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Generalized Volterra operators on polynomially generated Banach spaces

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Abstract. We study boundedness of generalized Volterra operators acting on certain Banach spaces of analytic functions generated by the polynomials on the open unit disc. The operators under study map into the weighted Banach spaces of analytic functions or Bloch type spaces. We also give some related results for the boundedness of continuous operators with respect to the topology of uniform convergence on compact subsets of the open unit disc.

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1. Introduction

Let $\mathbb{D}$ denote the open unit disc of the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ denote the space of all complex-valued analytic functions on $\mathbb{D}$. For an analytic selfmap $\varphi : \mathbb{D} \to \mathbb{D}$ and $u \in H(\mathbb{D})$, the *weighted composition operator* uC_φ is given by $(uC_\varphi f)(z) = u(z)f(\varphi(z))$ for all $f \in H(\mathbb{D})$ and $z \in \mathbb{D}$. In the special case of $u = 1$, we get the *composition operator* C_φ given by $C_\varphi f = f \circ \varphi$ for all $f \in H(\mathbb{D})$. Weighted composition operators appear in the study of dynamical systems. It is also known that isometries on many analytic function spaces are of the canonical forms of weighted composition operators. For more information about these operators the reader is referred to the monographs [5, 12] and references therein.

For $g \in H(\mathbb{D})$, the *Volterra operator* V_g is defined by

$$(V_g f)(z) = \int_0^z f(\zeta)g'(\zeta)d\zeta,$$

for all $f \in H(\mathbb{D})$ and $z \in \mathbb{D}$. Volterra operators were first considered by Pommerenke in [11] and, as it is mentioned in [1], interest in the study of these types of operators arose originally from studying semigroups of analytic composition

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operators (see also [13]). As a generalization of the Volterra operator V_g, for an analytic selfmap φ of $\mathbb{D}$, the product of composition operator C_φ and Volterra operator V_g, called *generalized Volterra operator*, is given by

$$(V_g^\varphi f)(z) = \int_0^{\varphi(z)} f(\xi)g'(\xi)d\xi,$$

for all $f \in H(\mathbb{D})$ and $z \in \mathbb{D}$ (see [8, 9]). Indeed, we have the decomposition

$$V_g^\varphi = C_\varphi \circ V_g,$$

and by letting $\varphi(z) = z$ in the generalized Volterra operator V_g^φ, we get the classic Volterra operator V_g.

Generalized Volterra operators and other similar integral type operators between different spaces of analytic functions have been intensively studied by many authors. See, for example, [1, 8, 9, 11, 13] and references therein. In this paper we study boundedness of generalized Volterra operator V_g^φ between certain subspaces of $H(\mathbb{D})$ defined as follows.

By a *weight* v we mean a strictly positive continuous function on $\mathbb{D}$ which is *radial*, that is $v(z) = v(|z|)$ for every $z \in \mathbb{D}$. Moreover, we assume that the weight v is decreasing with respect to $|z|$ and tends to zero at the boundary of $\mathbb{D}$, that is $\lim_{|z|\to 1} v(z) = 0$. A weight v is called *normal* if it satisfies properties (L1) and (L2)

$$\text{(L1)} \qquad \inf_k \frac{v(1-2^{-k-1})}{v(1-2^{-k})} > 0,$$

$$\text{(L2)} \qquad \limsup_n \frac{v(1-2^{-n-k})}{v(1-2^{-n})} < 1, \quad \text{for some } k \in \mathbb{N},$$

see [6, Lemma 1]. Note that for each $0 < \alpha < \infty$, the *standard weights* $v_\alpha(z) = (1-|z|^2)^\alpha$ are normal weights. For more information about normal weights, see [2, 3, 6] and references therein.

For a weight v, the *weighted Banach space of analytic functions* H_v^∞ is defined as

$$H_v^\infty = \left\{ f \in H(\mathbb{D}) : \sup_{z\in\mathbb{D}} v(z)|f(z)| < \infty \right\}.$$

The space H_v^∞ is a Banach space equipped with the norm

$$\|f\|_{H_v^\infty} = \sup_{z\in\mathbb{D}} v(z)|f(z)|, \quad (f \in H_v^\infty).$$

The *little* version of H_v^∞, denoted by H_v^0, is defined as

$$H_v^0 = \left\{ f \in H_v^\infty : \lim_{|z|\to 1^-} v(z)|f(z)| = 0 \right\}.$$

Indeed, the space H_v^0 is a closed subspace of H_v^∞.

For a weight v, the *Bloch type space* $\mathcal{B}_v^\infty$ is defined as

$$\mathcal{B}_v^\infty = \left\{ f \in H(\mathbb{D}) : \sup_{z\in\mathbb{D}} v(z)|f'(z)| < \infty \right\}.$$

The space $\mathcal{B}_v^\infty$ is a Banach space with the norm

$$\|f\|_{\mathcal{B}_v^\infty} = |f(0)| + \sup_{z\in\mathbb{D}} v(z)|f'(z)|, \quad (f \in \mathcal{B}_v^\infty).$$

The *little Bloch type space* $\mathcal{B}_v^0$, which is a closed subspace of $\mathcal{B}_v^\infty$, is defined as

$$\mathcal{B}_v^0 = \left\{ f \in \mathcal{B}_v^\infty : \lim_{|z|\to 1^-} v(z)|f'(z)| = 0 \right\}.$$

If v is a normal weight, then by using the weight

$$w(z) = (1 - |z|)v(z), \quad (z \in \mathbb{D}),$$

we can identify $H_v^\infty = \mathcal{B}_w^\infty$, which is briefly written as $H_v^\infty = \mathcal{B}_{(1-r)v}^\infty$. Similarly, for a normal weight v, we have the identification $H_v^0 = \mathcal{B}_{(1-r)v}^0$ (see [7, 10]).

Recall that for the Banach spaces X and Y, a linear operator $T : X \to Y$ is *bounded* if it takes each bounded set in X to a bounded set in Y. The space of all bounded operators $T : X \to Y$ is denoted by $\mathcal{B}(X, Y)$ and the *operator norm* of $T \in \mathcal{B}(X, Y)$ is denoted by $\|T\|_{X\to Y}$. There is a growing interest in characterizing boundedness of the integral type operators, like V_g^φ, in terms of their inducing functions. See, for example, [8, 9] and references therein for such results. In this paper, we investigate boundedness of the generalized Volterra operators V_g^φ on certain Banach spaces of analytic functions X mapping into the spaces H_v^∞, H_v^0, $\mathcal{B}_v^\infty$ or $\mathcal{B}_v^0$. The Banach spaces X that we consider are described as follows.

Let $(X, \|\cdot\|_X)$ be a Banach space of analytic functions on $\mathbb{D}$ containing the constant functions. Among several conditions described in [4, 7, 8] we consider the following conditions:

(I) The closed unit ball of X is compact with respect to the topology of uniform convergence on compact subsets of $\mathbb{D}$.

(II) For each $0 < r < 1$, the operator $T_r : X \to X$; $f \mapsto f_r$, is well-defined and

$$\sup_{0<r<1} \|T_r\|_{X\to X} < \infty,$$

where $f_r(z) = f(rz)$ for all $z \in \mathbb{D}$.

Note that Hardy spaces H^p and Bergman spaces A_α^p, for $1 \le p < \infty$ and $-1 < \alpha < \infty$, satisfy the assumptions (I) and (II). The spaces H_v^∞ and $\mathcal{B}_v^\infty$ satisfy condition (I) for any weight v. Also, for certain normal weights v, the spaces H_v^∞ satisfy condition (II). For more information about spaces satisfying conditions (I) and (II) see [4, 7, 8] and references therein.

Remark 1.1. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces of analytic functions on $\mathbb{D}$.

(i) If X satisfies condition (I), then norm convergence in X implies uniform convergence on compact subsets of $\mathbb{D}$. Consequently, norm convergence in X implies pointwise convergence on $\mathbb{D}$.

(ii) Assume that norm convergence in X implies uniform convergence on compact subsets of $\mathbb{D}$, and norm convergence in Y implies pointwise convergence on $\mathbb{D}$. If $T : X \to Y$ is a continuous operator with respect to the topology of uniform convergence on compact subsets of $\mathbb{D}$, then the closed graph theorem implies that $T : X \to Y$ is a bounded operator.

2. Main Results

The set of all polynomials $\mathcal{P}$ are contained in many Banach spaces of analytic functions $(X, \|\cdot\|_X)$. Therefore, one may consider $\overline{\mathcal{P}}^X$ as a Banach space contained in such spaces $(X, \|\cdot\|_X)$. In this section we investigate boundedness properties of operators on $\overline{\mathcal{P}}^X$. Besides giving some results for the general classes of operators T, we specially give some results for the generalized Volterra operators V_g^φ.

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces of analytic functions such that X contains $\mathcal{P}$. Clearly, each bounded operator $T : X \to Y$ induces a bounded operator $T : \overline{\mathcal{P}}^X \to Y$ by reducing its domain to the subspace $\overline{\mathcal{P}}^X$. In the next theorem we show that the converse is also valid under some general assumptions on the operator T and Banach spaces X and Y. Before stating the next theorem, we recall that for the real scalars R_1 and R_2, the notation $R_1 \lesssim R_2$ means $R_1 \leq cR_2$ for some positive constant c not depending on the variables in R_1 and R_2. Also, the notation $R_1 \asymp R_2$ means $R_1 \lesssim R_2$ and $R_2 \lesssim R_1$.

Theorem 2.1. *Let $X \subseteq H(\mathbb{D})$ be a Banach space containing the disc algebra A and satisfying conditions (I) and (II). Let $Y \subseteq H(\mathbb{D})$ be a Banach space satisfying condition (I). If $T : H(\mathbb{D}) \to H(\mathbb{D})$ is a continuous operator with respect to the topology of uniform convergence on compact subsets of $\mathbb{D}$, then the following statements are equivalent:*

(i) $T : X \to Y$ is bounded,

(ii) $T : \overline{\mathcal{P}}^X \to Y$ is bounded.

Moreover,

$$\|T\|_{X \to Y} \asymp \|T\|_{\overline{\mathcal{P}}^X \to Y}.$$

Proof. Clearly (i) implies (ii). Let $T : H(\mathbb{D}) \to H(\mathbb{D})$ be a continuous operator with respect to the topology of uniform convergence on compact subsets of $\mathbb{D}$. Let $T : \overline{\mathcal{P}}^X \to Y$ be a bounded operator. We next prove that the operator

$T : X \to Y$ is well-defined and therefore, by Remark 1.1, boundedness of $T : X \to Y$ will follow.

Let $f \in X$ and choose a sequence $(r_n) \subseteq (0,1)$ such that $r_n \to 1$. Then, for each $n \in \mathbb{N}$, $f_{r_n} \in A = \overline{\mathcal{P}}^{\|\cdot\|_\infty}$. Since X contains the disc algebra A and satisfies condition (I), the closed graph theorem and Remark 1.1 imply that the (well-defined) identity operator $id : A \to X$ is bounded. Therefore, for each $n \in \mathbb{N}$, we have $f_{r_n} \in \overline{\mathcal{P}}^X$ and hence $(Tf_{r_n}) \subseteq Y$. Note that (Tf_{r_n}) is a bounded sequence in Y, since by condition (II) on X, for each $n \in \mathbb{N}$ we have

$$\begin{aligned}\|Tf_{r_n}\|_Y &= \|T(T_{r_n} f)\|_Y \\ &\leq \|T\|_{\overline{\mathcal{P}}^X \to Y} \|T_{r_n}\|_{X \to X} \|f\|_X \\ &\leq \|T\|_{\overline{\mathcal{P}}^X \to Y} \sup_{0<r<1} \|T_r\|_{X \to X} \|f\|_X.\end{aligned}$$

This implies that the sequence (Tf_{r_n}) belongs to the closed ball $\overline{B}_Y(0,R)$, where

$$R = \|T\|_{\overline{\mathcal{P}}^X \to Y} \sup_{0<r<1} \|T_r\|_{X \to X} \|f\|_X.$$

Thus, since Y satisfies condition (I), there exist $g \in \overline{B}_Y(0,R)$ and a subsequence $(Tf_{r_{n_k}})$ such that $(Tf_{r_{n_k}})$ converges to g uniformly on compact subsets of $\mathbb{D}$. On the other hand, since (f_{r_n}) converges to f uniformly on compact subsets of $\mathbb{D}$, continuity of $T : H(\mathbb{D}) \to H(\mathbb{D})$ implies that (Tf_{r_n}) converges to Tf uniformly on compact subsets of $\mathbb{D}$. Consequently, $Tf = g \in Y$ meaning that $T : X \to Y$ is well-defined, which is the desired result.

In order to prove the norm estimate, first note that by the definition of operator norm, clearly we have $\|T\|_{\overline{\mathcal{P}}^X \to Y} \leq \|T\|_{X \to Y}$. On the other hand, for each $f \in X$, by considering the above mentioned argument, we have $g \in \overline{B}_Y(0,R)$ and therefore

$$\|Tf\|_Y = \|g\|_Y \leq R = \|T\|_{\overline{\mathcal{P}}^X \to Y} \sup_{0<r<1} \|T_r\|_{X \to X} \|f\|_X. \tag{2.1}$$

This implies that $\|T\|_{X \to Y} \lesssim \|T\|_{\overline{\mathcal{P}}^X \to Y}$ and completes the proof. □

Remark 2.2. By estimate (2.1) in the proof of Theorem 2.1, it follows that if the space X, instead of condition (II), satisfies the stronger condition

$$\sup_{0<r<1} \|T_r\|_{X \to X} \leq 1, \tag{2.2}$$

then

$$\|T\|_{X \to Y} = \|T\|_{\overline{\mathcal{P}}^X \to Y}.$$

Note that, condition (2.2) is satisfied for the spaces $X = H^\infty$ and $X = H_v^\infty$. It is also worth mentioning that Theorem 2.1 can be applied to the following special cases:

(i) $X = H^\infty$ and $\overline{\mathcal{P}}^X = A$.

(ii) $X = H_v^\infty$ and $\overline{\mathcal{P}}^X = H_v^0$.

(iii) $X = \mathcal{B}_v^\infty$ and $\overline{\mathcal{P}}^X = \mathcal{B}_v^0$.

Indeed, by applying Theorem 2.1 and Remark 2.2, we get the following corollary.

Corollary 2.3. *Let $Y \subseteq H(\mathbb{D})$ be a Banach space satisfying condition (I) and $T : H(\mathbb{D}) \to H(\mathbb{D})$ be a continuous operator with respect to the topology of uniform convergence on compact subsets of $\mathbb{D}$. Then,*

(i) $T : H^\infty \to Y$ is bounded if and only if $T : A \to Y$ is bounded. Moreover,

$$\|T\|_{H^\infty \to Y} = \|T\|_{A \to Y}.$$

(ii) $T : H_v^\infty \to Y$ is bounded if and only if $T : H_v^0 \to Y$ is bounded. Moreover,

$$\|T\|_{H_v^\infty \to Y} = \|T\|_{H_v^0 \to Y}.$$

(iii) $T : \mathcal{B}_v^\infty \to Y$ is bounded if and only if $T : \mathcal{B}_v^0 \to Y$ is bounded. Moreover,

$$\|T\|_{\mathcal{B}_v^\infty \to Y} = \|T\|_{\mathcal{B}_v^0 \to Y}.$$

In order to apply Theorem 2.1 and Remark 2.2 for the generalized Volterra operators $V_g^\varphi : H(\mathbb{D}) \to H(\mathbb{D})$, we next show that such operators are continuous with respect to the topology of uniform convergence on compact subsets of $\mathbb{D}$.

Example 2.4. Let $g \in H(\mathbb{D})$ and φ be an analytic selfmap of $\mathbb{D}$. Then, the generalized Volterra operator $V_g^\varphi : H(\mathbb{D}) \to H(\mathbb{D})$ is continuous with respect to the topology of uniform convergence on compact subsets of $\mathbb{D}$. In order to see this, let (f_n) be a sequence in $H(\mathbb{D})$ converging to 0 uniformly on compact subsets of $\mathbb{D}$. Let K be an arbitrary compact subset of $\mathbb{D}$. Then, one can find $0 < r_K < 1$ such that

$$\varphi(K) \subseteq \overline{D}(0, r_K) = \{z \in \mathbb{D} : |z| \le r_K\},$$

and consequently

$$\begin{aligned}\sup_{z \in K} |V_g^\varphi(f_n)(z)| &= \sup_{z \in K} \left| \int_0^{\varphi(z)} f_n(\xi) g'(\xi) d\xi \right| \\ &\le r_K \sup_{\omega \in \overline{D}(0, r_K)} |f_n(\omega)| \sup_{\omega \in \overline{D}(0, r_K)} |g'(\omega)|.\end{aligned}$$

Therefore, since (f_n) converges to 0 uniformly on $\overline{D}(0, r_K)$, we get the desired result, that is

$$\sup_{z \in K} |V_g^\varphi(f_n)(z)| \underset{n \to \infty}{\longrightarrow} 0.$$

As an immediate consequence of Example 2.4, we get the following corollaries for the generalized Volterra operators.

Corollary 2.5. *Let $g \in H(\mathbb{D})$ and φ be an analytic selfmap of $\mathbb{D}$. Let $X \subseteq H(\mathbb{D})$ be a Banach space containing the disc algebra A and satisfying conditions (I) and (II). Let $Y \subseteq H(\mathbb{D})$ be a Banach space satisfying condition (I). Then, the following statements are equivalent:*

(i) $V_g^\varphi : X \to Y$ *is bounded,*

(ii) $V_g^\varphi : \overline{\mathcal{P}}^X \to Y$ *is bounded.*

Moreover,

$$\|V_g^\varphi\|_{X\to Y} \asymp \|V_g^\varphi\|_{\overline{\mathcal{P}}^X \to Y}.$$

Corollary 2.6. *Let $g \in H(\mathbb{D})$ and φ be an analytic selfmap of $\mathbb{D}$. Let $Y \subseteq H(\mathbb{D})$ be a Banach space satisfying condition (I). Then,*

(i) $V_g^\varphi : H^\infty \to Y$ *is bounded if and only if* $V_g^\varphi : A \to Y$ *is bounded. Moreover,*

$$\|V_g^\varphi\|_{H^\infty \to Y} = \|V_g^\varphi\|_{A\to Y}.$$

(ii) $V_g^\varphi : H_v^\infty \to Y$ *is bounded if and only if* $V_g^\varphi : H_v^0 \to Y$ *is bounded. Moreover,*

$$\|V_g^\varphi\|_{H_v^\infty \to Y} = \|V_g^\varphi\|_{H_v^0\to Y}.$$

(iii) $V_g^\varphi : \mathcal{B}_v^\infty \to Y$ *is bounded if and only if* $V_g^\varphi : \mathcal{B}_v^0 \to Y$ *is bounded. Moreover,*

$$\|V_g^\varphi\|_{\mathcal{B}_v^\infty \to Y} = \|V_g^\varphi\|_{\mathcal{B}_v^0\to Y}.$$

We next give our results for the boundedness of generalized Volterra operators V_g^φ on the polynomially generated spaces $\overline{\mathcal{P}}^X$.

Lemma 2.7. *Let $g \in H(\mathbb{D})$ and φ be an analytic selfmap of $\mathbb{D}$. If v is a normal weight and $g\circ\varphi \in H_v^0$, then $V_g^\varphi(f_r) \in H_v^0$ for each $f \in H(\mathbb{D})$ and $0 < r < 1$.*

Proof. Since v is a normal weight, we have $H_v^0 = \mathcal{B}^0_{(1-r)v}$ (see [7, 10]). Thus, it suffices to prove that $V_g^\varphi(f_r) \in \mathcal{B}^0_{(1-r)v}$ or, equivalently, $(V_g^\varphi(f_r))' \in H^0_{(1-r)v}$. By the assumption we have $g\circ\varphi \in H_v^0 = \mathcal{B}^0_{(1-r)v}$ or, equivalently, $(g\circ\varphi)' \in H^0_{(1-r)v}$. On the other hand, for each $f \in H(\mathbb{D})$ and $0 < r < 1$, we have $f_r \circ \varphi \in H^\infty$. Consequently,

$$(V_g^\varphi(f_r))' = (f_r\circ\varphi)(g\circ\varphi)' \in H^0_{(1-r)v},$$

which is the desired result. □

Theorem 2.8. *Let $g \in H(\mathbb{D})$ and φ be an analytic selfmap of $\mathbb{D}$. Let $X \subseteq H(\mathbb{D})$ be a Banach space satisfying condition (I) and containing the polynomials $\mathcal{P}$. If v is a normal weight, then the following statements are equivalent:*

(i) $V_g^\varphi : \overline{\mathcal{P}}^X \to H_v^\infty$ *is bounded and* $g\circ\varphi \in H_v^0$,

(ii) $V_g^\varphi : \overline{\mathcal{P}}^X \to H_v^0$ *is bounded.*

Proof. First note that (ii) implies (i) since X contains the constant function 1 and also $g \circ \varphi = V_g^\varphi(1) + g(0)$. In order to show that (i) implies (ii), by Remark 1.1 and Example 2.4, it is enough to show that $V_g^\varphi : \overline{\mathcal{P}}^X \to H_v^0$ is well-defined. Let $f \in \overline{\mathcal{P}}^X$ and (p_n) be a sequence of polynomials such that $p_n \to f$ in X. Then, boundedness of $V_g^\varphi : \overline{\mathcal{P}}^X \to H_v^\infty$ implies that $V_g^\varphi(p_n) \to V_g^\varphi(f)$ in H_v^∞. Therefore, to prove that $V_g^\varphi(f) \in H_v^0$ it is enough to show that $V_g^\varphi(p) \in H_v^0$ for each polynomial $p \in \mathcal{P}$.

Choose an arbitrarily $p \in \mathcal{P}$ and define the polynomial $h(z) = p(2z)$ for all $z \in \mathbb{D}$. Then, $h \in \mathcal{P}$ and $p = h_{\frac{1}{2}}$. Hence, by Lemma 2.7, we have

$$V_g^\varphi(p) = V_g^\varphi(h_{\frac{1}{2}}) \in H_v^0,$$

which is the desired result. □

By applying Theorem 2.8 for the special case of $\varphi(z) = z$, we get the related result for the classic Volterra operator V_g. Indeed, using the next lemma, we prove that the result of Theorem 2.8 for the special case of Volterra operator V_g, is valid even if we remove the normality assumption of the weight v.

Lemma 2.9. *Let $g \in H(\mathbb{D})$ and v be an arbitrary weight. If $g \in H_v^0$ then $V_g(f_r) \in H_v^0$ for each $f \in H(\mathbb{D})$ and $0 < r < 1$.*

Proof. By adding and subtracting the term $f_r'(\omega)g(\omega)$ and defining $h(z) = \int_0^z f_r'(\omega)g(\omega)d\omega$ we get

$$\begin{aligned} V_g(f_r)(z) &= \int_0^z f_r(\omega)g'(\omega)d\omega \\ &= \int_0^z (f_r(\omega)g(\omega))'d\omega - \int_0^z f_r'(\omega)g(\omega)d\omega \\ &= f_r(z)g(z) - f_r(0)g(0) - h(z). \end{aligned} \tag{2.3}$$

Note that since $f_r \in H^\infty$ and $g \in H_v^0$, we have $f_r g \in H_v^0$. Therefore, in order to prove that $V_g(f_r) \in H_v^0$, by (2.3), it is enough to prove that $h \in H_v^0$.

Let $\varepsilon > 0$. Since $g \in H_v^0$, one can choose $0 < r_1 < 1$ such that

$$v(z)|g(z)| < \frac{\varepsilon}{2\|f'\|_{\overline{r\mathbb{D}}}}, \tag{2.4}$$

for every $z \in \mathbb{D}$ with $r_1 < |z| < 1$. Moreover, since $\lim_{|z|\to 1^-} v(z) = 0$, there exists $r_1 < r_2 < 1$ such that

$$v(z) < \frac{\varepsilon}{2\|f'\|_{\overline{r\mathbb{D}}}\|g\|_{\overline{\sqrt{r_1}\mathbb{D}}}}, \tag{2.5}$$

whenever $r_2 < |z| < 1$. Now, if $\sqrt{r_2} < |z| < 1$ then by (2.4) and (2.5) we have

$$\begin{aligned} v(z)|h(z)| &= v(z)\left|\int_0^z f_r'(\omega)g(\omega)d\omega\right| = v(z)\left|\int_0^1 rf'(rtz)g(tz)zdt\right| \\ &\le v(z)\|f'\|_{\overline{r\mathbb{D}}}\int_0^1 |g(tz)|dt \\ &= v(z)\|f'\|_{\overline{r\mathbb{D}}}\int_0^{\sqrt{r_1}} |g(tz)|dt + v(z)\|f'\|_{\overline{r\mathbb{D}}}\int_{\sqrt{r_1}}^1 |g(tz)|dt \\ &\le v(z)\|f'\|_{\overline{r\mathbb{D}}}\|g\|_{\overline{\sqrt{r_1}\mathbb{D}}} + \|f'\|_{\overline{r\mathbb{D}}}\int_{\sqrt{r_1}}^1 v(tz)|g(tz)|dt < \varepsilon, \end{aligned}$$

which completes the proof. □

Theorem 2.10. *Let $g \in H(\mathbb{D})$ and $X \subseteq H(\mathbb{D})$ be a Banach space satisfying condition (I) and containing the polynomials $\mathcal{P}$. Then, for each weight v the following statements are equivalent:*

(i) $V_g : \overline{\mathcal{P}}^X \to H_v^\infty$ is bounded and $g \in H_v^0$,

(ii) $V_g : \overline{\mathcal{P}}^X \to H_v^0$ is bounded.

Proof. Clearly, (ii) implies (i) since X contains the constant function 1 and $g = V_g(1) + g(0)$. By applying Lemma 2.9 and using a similar approach as in the proof of Theorem 2.8, one can also see that (i) implies (ii). □

We finally prove the result of Theorem 2.8 in the case of Bloch type spaces.

Lemma 2.11. *Let $g \in H(\mathbb{D})$ and φ be an analytic selfmap of $\mathbb{D}$. If v is an arbitrary weight and $g \circ \varphi \in \mathcal{B}_v^0$, then $V_g^\varphi(f_r) \in \mathcal{B}_v^0$ for each $f \in H(\mathbb{D})$ and $0 < r < 1$.*

Proof. Note that

$$\begin{aligned} v(z)|(V_g^\varphi(f_r))'(z)| &= v(z)|f_r(\varphi(z))g'(\varphi(z))\varphi'(z)| \\ &\le \|f\|_{\overline{r\mathbb{D}}}\, v(z)|(g\circ\varphi)'(z)|, \end{aligned}$$

for every $z \in \mathbb{D}$. Therefore, the fact that $f_r \in H^\infty$ and $g \circ \varphi \in \mathcal{B}_v^0$ implies $V_g^\varphi(f_r) \in \mathcal{B}_v^0$. □

Theorem 2.12. *Let $g \in H(\mathbb{D})$ and φ be an analytic selfmap of $\mathbb{D}$. Let $X \subseteq H(\mathbb{D})$ be a Banach space satisfying condition (I) and containing the polynomials $\mathcal{P}$. Then, for each weight v the following statements are equivalent:*

(i) $V_g^\varphi : \overline{\mathcal{P}}^X \to \mathcal{B}_v^\infty$ is bounded and $g \circ \varphi \in \mathcal{B}_v^0$,

(ii) $V_g^\varphi : \overline{\mathcal{P}}^X \to \mathcal{B}_v^0$ is bounded.

Proof. The proof follows by applying a similar argument as in the proof of Theorem 2.8 and using Remark 1.1, Example 2.4 and Lemma 2.11. □

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