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On a product of universal relational systems

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Abstract. In this paper, arities of relations are considered to be arbitrary sets. We introduce and study a new operation of product of universal relational systems, which lies between their direct product and the direct product of their reflexive hulls. For the new operation of product and the direct sum of universal relational systems, the validity of the distributive law is shown. Moreover, we define a new power of universal relational systems by combining their direct power and structural power. Then, all the three powers are discussed. It is shown that the introduced power of universal relational systems satisfies the first exponential law with respect to the combined product. Further, we show that the weak forms of the second and third exponential laws for each of the three powers of universal relational systems with respect to the new operation of product are satisfied.

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1. Introduction

G. Birkhoff introduced the cardinal (i.e., direct) arithmetic of partially ordered sets and showed that it behaves analogously to the arithmetic of natural numbers in [1] and [2], i.e., satisfies

$(\mathbf{A}^{\mathbf{B}})^{\mathbf{C}} \cong \mathbf{A}^{\mathbf{B}\times\mathbf{C}}$ - *the first exponential law*,

$\prod_{i\in I} \mathbf{A}_i^{\mathbf{B}} \cong (\prod_{i\in I} \mathbf{A}_i)^{\mathbf{B}}$ - *the second exponential law*,

$\prod_{i\in I} \mathbf{A}^{\mathbf{B}_i} \cong \mathbf{A}^{\sum_{i\in I} \mathbf{B}_i}$ (if $\mathbf{B}_i$, $i \in I$, are pair-wise disjoint) - *the third exponential law.*

In [5], Birkhoff's arithmetic of ordered sets was generalized. The cardinal arithmetic was extended to relational systems e.g. in [6],[9],[10] and [11]. Conversely, the cardinal arithmetic was restricted from relational systems to algebras in [12], to partial algebras in [13] and to hyperalgebras in [4]. In the present paper, the study of the power of relational systems will be continued. In accordance with [11], relations are considered to have arbitrary sets as arities.

Generally, the partial algebras and the hyperalgebras lie between relational systems and algebras because algebras are the relational systems that are both partial algebras and hyperalgebras.

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M. Novotný and J. Šlapal introduced and studied a new operation of power of n-ary relational systems by combining the direct power and the structural power of the systems in [7]. In this note, we will extend the operation of power discussed in [7] to universal relational systems.

In [4], N. Chaisansuk and J. Šlapal introduced and studied a new operation of product of universal hyperalgebras which is combination of the direct sum and the direct product of the systems. The operation of product discussed in [4] will be extended from universal hyperalgebras to universal relational systems in the present paper.

This note contributes to the development of the arithmetic of universal relational systems. The new power and product of universal relational systems will be studied and it will be proved first that the distributive law is valid for the new product and the direct sum of universal relational systems. Further, the validity of the first exponential law for the new power and the new product of universal relational systems will be proved. For the new operation of product, and each of the three powers of universal relational systems, the weak forms of the second and third exponential laws will also be proved.

2. Combined Product of Universal Relational Systems

Given sets A, B, we denote by A^B the set of all mappings of B into A. It is easy to see that there is a bijection $\varphi : (A^B)^C \to A^{B\times C}$(where $\times$ denotes the Cartesian product) given by $\varphi(h)(b, c) = h(c)(b)$ whenever $h \in (A^B)^C, b \in B$ and $c \in C$. The bijection φ will be called *canonical*.

Throughout the paper, maps $f : B \to A$ (A, B sets) will often be denoted as indexed sets $(f_i; i \in B)$, where $f_i \in A$ for every $i \in B$. Of course, then f_i means $f(i)$ for every $i \in B$.

Let Ω be a nonempty set. A family $\tau = (K_\lambda; \lambda \in \Omega)$ of sets will be called a *type*. By a *universal relational system* (briefly, a relational system) of type τ we understand a pair $\mathbf{A} = \langle A, (p_\lambda; \lambda \in \Omega)\rangle$, where A is a nonempty set, the so-called *carrier* of $\mathbf{A}$, and, for every $\lambda \in \Omega$, p_λ is K_λ-*ary relation*, i.e., $p_\lambda \subseteq A^{K_\lambda}$. Of course, if $K_\lambda = \emptyset$, then p_λ is nothing but a nonempty subset of A. In the case $\tau = (K)$ where card $K = n$ and card $K = 2$ relational systems of type τ are usually called *n-ary relations* and *binary relations*, respectively.

Let $\mathbf{B} = \langle B, (q_\lambda; \lambda \in \Omega)\rangle, \mathbf{A} = \langle A, (p_\lambda; \lambda \in \Omega)\rangle$ be a pair of relational systems of type τ. Then $\mathbf{A}$ is called a *relational subsystem* of $\mathbf{B}$ provided that $A \subseteq B$ and $p_\lambda \subseteq q_\lambda$ for every $\lambda \in \Omega$. A map $f : B \to A$ is called a *homomorphism* of $\mathbf{B}$ into $\mathbf{A}$ if, for each $\lambda \in \Omega$, $(b_i; i \in K_\lambda) \in q_\lambda$ implies $(f(b_i); i \in K_\lambda) \in p_\lambda$. The set of all homomorphisms of $\mathbf{B}$ into $\mathbf{A}$ will be denoted by $Hom(\mathbf{B}, \mathbf{A})$. If f is a bijection of B onto A and both $f : B \to A$ and $f^{-1} : A \to B$ are homomorphisms, then f is called an *isomorphism* of $\mathbf{B}$ onto $\mathbf{A}$ and we say that $\mathbf{B}$ and $\mathbf{A}$ are *isomorphic*, in symbols $\mathbf{B} \cong \mathbf{A}$. We say that $\mathbf{B}$ may be *embedded* into $\mathbf{A}$ and write $\mathbf{B} \preccurlyeq \mathbf{A}$ if there exists a relational subsystem $\mathbf{A}'$ of $\mathbf{A}$ such that $\mathbf{B} \cong \mathbf{A}'$.

The *direct product* of a family $\mathbf{A}_i = \langle A_i, (p^i_\lambda; \lambda \in \Omega)\rangle, i \in I$, of relational systems of type $\tau = (K_\lambda; \lambda \in \Omega)$ is the relational system $\prod_{i\in I} \mathbf{A}_i =$

$\langle \prod_{i\in I} A_i, (q_\lambda; \lambda \in \Omega)\rangle$ of type τ, where $\prod_{i\in I} A_i$ denotes the Cartesian product of sets and, for any $\lambda \in \Omega$ and any $(f_k; k \in K_\lambda) \in (\prod_{i\in I} A_i)^{K_\lambda}$, $(f_k; k \in K_\lambda) \in q_\lambda$ if and only if $(f_k(i); k \in K_\lambda) \in p^i_\lambda$. If the set I is finite, say $I = \{1, \ldots, m\}$, then we write $\mathbf{A}_1 \times \ldots \times \mathbf{A}_m$ instead of $\prod_{i\in I} \mathbf{A}_i$. If $\mathbf{A}_i = \mathbf{A}$ for every $i \in I$, then we write $\mathbf{A}^I$ instead of $\prod_{i\in I} \mathbf{A}_i$.

The *direct sum* of a family $\mathbf{A}_i = \langle A_i, (p^i_\lambda; \lambda \in \Omega)\rangle, i \in I$, of relational systems of type τ is the relational system $\sum_{i\in I} \mathbf{A}_i = \langle \bigcup_{i\in I} A_i, (\bigcup_{i\in I} p^i_\lambda; \lambda \in \Omega)\rangle$. If the set I is finite, say $I = \{1, \ldots, m\}$, then we write $\mathbf{A}_1 \uplus \ldots \uplus \mathbf{A}_m$ instead of $\sum_{i\in I} \mathbf{A}_i$.

Let $\mathbf{A} = \langle A, (p_\lambda; \lambda \in \Omega)\rangle$ be a relational system of type $\tau = (K_\lambda; \lambda \in \Omega)$ and let $\lambda_0 \in \Omega$. An element $a \in A$ is called *reflexive* with respect to the relation p_{λ_0} if $(a_i; i \in K_{\lambda_0}) \in p_{\lambda_0}$ whenever $a_i = a$ for every $i \in K_{\lambda_0}$. If every element of A is a reflexive with respect to each relation $p_\lambda, \lambda \in \Omega$, then the relational system $\mathbf{A}$ is said to be *reflexive*.

Let $\mathbf{A} = \langle A, (p_\lambda; \lambda \in \Omega)\rangle$ be a relational system of type $\tau = (K_\lambda; \lambda \in \Omega)$. For every $\lambda \in \Omega$, we denote by $\bar{p}_\lambda$ the relation on $\mathbf{A}$ such that, for every $(a_i; i \in K_\lambda) \in A^{K_\lambda}$, $(a_i; i \in K_\lambda) \in \bar{p}_\lambda$ if and only if $a_i = a$ for every $i \in K_\lambda$ or $(a_i; i \in K_\lambda) \in p_\lambda$. The relational system $\langle A, (\bar{p}_\lambda; \lambda \in \Omega)\rangle$ is called the *reflexive hull* of $\mathbf{A}$ and is denoted by $\bar{\mathbf{A}}$.

Let $\mathbf{A}_i = \langle A_i, (p^i_\lambda; \lambda \in \Omega)\rangle, i \in I$, be a family of relational systems of type $\tau = (K_\lambda; \lambda \in \Omega)$. The *combined product* of the family $\mathbf{A}_i, i \in I$, is the relational system $\bigotimes_{i\in I} \mathbf{A}_i = \langle \prod_{i\in I} A_i, (r_\lambda; \lambda \in \Omega)\rangle$ of type τ given by $\bigotimes_{i\in I} \mathbf{A}_i = \sum_{i\in I} \prod_{j\in I} \mathbf{A}_{ij}$, where

$$\mathbf{A}_{ij} = \begin{cases} \bar{\mathbf{A}}_j & \text{if } i = j, \\ \mathbf{A}_j & \text{if } i \neq j. \end{cases}$$

Thus, for any $\lambda \in \Omega$ and any $((a^i_k; i \in I); k \in K_\lambda) \in (\prod_{i\in I} \mathbf{A}_i)^{K_\lambda}$, we have $((a^i_k; i \in I); k \in K_\lambda) \in r_\lambda$ if and only if there exists a subset $J \subseteq I$, card $J \leq 1$, such that $a^i \in p^i_\lambda(a^i_k; k \in K_\lambda)$ for every $i \in I \setminus J$ and $a^i_k = a^i$ for every $k \in K_\lambda$ and every $i \in J$.

If the set I is finite, say $I = \{1, \ldots, m\}$, we write $\mathbf{A}_1 \otimes \ldots \otimes \mathbf{A}_m$ instead of $\bigotimes_{i\in I} \mathbf{A}_i$. We then clearly have $\mathbf{A}_1 \otimes \ldots \otimes \mathbf{A}_m = (\mathbf{A}_{11} \times \mathbf{A}_{12} \times \ldots \times \mathbf{A}_{1m}) \uplus (\mathbf{A}_{21} \times \mathbf{A}_{22} \times \ldots \times \mathbf{A}_{2m}) \uplus \ldots \uplus (\mathbf{A}_{m1} \times \mathbf{A}_{m2} \ldots, \times \mathbf{A}_{mm}) = (\bar{\mathbf{A}}_1 \times \mathbf{A}_2 \times \ldots \times \mathbf{A}_m) \uplus (\mathbf{A}_1 \times \bar{\mathbf{A}}_2 \times \ldots \times \mathbf{A}_m) \uplus \ldots \uplus (\mathbf{A}_1 \times \mathbf{A}_2 \times \ldots \times \bar{\mathbf{A}}_m)$.

In particular, if $I = \{1, 2\}$, then, for every $\lambda \in \Omega$ and every $((a_k, b_k); k \in K_\lambda) \in (\mathbf{A}_1 \times \mathbf{A}_2)^{K_\lambda}$, $((a_k, b_k); k \in K_\lambda) \in r_\lambda$ if and only if one of the following three conditions is satisfied:

(i) $(a_k; k \in K_\lambda) \in p^1_\lambda$ and $(b_k; k \in K_\lambda) \in p^2_\lambda$,

(ii) $a = a_k$ for every $k \in K_\lambda$ and $(b_k; k \in K_\lambda) \in p^2_\lambda$,

(iii) $(a_k; k \in K_\lambda) \in p^1_\lambda$ and $b = b_k$ for every $k \in K_\lambda$.

Example 2.1. Let $\mathbf{A} = \langle A, p\rangle, \mathbf{B} = \langle B, q\rangle$ be binary relational systems, where $A = \{a, b\}, B = \{x, y\}, p = \{(a, b)\}$ and $q = \{(x, y)\}$. Then $\mathbf{A} \otimes \mathbf{B} = (A \times B, r)$, where

$$r = \{((a, x), (b, y)), ((a, x), (b, x)), ((a, y), (b, y)), ((a, x), (a, y)), ((b, x), (b, y))\}.$$

Example 2.2. Let $\mathbf{A} = \langle A, p\rangle, \mathbf{B} = \langle B, q\rangle$ be binary relational systems, where $A = \{a\}, B = \{x, y\}, p = \{(a, a)\}$ and $q = \{(x, y)\}$. Then $\mathbf{A} \otimes \mathbf{B} = (A \times B, r)$, where $r = \{((a, x), (a, y)), ((a, x), (a, x)), ((a, y), (a, y))\}$. Thus, $\mathbf{A} \otimes \mathbf{B}$ is reflexive.

Remark 2.3. Let $\mathbf{A}_i, i \in I$, be a family of relational systems. If $\mathbf{A}_i$ is reflexive for every $i \in I$, then $\prod_{i\in I} \mathbf{A}_i = \bigotimes_{i\in I} \mathbf{A}_i$. If $\mathbf{A}_i, i \in I$, are reflexive with the exception of at most one of them, then $\bigotimes_{i\in I} \mathbf{A}_i$ is reflexive.

The combined product of relational systems distributes over their direct sum, which is shown as follows.

Theorem 2.4. *Let $\mathbf{A}_i, i \in I$, be a nonempty family of relational systems of the same type τ and let $\mathbf{B}$ be a relational system of type τ. Then $\sum_{i\in I}(\mathbf{B} \otimes \mathbf{A}_i) = \mathbf{B} \otimes \sum_{i\in I} \mathbf{A}_i$.*

Proof. Let $\mathbf{A}_i = \langle A_i, (p^i_\lambda; \lambda \in \Omega)\rangle, i \in I$, $\mathbf{B} = \langle B, (q_\lambda; \lambda \in \Omega)\rangle$ and let $\tau = (K_\lambda; \lambda \in \Omega)$. Let $\sum_{i\in I} \mathbf{A}_i = \langle \bigcup_{i\in I} A_i, (s_\lambda; \lambda \in \Omega)\rangle, \mathbf{B} \otimes \mathbf{A}_i = \langle B \times A_i, (r^i_\lambda; \lambda \in \Omega)\rangle$ for each $i \in I$, $\mathbf{B} \otimes \sum_{i\in I} \mathbf{A}_i = \langle B \times \bigcup_{i\in I} A_i, (u_\lambda; \lambda \in \Omega)\rangle$ and $\sum_{i\in I}(\mathbf{B} \otimes \mathbf{A}_i) = \langle \bigcup_{i\in I}(B \times A_i), (v_\lambda; \lambda \in \Omega)\rangle$. We will show that $((b_k, a_k); k \in K_\lambda) \in u_\lambda$ if and only if $((b_k, a_k); k \in K_\lambda) \in v_\lambda$ for every $(b_k, a_k) \in B \times \bigcup_{i\in I} A_i = \bigcup_{i\in I}(B \times A_i)$ and for every $k \in K_\lambda$.

It is easy to see that the following conditions satisfy $(a) \Leftrightarrow (b) \Leftrightarrow (c) \Leftrightarrow (d) \Leftrightarrow (e)$:

(a) $((b_k, a_k); k \in K_\lambda) \in u_\lambda$;

(b) one of the following three cases occurs:

- (i) $(b_k; k \in K_\lambda) \in q_\lambda$ and $(a_k; k \in K_\lambda) \in s_\lambda$;
- (ii) $b = b_k$ for every $k \in K_\lambda$ and $(a_k; k \in K_\lambda) \in s_\lambda$;
- (iii) $(b_k; k \in K_\lambda) \in q_\lambda$ and $a = a_k$ for every $k \in K_\lambda$;

(c) one of the following three cases occurs:

- (i) $(b_k; k \in K_\lambda) \in q_\lambda$ and $(a_k; k \in K_\lambda) \in p^i_\lambda$ for some $i \in I$;
- (ii) $b = b_k$ for every $k \in K_\lambda$ and $(a_k; k \in K_\lambda) \in p^i_\lambda$ for some $i \in I$;
- (iii) $(b_k; k \in K_\lambda) \in q_\lambda$ and $a = a_k$ for every $k \in K_\lambda$;

(d) $((b_k, a_k); k \in K_\lambda) \in r^i_\lambda$ for some $i \in I$;

(e) $((b_k, a_k); k \in K_\lambda) \in v_\lambda$.

This proves the statement. □

3. The power of universal relational systems

Let $\mathbf{B} = \langle B, (q_\lambda; \lambda \in \Omega)\rangle$ and $\mathbf{A} = \langle A, (p_\lambda; \lambda \in \Omega)\rangle$ be relational systems of the same type $\tau = (K_\lambda; \lambda \in \Omega)$.

The *direct power* of relational systems $\mathbf{A}$ and $\mathbf{B}$ is the relational system $\mathbf{A}^{\diamond\mathbf{B}} = \langle Hom(\mathbf{B}, \mathbf{A}), (t_\lambda; \lambda \in \Omega)\rangle$ of type τ where, for any $\lambda \in \Omega$ and any $(f_i; i \in K_\lambda) \in (Hom(\mathbf{B}, \mathbf{A}))^{K_\lambda}$, $(f_i; i \in K_\lambda) \in t_\lambda$ if and only if $(f_i(b); i \in K_\lambda) \in p_\lambda$ for each $b \in B$.

The *structural power* of relational systems $\mathbf{A}$ and $\mathbf{B}$ is the relational system $\mathbf{A}^{\circ\mathbf{B}} = \langle Hom(\mathbf{B}, \mathbf{A}), (s_\lambda; \lambda \in \Omega)\rangle$ of type τ where, for any $\lambda \in \Omega$ and any $(f_i; i \in K_\lambda) \in (Hom(\mathbf{B}, \mathbf{A}))^{K_\lambda}$, $(f_i; i \in K_\lambda) \in s_\lambda$ if and only if $(f_i(b_i); i \in K_\lambda) \in p_\lambda$ whenever $(b_i; i \in K_\lambda) \in q_\lambda$.

The combined power of relational systems is defined by combining the direct power and the structural power of relational systems.

Definition 3.1. Let $\mathbf{B} = \langle B, (q_\lambda; \lambda \in \Omega)\rangle$ and $\mathbf{A} = \langle A, (p_\lambda; \lambda \in \Omega)\rangle$ be relational systems of the same type $\tau = (K_\lambda; \lambda \in \Omega)$. The *combined power* of relational systems $\mathbf{A}$ and $\mathbf{B}$ is the relational system $\mathbf{A}^{\mathbf{B}}$ given by:

$$\mathbf{A}^{\mathbf{B}} = \mathbf{A}^{\diamond\mathbf{B}} \cap \mathbf{A}^{\circ\mathbf{B}}$$

Thus, $\mathbf{A}^{\mathbf{B}} = \langle Hom(\mathbf{B}, \mathbf{A}), (r_\lambda; \lambda \in \Omega)\rangle$ is the relational system of type τ where, for any $\lambda \in \Omega$ and any $(f_i; i \in K_\lambda) \in (Hom(\mathbf{B}, \mathbf{A}))^{K_\lambda}$, $(f_i; i \in K_\lambda) \in r_\lambda$ if and only if $(f_i(b_i); i \in K_\lambda) \in p_\lambda$ whenever $(b_i; i \in K_\lambda) \in \bar{q}_\lambda$.

Remark 3.2. a) Given relational systems $\mathbf{A}$ and $\mathbf{B}$, it may easily be seen that $\mathbf{A}^{\circ\mathbf{B}} \subseteq \mathbf{A}^{\diamond\mathbf{B}}$ if $\mathbf{B}$ is reflexive, and then $\mathbf{A}^{\circ\mathbf{B}} = \mathbf{A}^{\mathbf{B}}$.

b) Let K, L, A be nonempty sets. By a $K \times L$-matrix M over A, we understand any map $M : K \times L \to A$, i.e., the indexed sets $(a_{ij}; i \in K, j \in L)$ denoted briefly by (a_{ij}). In [11], a relational system $\langle A, (p_\lambda; \lambda \in \Omega)\rangle$ of type τ is called *diagonal* if, for every $\lambda \in \Omega$ and every $K_\lambda \times K_\lambda$-matrix (a_{ij}) over A, from $(a_{ij}; i \in K_\lambda) \in p_\lambda$ for each $j \in K_\lambda$ and $(a_{ij}; j \in K_\lambda) \in p_\lambda$ for each $i \in K_\lambda$ it follows that $(a_{ii}; i \in K_\lambda) \in p_\lambda$ for each $i \in K_\lambda$. For idempotent algebras with one finitary operation, the diagonality introduced coincides with the diagonality studied in [8]. Given relational systems $\mathbf{A}$ and $\mathbf{B}$ of the same type, if $\mathbf{A}$ is diagonal, then $\mathbf{A}^{\diamond\mathbf{B}} \subseteq \mathbf{A}^{\circ\mathbf{B}}$, and then $\mathbf{A}^{\diamond\mathbf{B}} = \mathbf{A}^{\mathbf{B}}$.

c) Let $\mathbf{A}$ and $\mathbf{B}$ be relational systems. If $\mathbf{A}$ is diagonal and $\mathbf{B}$ is reflexive, then $\mathbf{A}^{\diamond\mathbf{B}} = \mathbf{A}^{\circ\mathbf{B}} = \mathbf{A}^{\mathbf{B}}$.

In general, the direct power and the structural power of relational systems need not fulfill the first exponential law. The validity of the first exponential law for the combined power of relational systems with respect to the combined product is shown as follows.

Lemma 3.3. *Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be relational systems of the same type. Then the canonical bijection $\varphi : (A^B)^C \to A^{B\times C}$ restricted to $Hom(\mathbf{C}, \mathbf{A}^{\mathbf{B}})$ is a bijection of $Hom(\mathbf{C}, \mathbf{A}^{\mathbf{B}})$ onto $Hom(\mathbf{B} \otimes \mathbf{C}, \mathbf{A})$.*

Proof. Let $\mathbf{B} = \langle B, (q_\lambda; \lambda \in \Omega)\rangle, \mathbf{A} = \langle A, (p_\lambda; \lambda \in \Omega)\rangle, \mathbf{C} = \langle C, (s_\lambda; \lambda \in \Omega)\rangle$. Let $\tau = (K_\lambda; \lambda \in \Omega)$ be the type of $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$. Let

$$\mathbf{A}^{\mathbf{B}} = \langle Hom(\mathbf{B}, \mathbf{A}), (r_\lambda; \lambda \in \Omega)\rangle, \quad \mathbf{B} \otimes \mathbf{C} = \langle B \times C, (v_\lambda; \lambda \in \Omega)\rangle,$$

$$(\mathbf{A}^{\mathbf{B}})^{\mathbf{C}} = \langle Hom(\mathbf{C}, \mathbf{A}^{\mathbf{B}}), (t_\lambda; \lambda \in \Omega)\rangle \text{ and}$$
$$\mathbf{A}^{\mathbf{B}\otimes\mathbf{C}} = \langle Hom(\mathbf{B} \otimes \mathbf{C}, \mathbf{A}), (u_\lambda; \lambda \in \Omega)\rangle.$$

Let $\lambda \in \Omega$, $h \in Hom(\mathbf{C}, \mathbf{A}^{\mathbf{B}})$ and let $((b_i, c_i); i \in K_\lambda) \in v_\lambda$ for every $i \in K_\lambda$. We will show that $(\varphi(h)(b_i, c_i); i \in K_\lambda) \in p_\lambda$. Clearly, one of the following three conditions is satisfied:

(i) $(b_i; i \in K_\lambda) \in q_\lambda$ and $(c_i; i \in K_\lambda) \in s_\lambda$;

(ii) $b = b_i$ for every $i \in K_\lambda$ and $(c_i; i \in K_\lambda) \in s_\lambda$;

(iii) $(b_i; i \in K_\lambda) \in q_\lambda$ and $c = c_i$ for every $i \in K_\lambda$.

Suppose that (i) is satisfied. Since $(c_i; i \in K_\lambda) \in s_\lambda$, by the above considerations we have $(h(c_i); i \in K_\lambda) \in r_\lambda$. Then $(h(c_i)(b_i); i \in K_\lambda) \in r_\lambda$ because $(b_i; i \in K_\lambda) \in q_\lambda$. Therefore, $(\varphi(h)(b_i, c_i); i \in K_\lambda) \in p_\lambda$ provided that (i) is satisfied.

If $(c_i; i \in K_\lambda) \in s_\lambda$, then $(h(c_i); i \in K_\lambda) \in r_\lambda$, hence $(h(c_i)(b); i \in K_\lambda)) = p_\lambda(\varphi(h)(b, c_i); i \in K_\lambda) \in p_\lambda$ for every $b \in B$. Thus, $(\varphi(h)(b, c_i); i \in K_\lambda) \in p_\lambda$ provided that (ii) is satisfied.

If $(b_i; i \in K_\lambda) \in q_\lambda$, then $(h(c)(b_i); i \in K_\lambda) = (\varphi(h)(b_i, c); i \in K_\lambda) \in p_\lambda$ because $h(c) \in Hom(\mathbf{B}, \mathbf{A})$ for every $c \in C$. So we have, $(\varphi(h)(b_i, c); i \in K_\lambda) \in p_\lambda$ provided that (iii) is satisfied.

We have shown that $\varphi(h) \in Hom(\mathbf{B} \otimes \mathbf{C}, \mathbf{A})$.

Let $g \in Hom(\mathbf{B} \otimes \mathbf{C}, \mathbf{A})$, $c \in C$ and $(b_i; i \in K_\lambda) \in q_\lambda$. Consequently, $((b_i, c); i \in K_\lambda) \in v_\lambda$. Since g is a homomorphism, we have $(g(b_i, c); i \in K_\lambda) \in p_\lambda$. Thus, $(\varphi^{-1}(g)(c)(b_i); i \in K_\lambda) \in p_\lambda$. Therefore, $\varphi^{-1}(g)(c) \in Hom(\mathbf{B}, \mathbf{A})$ for each $c \in C$.

Next, let $(c_i; i \in K_\lambda) \in s_\lambda$. We will show that $(\varphi^{-1}(g)(c_i); i \in K_\lambda) \in r_\lambda$. Assume $(b_i; i \in K_\lambda) \in \bar{q}_\lambda$, then we have $((b_i, c_i); i \in K_\lambda) \in v_\lambda$. Since g is a homomorphism, we get $(g(b_i, c_i); i \in K_\lambda) \in p_\lambda$. So $(\varphi^{-1}(g)(b_i)(c_i); i \in K_\lambda) = (g(b_i, c_i); i \in K_\lambda) \in p_\lambda$. Consequently, $(\varphi^{-1}(g)(c_i); i \in K_\lambda) \in r_\lambda$, which yields $\varphi^{-1}(g) \in Hom(\mathbf{C}, \mathbf{A}^{\mathbf{B}})$. □

Theorem 3.4. *Let* $\mathbf{A}, \mathbf{B}, \mathbf{C}$ *be relational systems of the same type, then*

$$(\mathbf{A}^{\mathbf{B}})^{\mathbf{C}} \cong \mathbf{A}^{\mathbf{B}\otimes\mathbf{C}}.$$

Proof. Let $\mathbf{B} = \langle B, (q_\lambda; \lambda \in \Omega)\rangle, \mathbf{A} = \langle A, (p_\lambda; \lambda \in \Omega)\rangle, \mathbf{C} = \langle C, (s_\lambda; \lambda \in \Omega)\rangle$ and let $\tau = (K_\lambda; \lambda \in \Omega)$ be the type of $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$. Let

$$\mathbf{A}^{\mathbf{B}} = \langle Hom(\mathbf{B}, \mathbf{A}), (r_\lambda; \lambda \in \Omega)\rangle, \quad \mathbf{B} \otimes \mathbf{C} = \langle B \times C, (v_\lambda; \lambda \in \Omega)\rangle,$$

$$(\mathbf{A}^{\mathbf{B}})^{\mathbf{C}} = \langle Hom(\mathbf{C}, \mathbf{A}^{\mathbf{B}}), (t_\lambda; \lambda \in \Omega)\rangle \text{ and}$$

$$\mathbf{A}^{\mathbf{B}\otimes\mathbf{C}} = \langle Hom(\mathbf{B}\otimes\mathbf{C},\mathbf{A}), (u_\lambda; \lambda \in \Omega)\rangle.$$

Because of Lemma 3.3, we only have to show that φ and φ^{-1} are homomorphisms.

Let $(h_i; i \in K_\lambda) \in t_\lambda$ whenever $h_i \in Hom(\mathbf{C}, \mathbf{A}^{\mathbf{B}})$ and let $((b_i, c_i); i \in K_\lambda) \in v_\lambda$ for every $i \in K_\lambda$. We will show that $(\varphi(h_i)(b_i, c_i); i \in K_\lambda) \in p_\lambda$. Clearly, one of the following three conditions is satisfied:

(i) $(b_i; i \in K_\lambda) \in q_\lambda$ and $(c_i; i \in K_\lambda) \in s_\lambda$;

(ii) $b = b_i$ for every $i \in K_\lambda$ and $(c_i; i \in K_\lambda) \in s_\lambda$;

(iii) $(b_i; i \in K_\lambda) \in q_\lambda$ and $c = c_i$ for every $i \in K_\lambda$.

Suppose that (i) is satisfied. Since $(c_i; i \in K_\lambda) \in s_\lambda$, we have $(h_i(c_i); i \in K_\lambda) \in r_\lambda$. Then $(h_i(c_i); i \in K_\lambda) \in r_\lambda$ because $(b_i; i \in K_\lambda) \in q_\lambda$. Therefore, $(\varphi(h_i)(b_i, c_i); i \in K_\lambda) \in p_\lambda$ provided that (i) is satisfied.

If $(c_i; i \in K_\lambda) \in s_\lambda$, then $(h_i(c_i); i \in K_\lambda) \in r_\lambda$, hence $(h_i(c_i)(b); i \in K_\lambda)) = (\varphi(h_i)(b, c_i); i \in K_\lambda) \in p_\lambda$ for every $b \in B$. Thus, $(\varphi(h_i)(b, c_i); i \in K_\lambda) \in p_\lambda$ for every $b \in B$ provided that (ii) is satisfied.

If $(b_i; i \in K_\lambda) \in q_\lambda$, then $(h_i(c)(b_i); i \in K_\lambda) = (\varphi(h_i)(b_i, c); i \in K_\lambda) \in p_\lambda$ because $(h_i(c); i \in K_\lambda) \in r_\lambda$ for every $c \in C$. So, $(\varphi(h_i)(b_i, c_); i \in K_\lambda) \in p_\lambda$ provided that (iii) is satisfied. Hence φ is a homomorphism of $(\mathbf{A}^{\mathbf{B}})^{\mathbf{C}}$ onto $\mathbf{A}^{\mathbf{B}\otimes\mathbf{C}}$.

Next, we will show that φ^{-1} is a homomorphism of $\mathbf{A}^{\mathbf{B}\otimes\mathbf{C}}$ onto $(\mathbf{A}^{\mathbf{B}})^{\mathbf{C}}$. Let $(g_i; i \in K_\lambda) \in u_\lambda$, $(c_i; i \in K_\lambda) \in \bar{s}_\lambda$ and $(b_i; i \in K_\lambda) \in \bar{q}_\lambda$. Then we have $((b_i, c_i); i \in K_\lambda) \in v_\lambda$ and, consequently, $(g_i(b_i, c_i); i \in K_\lambda) \in p_\lambda$. Therefore, $(\varphi^{-1}(g_i)(c_i)(b_i); i \in K_\lambda) = (g_i(b_i, c_i); i \in K_\lambda) \in p_\lambda$. Then, $(\varphi^{-1}(g_i)(c_i); i \in K_\lambda) \in r_\lambda$. Since $(g_i; i \in K_\lambda) \in u_\lambda$, we have $(g_i(b, c); i \in K_\lambda) \in p_\lambda$ for every $(b, c) \in B \times C$. Hence, $(\varphi^{-1}(g_i); i \in K_\lambda) \in t_\lambda$, so that φ^{-1} is a homomorphism.

We have shown that $(\mathbf{A}^{\mathbf{B}})^{\mathbf{C}}$ is isomorphic to $\mathbf{A}^{\mathbf{B}\otimes\mathbf{C}}$. □

Corollary 3.5. *Let* $\mathbf{A}, \mathbf{B}, \mathbf{C}$ *be relational systems of the same type. If* $\mathbf{B}$ *and* $\mathbf{C}$ *are reflexive, then*

$$(\mathbf{A}^{\circ\mathbf{B}})^{\circ\mathbf{C}} \cong \mathbf{A}^{\circ\mathbf{B}\times\mathbf{C}}.$$

Proof. Obviously, if $\mathbf{B}$ and $\mathbf{C}$ are reflexive, then $\mathbf{B}\times\mathbf{C}$ is reflexive. Therefore, $\mathbf{B}\times\mathbf{C} = \mathbf{B}\otimes\mathbf{C}$. By Remark 3.2(a), we have $(\mathbf{A}^{\circ\mathbf{B}})^{\circ\mathbf{C}} = (\mathbf{A}^{\mathbf{B}})^{\mathbf{C}}$ and $\mathbf{A}^{\circ(\mathbf{B}\times\mathbf{C})} = \mathbf{A}^{\mathbf{B}\times\mathbf{C}} = \mathbf{A}^{\mathbf{B}\otimes\mathbf{C}}$. Thus, the assertion follows from Theorem 3.4.

□

Corollary 3.6. *Let* $\mathbf{A}, \mathbf{B}, \mathbf{C}$ *be relational systems of the same type. If* $\mathbf{A}$ *is diagonal and* $\mathbf{B}$ *and* $\mathbf{C}$ *are reflexive, then*

$$(\mathbf{A}^{\diamond\mathbf{B}})^{\diamond\mathbf{C}} \cong \mathbf{A}^{\diamond(\mathbf{B}\times\mathbf{C})}.$$

Proof. By Remark 3.2 (b) and Remark 3.2(c), we have $(\mathbf{A}^{\diamond\mathbf{B}})^{\diamond\mathbf{C}} = (\mathbf{A}^{\mathbf{B}})^{\mathbf{C}}$ and $\mathbf{A}^{\diamond(\mathbf{B}\times\mathbf{C})} = \mathbf{A}^{\mathbf{B}\times\mathbf{C}}$. Since $\mathbf{B}$ and $\mathbf{C}$ are reflexive, $\mathbf{B}\times\mathbf{C}$ is reflexive. So, we have $\mathbf{B}\times\mathbf{C} = \mathbf{B}\otimes\mathbf{C}$. Thus, the assertion follows from Theorem 3.4. □

For the direct power and the structural power of relational systems and their direct product, the second and third exponential laws are valid. These results were proved in [9] and [3]. Then, we get the second and third exponential laws also for the combined power. However, these laws are not generally true for the combined product of relational systems. Next, we will show that, for the direct power, the structural power and combined power of relational systems, weak forms of the second and third exponential laws hold with respect to combined product.

Theorem 3.7. *Let $\mathbf{A}_i, i \in I$ be a nonempty family of relational systems of the same type τ and let $\mathbf{B}$ be a relational system of type τ. Then the second exponential law holds for the direct power, the structural power and the combined power:*

$$(1)\ \bigotimes_{i\in I}(\mathbf{A}_i^{\diamond\mathbf{B}}) \preccurlyeq (\bigotimes_{i\in I}\mathbf{A}_i)^{\diamond\mathbf{B}};$$

$$(2)\ \bigotimes_{i\in I}(\mathbf{A}_i^{\circ\mathbf{B}}) \preccurlyeq (\bigotimes_{i\in I}\mathbf{A}_i)^{\circ\mathbf{B}};$$

$$(3)\ \bigotimes_{i\in I}(\mathbf{A}_i^{\mathbf{B}}) \preccurlyeq (\bigotimes_{i\in I}\mathbf{A}_i)^{\mathbf{B}}.$$

Proof. Let $\mathbf{A} = \langle A_i, (p^i_\lambda; \lambda \in \Omega)\rangle$ for every $i \in I$, $\mathbf{B} = \langle B, (q_\lambda; \lambda \in \Omega)\rangle$ and let $\tau = (K_\lambda; \lambda \in \Omega)$. Let $\bigotimes_{i\in I}\mathbf{A}_i = \langle \prod_{i\in I} A_i, (r_\lambda; \lambda \in \Omega)\rangle$, $\mathbf{A}_i^{\mathbf{B}} = \langle Hom(\mathbf{B}, \mathbf{A}_i), (u^i_\lambda; \lambda \in \Omega)\rangle$ for every $i \in I$, $\bigotimes_{i\in I}(\mathbf{A}_i^{\mathbf{B}}) = \langle \prod_{i\in I} Hom(\mathbf{B}, \mathbf{A}_i), (s_\lambda; \lambda \in \Omega)\rangle$ and $(\bigotimes_{i\in I}\mathbf{A}_i)^{\mathbf{B}} = \langle Hom(\mathbf{B}, \bigotimes_{i\in I}\mathbf{A}_i), (t_\lambda; \lambda \in \Omega)\rangle$.

We define the map $\alpha : \prod_{i\in I} Hom(\mathbf{B}, \mathbf{A}_i) \to (\prod_{i\in I} A_i)^B$ by $\alpha(f^i;\ i \in I)(b) = (f^i(b);\ i \in I)$ for each $b \in B$.

Let $(f^i;\ i \in I) \in \prod_{i\in I} Hom(\mathbf{B}, \mathbf{A}_i)$ and $(b_k; k \in K_\lambda) \in q_\lambda$. Since $f^i \in Hom(\mathbf{B}, \mathbf{A}_i)$, we have $(f^i(b_k); k \in K_\lambda) \in p_i$ for every $i \in I$. Then $((f^i(b_k); i \in I); k \in K_\lambda) \in r_\lambda$ and we have $(\alpha(f^i;\ i \in I)(b_k); k \in K_\lambda) \in r_\lambda$. Therefore, $\alpha(f^i;\ i \in I) \in Hom(\mathbf{B}, \bigotimes_{i\in I}\mathbf{A}_i)$.

Suppose that $\alpha(f^i;\ i \in I) = \alpha(g^i;\ i \in I)$, where $(f^i;\ i \in I), (g^i;\ i \in I) \in \prod_{i\in I} Hom(\mathbf{B}, \mathbf{A}_i)$. Then $(f^i(b);\ i \in I) = \alpha(f^i;\ i \in I)(b) = \alpha(g^i;\ i \in I)(b) = (g^i(b);\ i \in I)$ for every $b \in B$. Therefore, $f^i(b) = g^i(b)$ for every $i \in I$ and every $b \in B$. Hence, $f^i = g^i$ for every $i \in I$. Thus, α: $\prod_{i\in I} Hom(\mathbf{B}, \mathbf{A}_i) \to Hom(\mathbf{B}, \bigotimes_{i\in I}\mathbf{A}_i)$ is an injection.

We will show that α is a homomorphism. Let $((f^i_k; i \in I); k \in K_\lambda) \in s_\lambda$ and let $J \subseteq I$, card $J \leq 1$. Consequently, $(f^i_k; k \in K_\lambda) \in u^i_\lambda$ for every $i \in I \setminus J$ and $f^i = f^i_k$ for every $k \in K_\lambda$ and $i \in J$.

Hence, $(f^i_k(b); k \in K_\lambda) \in p^i_\lambda$ for every $i \in I \setminus J$ and every $b \in B$ and $f^i_k(b) = f^i(b)$ for every $k \in K_\lambda$, $i \in J$ and every $b \in B$. Therefore, $((f^i_k(b);\ i \in I); k \in K_\lambda) \in r_\lambda$ for every $b \in B$. This gives the condition (1).

If $(b_k; k \in K_\lambda) \in q_\lambda$, then we have $(f^i_k(b_k); k \in K_\lambda) \in p^i_\lambda$ for every $i \in I \setminus J$ and $f^i_k(b_k) = f^i(b)$ for every $k \in K_\lambda$, $i \in J$. Therefore, $((f^i_k(b_k);\ i \in I); k \in K_\lambda) \in r_\lambda$. Thus $(\alpha(f^i_k;\ i \in I); k \in K_\lambda) \in t_\lambda$. This gives the condition (2).

Since

$$\bigotimes_{i\in I}(\mathbf{A}_i^{\mathbf{B}}) = \bigotimes_{i\in I}(\mathbf{A}_i^{\diamond\mathbf{B}}) \ \cap\ \bigotimes_{i\in I}(\mathbf{A}_i^{\circ\mathbf{B}}) \text{ and}$$

$$(\bigotimes_{i\in I}\mathbf{A}_i)^{\diamond\mathbf{B}} \,\cap\, (\bigotimes_{i\in I}\mathbf{A}_i)^{\circ\mathbf{B}} = (\bigotimes_{i\in I}\mathbf{A}_i)^{\mathbf{B}},$$

the condition (3) is satisfied. □

It may be simply shown that, in Theorem 3.7, $\cong$ may be written instead of $\preccurlyeq$ if $\mathbf{A}_i$ is reflexive for every $i \in I$. By using Remark 2.3, we obtain the second exponential law for the three powers with respect to the direct product

$$(1)\ \prod_{i\in I}\mathbf{A}_i^{\diamond\mathbf{B}} \cong (\prod_{i\in I}\mathbf{A}_i)^{\diamond\mathbf{B}};$$

$$(2)\ \prod_{i\in I}\mathbf{A}_i^{\circ\mathbf{B}} \cong (\prod_{i\in I}\mathbf{A}_i)^{\circ\mathbf{B}};$$

$$(3)\ \prod_{i\in I}\mathbf{A}_i^{\mathbf{B}} \cong (\prod_{i\in I}\mathbf{A}_i)^{\mathbf{B}}.$$

Theorem 3.8. *Let* $\mathbf{A}$ *be a relational system of type* τ *and let* $\mathbf{B}_i$, $i \in I$, *be a family of pair-wise disjoint relational systems of the same type* τ. *Then the third exponential law holds for the direct power, the structural power and the combined power:*

$$(1)\ \mathbf{A}^{\sum_{i\in I}\diamond\mathbf{B}_i} \preccurlyeq \bigotimes_{i\in I}\mathbf{A}^{\diamond\mathbf{B}_i};$$

$$(2)\ \mathbf{A}^{\sum_{i\in I}\circ\mathbf{B}_i} \preccurlyeq \bigotimes_{i\in I}\mathbf{A}^{\circ\mathbf{B}_i};$$

$$(3)\ \mathbf{A}^{\sum_{i\in I}\mathbf{B}_i} \preccurlyeq \bigotimes_{i\in I}\mathbf{A}^{\mathbf{B}_i}.$$

Proof. Let $\mathbf{A} = \langle A, (p_\lambda; \lambda \in \Omega)\rangle$, $\mathbf{B}_i = \langle B_i, (q^i_\lambda; \lambda \in \Omega)\rangle$ for every $i \in I$ and let $\tau = (K_\lambda; \lambda \in \Omega)$. Let $\sum_{i\in I}\mathbf{B}_i = \langle\bigcup_{i\in I} B_i, (v_\lambda; \lambda \in \Omega)\rangle$, $\mathbf{A}^{\mathbf{B}_i} = \langle Hom(\mathbf{B}_i, \mathbf{A}), (r^i_\lambda; \lambda \in \Omega)\rangle$ for every $i \in I$, $\prod_{i\in I}(\mathbf{A}^{\mathbf{B}_i}) = \langle\prod_{i\in I} Hom(\mathbf{B}_i, \mathbf{A}), (t_\lambda; \lambda \in \Omega)\rangle$ and $\mathbf{A}^{\sum_{i\in I}\mathbf{B}_i} = \langle Hom(\sum_{i\in I}\mathbf{B}_i,\mathbf{A}), (u_\lambda; \lambda \in \Omega)\rangle$.

We define the map $\alpha : Hom(\sum_{i\in I}\mathbf{B}_i, \mathbf{A}) \to \prod_{i\in I} Hom(\mathbf{B}_i, \mathbf{A})$ by $\alpha(h) = (f^i;\ i \in I)$ whenever $h \in Hom(\sum_{i\in I}\mathbf{B}_i, \mathbf{A})$, where $f^i = h|_{B_i}$ for every $i \in I$. It may easily be seen that f_i is a homomorphism of $\mathbf{B}_i$ into $\mathbf{A}$ for every $i \in I$. It follows that $\alpha(h) \in \prod_{i\in I} Hom(\mathbf{B}_i, \mathbf{A})$. Clearly, α is an injection of the set $Hom(\sum_{i\in I}\mathbf{B}_i, \mathbf{A})$ into $\prod_{i\in I} Hom(\mathbf{B}_i, \mathbf{A})$

We will show that α is a homomorphism. Let $(h_k; k \in K_\lambda) \in u_\lambda$. So, $(h_k|_{B_i}(b^i); k \in K_\lambda) = (f^i_k(b^i); k \in K_\lambda) \in p_\lambda$ for any $b^i \in B_i$ and for any $i \in I$. This gives the condition (1). If $(b^i_k; k \in K_\lambda) \in q^i_\lambda$, then $(h_k|_{B_i}(b^i_k); k \in K_\lambda) = (f^i_k(b^i_k); k \in K_\lambda) \in p_\lambda$ for any $i \in I$. This gives the condition (2). And, the condition (3) is satisfied because $\mathbf{A}^{\sum_{i\in I}\mathbf{B}_i} = \prod_{i\in I}\mathbf{A}_i^{\diamond\mathbf{B}} \cap \mathbf{A}^{\sum_{i\in I}\circ\mathbf{B}_i}$ and $\bigotimes_{i\in I}\mathbf{A}^{\mathbf{B}_i} = \bigotimes_{i\in I}\mathbf{A}^{\diamond\mathbf{B}_i} \cap \bigotimes_{i\in I}\mathbf{A}^{\circ\mathbf{B}_i}$.

□

It may be simply shown that, in Theorem 3.8, $\cong$ may be written instead of $\preccurlyeq$ if $\mathbf{B}_i$ is reflexive for every $i \in I$. It may be seen that $\mathbf{A}^{\diamond\mathbf{B}_i}$, $\mathbf{A}^{\circ\mathbf{B}_i}$ and $\mathbf{A}^{\mathbf{B}_i}$ are reflexive for every $i \in I$ because the underlying set of the three powers is a set of all homomorphisms from $\mathbf{B}_i$ into $\mathbf{A}$ for every $i \in I$. By using Remark 2.3, we obtain the third exponential law for the three powers and the direct product

(1) $\prod_{i\in I}\mathbf{A}^{\diamond\mathbf{B}_i}\cong\mathbf{A}^{\sum_{i\in I}\diamond\mathbf{B}_i}$;

(2) $\prod_{i\in I}\mathbf{A}^{\circ\mathbf{B}_i}\cong\mathbf{A}^{\sum_{i\in I}\circ\mathbf{B}_i}$;

(3) $\prod_{i\in I}\mathbf{A}^{\mathbf{B}_i}\cong\mathbf{A}^{\sum_{i\in I}\mathbf{B}_i}$.

Remark 3.9. As for generality, hyperalgebras lie between relational systems and algebras. Let Ω be a nonempty set and $\tau=(K_\lambda;\lambda\in\Omega)$. An *universal hyperalgebra* $\mathbf{A}=\langle A,(p_\lambda;\lambda\in\Omega)\rangle$ of type $\tau=(K_\lambda;\lambda\in\Omega)$ is a relational system $\mathbf{A}=\langle A,(q_\lambda;\lambda\in\Omega)\rangle$ of type $(K_\lambda\cup\{k_\lambda\};\lambda\in\Omega)$, where $\{k_\lambda\}$ is a singleton set and $k_\lambda\notin K_\lambda$ for every $\lambda\in\Omega$, such that for each $f=(a_i;i\in K_\lambda)\in A^{K_\lambda}$ there exists $g=(a_i;i\in K_\lambda\cup\{k_\lambda\})\in q_\lambda$ such that $g|_{K_\lambda}=f$. If such $g(k_\lambda)$ is unique whenever $(a_i;i\in K_\lambda)\in A^{K_\lambda}$, then the universal relational system $\mathbf{A}$ is nothing but a *universal algebra* (cf. [3]). The power of hyperalgebras was studied in [4] and that of algebras in [3]. A combined power $\mathbf{A}^{\mathbf{B}}$ of universal algebras (hyperalgebras) is a direct power if $\mathbf{A}$ is both diagonal and commutative and $\mathbf{B}$ is idempotent. Of course, if $\mathbf{A},\mathbf{B}$ are universal algebras (hyperalgebras), then the power $\mathbf{A}^{\mathbf{B}}$ need not be a universal algebra (hyperalgebra). Therefore, it is an open problem to find conditions under which the three exponential laws or at least their weak forms are satisfied for universal algebras (hyperalgebras).

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