Tamkang J. Math. 37*, 1 (2006), 27–46.

[17] Hedayati, V., and Rezapour, S. The existence of solution for a k-dimensional system of fractional differential inclusions with anti-periodic boundary value conditions. *Filomat 30*, 6 (2016), 1601–1613.

[18] Kilbas, A., Srivastava, H., and Trujillo, J. Theory and applications of fractional differential equations. vol. 204 of *North-Holland Mathematics Studies.* Elsevier Science B.V., Amsterdam, 2006.

[19] Kisielewicz, M. Differential inclusions and optimal control. vol. 44 of *East European Series.* Kluwer Academic Publishers Group, Dordrecht, 1991.

[20] Lasota, A., and Opial, Z. An application of the Kakutani-Ky Fan theorem in the theory of ordinary differential equations. *Bull. Acad. Pol. Sci. Set. Sci. Math. Astronom. Phy. 13* (1965), 781–786.

[21] Nieto, J., Ouahab, A., and P., P. Extremal solutions and relaxation problems for fractional differential inclusions. *Abst. Appl. Anal. 2013*, Article ID 292643 (2013), 9 pages.

[22] Wang, G., Zhang, L., and Song, G. Boundary value problem of a nonlinear Langevin equation with two different fractional orders and impulses. *Fixed Point Theory Appl. 2012* (2012), 2012:200.

[23] Wang, J., and Ibrahim, A. Existence and controllability results for nonlocal fractional impulsive differential inclusions in Banach spaces. *J. Function Sp. 2013*, Article ID 518306 (2013), 16 pages.

[24] Wax, N. Selected papers on noise and stochastic processes. Dover Publications Inc., New York, 1954.

*Received by the editors October 20, 2017*
*First published online November 17, 2019*