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A homological approach to $\oplus$-supplemented modules

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Abstract. $\oplus$-supplemented modules as a famous generalization of lifting (projective supplemented) modules were widely studied in the last decades. In this paper, we peruse a homological approach to $\oplus$-supplemented modules. Let R be a ring, M a right R-module and $S = End_R(M)$. We say that M is endomorphism $\oplus$-supplemented (briefly, E-$\oplus$-supplemented) provided that for every $f \in S$, there exists a direct summand D of M such that $Imf + D = M$ and $Imf \cap D \ll D$. We investigate some general properties of E-$\oplus$-supplemented modules and try to consider their relation with some known classes of modules such as dual Rickart modules, H-supplemented modules and $\oplus$-supplemented modules.

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1. Introduction

Recently, after the introduction of dual Rickart modules in [10], generalizations of dual Rickart modules seem to be interesting for researchers in Ring and Module Theory. In particular, making a connection between the ring of endomorphisms of a module M and the concepts of lifting modules, H-supplemented modules and others may help us describe their structures better. Let M be a module. Then M is called *dual Rickart*, if the image in M of any single element of S is generated by an idempotent of S, equivalently, for any $f \in S$, Imf is a direct summand of M. In [1], the author studied a new generalization of both lifting and dual Rickart modules namely $\mathcal{I}$-lifting modules. A module M is called *$\mathcal{I}$-lifting* provided that for every nonzero endomorphism f of M, there exists a direct summand D of M such that Imf/D is small in M/D (recall that a submodule N of a module M is small in M, denoted by $N \ll M$ in case $N + K = M$ implies $K = M$). In [1], some properties of $\mathcal{I}$-lifting modules were investigated.

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In [12], the authors introduced a new proper generalization of both $\mathcal{I}$-lifting modules and H-supplemented modules via homomorphisms which they called E-H-supplemented modules. A module M is called E-H-supplemented provided that for every $f \in End_R(M)$, there is a direct summand D of M such that $Imf + X = M$ if and only if $D + X = M$ for every $X \leq M$. E-H-supplemented modules from various points of view were studied in [12]. The authors investigated the relation of E-H-supplemented modules with the famous classes of modules such as dual Rickart modules, $\mathcal{I}$-lifting modules and H-supplemented modules.

Inspired by [1] and [12], it is natural to define $\oplus$-supplemented modules using homomorphisms. So in this work we call a module M *endomorphism $\oplus$-supplemented* in the case when for every nonzero endomorphism f of M, there is a direct summand D of M such that $M = Imf + D$ and $Imf \cap D \ll D$. In Section 2, we investigate some properties of endomorphism $\oplus$-supplemented modules. We observe that endomorphism $\oplus$-supplemented modules generalize the dual Rickart modules. This relation makes the endomorphism $\oplus$-supplemented property more impressive. We also present conditions under which these two concepts coincide.

In what follows, $J(R)$ denotes the Jacobson radical of a ring R and $Rad(M)$ stands for the radical of a module M. Also, S denotes the endomorphism ring $\mathrm{End}_R(M)$ of an R-module M. For any unexplained terminologies we refer to [2, 11, 17].

2. E-$\oplus$-supplemented modules

We shall list some basic definitions which we use freely throughout the paper.

Definition 2.1. Let M be a module. Then M is called:

(1) lifting in the case when for every submodule N of M there is a direct summand D of M contained in N such that $N/D \ll M/D$.

(2) H-supplemented provided that for every submodule N of M there exists a direct summand D of M such that $M = N + X$ if and only if $M = D + X$ for every submodule X of M.

(3) $\oplus$-supplemented if for every submodule N of M there exists a direct summand K of M such that $M = N + K$ and $N \cap K \ll K$.

(4) dual Rickart in the case when for every endomorphism f of M, Imf is a direct summand of M.

(5) $\mathcal{I}$-lifting if the image of any endomorphism f of M contains a direct summand D of M such that $Imf/D \ll M/D$.

(6) E-H-supplemented provided that for every $\varphi \in End_R(M)$, there exists a direct summand D of M such that $M = Im\varphi + X$ if and only if $M = D + X$ for every submodule X of M.

By the definitions we have the following hierarchies:

$$\begin{array}{ccccc} lifting & \Rightarrow & H-supplemented & \Rightarrow & \oplus - supplemented \\ \Downarrow & & \Downarrow & & \\ \mathcal{I}-lifting & \Rightarrow & E-H-supplemented & & \end{array}$$

We start this section by introducing a new class of modules which is a proper generalization of ⊕-supplemented modules.

Definition 2.2. A module M is called *endomorphism ⊕-supplemented* (E-⊕-*supplemented*, for short) in the case when for every $f \in S$, there exists a direct summand D of M such that $Imf + D = M$ and $Imf \cap D \ll D$.

First of all, we prefer to emphasize that the class of E-⊕-supplemented modules contains properly the class of (⊕-)supplemented modules.

Example 2.3. (1) It is obvious that every dual Rickart module is E-⊕-supplemented. So, every injective module over a right hereditary ring is E-⊕-supplemented by [10, Theorem 2.29]. Consider the $\mathbb{Z}$-module $M = \mathbb{Q}^{(I)}$ where I is an arbitrary index set. Since M is injective, M is E-⊕-supplemented. Also, it is well-known that $\mathbb{Q}$ is not (⊕-)supplemented, hence M is not ⊕-supplemented. Generally, every non-supplemented injective module over a right hereditary ring is E-⊕-supplemented but not ⊕-supplemented.

(2) Let M be any $\mathbb{Z}$-module and p be an arbitrary prime number. Set $M(p) = \{m \in M \mid \exists n \in \mathbb{N}, p^n m = 0\}$. Zöchinger in [18] proved that M is supplemented if and only if M is a torsion $\mathbb{Z}$-module and for any prime number p, the $\mathbb{Z}$-module $M(p)$ is a direct sum of an Artinian module and a module with bounded order. If M is an infinite direct sum of copies of the Prüfer p-group $\mathbb{Z}_{p^\infty}$, then M is not supplemented while M is a direct sum of supplemented modules. Note that M is injective and M is dual Rickart by [10, Theorem 2.29]. Hence M is E-⊕-supplemented.

The following provides a rich source of E-⊕-supplemented modules.

Proposition 2.4. *Every E-H-supplemented module is E-⊕-supplemented.*

Proof. Let M be an E-H-supplemented module and $f \in S$. Then there is a decomposition $M = K \oplus K'$ such that $Imf + X = M$ if and only if $K + X = M$ for every $X \leq M$. Now, $Imf + K' = M$. We shall verify that $Imf \cap K' \ll K'$. To prove the last assertion, suppose that $(Imf \cap K') + L = K'$ for some submodule of K'. Then $Imf + L = Imf + K' = M$. M being an E-H-supplemented module implies $K + L = M$. Hence, by the modular law we conclude that $L = K'$, as required. □

Recall from [3] that a module M is called *epi-retractable* in the case when every submodule of M is a homomorphic image of M. It is not hard to check that for an epi-retractable module, the two concepts, H-supplemented and E-H-supplemented, coincide. The same assertion holds for the concepts of ⊕-supplemented and E-⊕-supplemented. Similarly, an epi-retractable module M is lifting if and only if M is $\mathcal{I}$-lifting.

We show that the class of E-⊕-supplemented modules contains properly the class of $\mathcal{I}$-lifting modules.

Example 2.5. (1) Let p be a prime number. Consider the $\mathbb{Z}$-module $M_1 = \mathbb{Z}_{p^3}$. Then by [7, Example 4.6], the $\mathbb{Z}$-module $M = M_1 \oplus \frac{M_1}{(p)} \oplus \frac{(p)}{(p^2)} \oplus \frac{(p^2)}{(0)}$ is H-supplemented, so that M is E-H-supplemented. Since $\mathbb{Z}_p \oplus \mathbb{Z}_{p^3}$ is isomorphic to a direct summand of M, M is not lifting from [6, Corollary 2]. M being a finitely generated $\mathbb{Z}$-module implies that M is epi-retractable by [3, Example 2.4]. Hence M is not $\mathcal{I}$-lifting. On the other hand, M is E-$\oplus$-supplemented as well as E-H-supplemented (Proposition 2.4).

(2) (see [14, Example 2.3]) Let I and J be two ideals of a commutative local ring R with maximal ideal m such that $I \subset J \subseteq m$ and $mJ \nsubseteq I$ (e.g., R is a DVR with maximal ideal m, $I = m^4$ and $J = m^2$). We consider the module $M = R/I \times R/J$. From [14, Proposition 2.1] it follows that M is H-supplemented and so M is E-$\oplus$-supplemented, as M is E-H-supplemented. In other words, from [14, Example 2.3], M is not lifting. M being an epi-retractable module implies M is not $\mathcal{I}$-lifting.

The converse of Proposition 2.4 does not hold in general.

Example 2.6. Let R be a discrete valuation ring and let $I_1, \dots, I_n$ be some ideals of R. Consider the R-module $M \cong R/I_1 \times \cdots \times R/I_n$. By [11, Lemma A.4], M is $\oplus$-supplemented and hence M is E-$\oplus$-supplemented. If $I_1 \subseteq \cdots \subseteq I_n \subset R$, then M is H-supplemented by [14, Proposition 2.1]. Otherwise, i.e., if the condition $I_1 \subseteq \cdots \subseteq I_n \subset R$ does not hold, M is not H-supplemented. Note also that M is an epi-retractable R-module by [3, Example 2.4(3)]. It means that in this case M is not E-H-supplemented.

We provide an assumption under which the $\oplus$-supplemented and E-$\oplus$-supplemented properties coincide.

Recall from [12, Definition 2.19] that a module M is *s-retractable* in the case when for every submodule N of M, there exists a nonzero homomorphism $f : M \to N$ such that $N/Imf \ll M/Imf$. By definition, every s-retractable module is retractable. In other words, every retractable hollow module is s-retractable. In particular, every local ring R over itself is an s-retractable module.

Proposition 2.7. *In each of the following cases a module M is $\oplus$-supplemented if and only if M is E-$\oplus$-supplemented.*

(1) *M is epi-retractable.*

(2) *M is s-retractable.*

Proof. (1) It is clear by definitions.

(2) The necessity is clear. For the converse, let N be a submodule of M. M being an s-retractable module implies that there is an endomorphism f of M with $Imf \subseteq N$ and $N/Imf \ll M/Imf$. Since M is E-$\oplus$-supplemented, there is a direct summand K of M such that $Imf + K = M$ and $Imf \cap K \ll K$. Now, $N + K = M$. It remains to show that $N \cap K \ll K$. To verify this assertion, suppose that $(N \cap K) + L = K$ for some submodule L of K. Then $N + L = N + K = M$. It follows that $N/Imf + (L + Imf)/Imf = M/Imf$, which implies that $L + Imf = M$ as $N/Imf \ll M/Imf$. By modularity, we

conclude that $L + (Imf \cap K) = K$. Hence $L = K$, since K is a supplement of Imf in M. This completes the proof. □

We next show that for a projective s-retractable module, the concepts of H-supplemented, E-H-supplemented, ⊕-supplemented and E-⊕-supplemented module coincide.

Corollary 2.8. *Let M be an s-retractable module. Consider the following:*
(1) *M is H-supplemented;*
(2) *M is E-H-supplemented;*
(3) *M is E-⊕-supplemented;*
(4) *M is ⊕-supplemented.*
Then (1) ⇔ (2) ⇒ (3) ⇔ (4). *They are equivalent in the case when M is self-projective.*

Proof. (1) ⇔ (2) By [12, Proposition 2.20].
(2) ⇒ (3) It follows from Proposition 2.4.
(3) ⇔ (4) Follows by Proposition 2.7.
(4) ⇒ (1) This follows from the assumption that M is self-projective and [9, Proposition 2.6]. □

Let M be a module. Then by [15], M is called *(non)cosingular* if $(\overline{Z}(M) = M)$ $\overline{Z}(M) = 0$, in which $\overline{Z}(M) = \cap\{Kerf \mid f : M \to L, L \in \mathcal{U}\}$ where $\mathcal{U}$ denotes the class of all small right R-modules. If we consider $M = R_R$, then $\overline{Z}(R_R)$ is a two-sided ideal of R.

Proposition 2.9. *Let R be a commutative ring and M a torsion-free E-⊕-supplemented R-module with $\overline{Z}(M) \neq M$. Then $\overline{Z}(R) \ll R$. In addition, if $J(R) = 0$, then R is a cosingular ring.*

Proof. Let $0 \neq a \in \overline{Z}(R)$. Consider the homomorphism $f : M \to M$ defined by $f(m) = ma$ for every $m \in M$. Then $Imf = Ma$. Now by the assumption there is a direct summand K of M such that $Ma+K = M$ and $Ma\cap K \ll K$. As K is a summand of M, we have $Ma\cap K = Ka \ll K$ (note that if M is noncosingular, then Ma is noncosingular as a homomorphic image of M. Therefore, Ka as a direct summand of Ma must be both cosingular and noncosingular, which implies that $a = 0$). So that $Ka \subseteq Rad(M)$. From [16, Proposition 2.1], $Rad(M)\overline{Z}(R) = 0$. It follows that $Ka^2 = 0$. Since M is torsion-free, we have $a^2 = 0$. Therefore, $a \in J(R)$. This implies that $\overline{Z}(R) \subseteq J(R)$. This completes the proof. □

In [8], the module M is called *$\mathcal{T}$-noncosingular* if for any $f \in S$, Imf is small in M implies $f = 0$. Note that a noncosingular module is clearly $\mathcal{T}$-noncosingular. Recall that a module M satisfies D_2 in the case when $M/N \cong D$ with D a direct summand of M, implies N is a direct summand of M.

Proposition 2.10. *Let R be a commutative ring and M an E-⊕-supplemented module. If M is $\mathcal{T}$-noncosingular, then for each $0 \neq a \in R$, aM is a direct summand of M. If, in addition, M satisfies D_2 then for each $a \in R$, $r_M(a)$ is a direct summand of M.*

Proof. Let $0 \neq a \in R$. Consider the homomorphism $f : M \to M$ with $f(m) = am$. Then there is a direct summand K of M such that $aM + K = M$ and $aM \cap K = aK \ll K$. Now the image of $jo\pi of : M \to M$ is aK, which is a small submodule of M. As M is $\mathcal{T}$-noncosingular, $aK = 0$. Hence, $aM \oplus K = M$. Now, suppose that M satisfies D_2. Since $M/r_M(a) \cong aM$ and aM is a direct summand of M, then $r_M(a)$ is a direct summand of M. □

Proposition 2.11. *Let M be a module with $Rad(M) = 0$. Then the following are equivalent:*

(1) *M is E-H-supplemented;*
(2) *M is E-$\oplus$-supplemented;*
(3) *M is dual Rickart;*
(4) *M is $\mathcal{I}$-lifting.*

Proof. (1) ⇒ (2) It follows from Proposition 2.4.

(2) ⇒ (3) Let M be E-$\oplus$-supplemented and $f \in S$. Then $Imf + K = M$ and $Imf \cap K \ll K$, where K is a direct summand of M. Then $Imf \cap K = 0$, as $Rad(M) = 0$, which shows that M is dual Rickart.

(3) ⇒ (4) and (4) ⇒ (1) are straightforward. □

Let R be a ring. Then R is called a right *V-ring* in the case when every simple right R-module is injective. It is well-known that R is a right V-ring if and only if $Rad(M) = 0$ for every right R-module M. It follows from [15, Proposition 2.5 and Corollary 2.6] that all modules over a right V-ring R are noncosingular.

Corollary 2.12. *The following statements are equivalent for a module M over a V-ring R:*

(1) *M is E-H-supplemented;*
(2) *M is E-$\oplus$-supplemented;*
(3) *M is dual Rickart;*
(4) *M is $\mathcal{I}$-lifting.*

It follows from [8, Corollary 2.7] that a ring R is right (left) $\mathcal{T}$-noncosingular if and only if $J(R) = 0$. Now from this fact and Propositions 2.10 and 2.11 we have the following:

Corollary 2.13. *Let R be a commutative ring with $J(R) = 0$. Then the following conditions are equivalent:*

(1) *R is E-H-supplemented;*
(2) *R is E-$\oplus$-supplemented;*
(3) *R is a von Neumann regular ring.*

Proof. (1) ⇔ (2) Follows from Proposition 2.11.

(1) ⇔ (3) It is proved in [12, Corollary 2.5]. □

Let M be a module. Following [1], M is called *$\mathcal{I}$-supplemented* provided that the image of every endomorphism of M has a supplement in M, i.e. for every $f \in S$, there is a submodule N of M such that $M = Imf + N$ and

$Imf \cap N \ll N$. Also, the module M is called *amply $\mathcal{I}$-supplemented* in the case that whenever $M = Imf + B$, then B contains a supplement of Imf in M. By definitions, every amply $\mathcal{I}$-supplemented module is $\mathcal{I}$-supplemented. It is clear by definitions that every E-⊕-supplemented module is $\mathcal{I}$-supplemented. The following is an analogue of [5, Lemma 1.1] for E-⊕-supplemented (the techniques of the proof are the same as [5, Lemma 1.1]).

Proposition 2.14. *Let R be a commutative prime ring and M a torsion-free injective R-module. Then M is $\mathcal{I}$-supplemented if and only if M is E-⊕-supplemented.*

Proof. Let M be $\mathcal{I}$-supplemented and $f \in S$. Suppose that L is a supplement of Imf in M. Therefore, $Imf + L = M$ and $Imf \cap L \ll L$. Let c be a nonzero element of R. Since M is injective, it will be divisible by [4, Proposition 6.12]. Hence $M = Mc = (Imf)c + Lc$ which is equal to $Imf + Lc$ since $(Imf)c = f(M)c = f(Mc) = f(M) = Imf$. It follows that $M = Imf + Lc$. By the modular law, $L = Imf \cap L + Lc$. As $Imf \cap L \ll L$, we conclude that $L = Lc$. Now, the fact that $L = Lc$ holds for every nonzero element c of R implies that L is a divisible R-module by [4, Theorem 6.4]. Hence by [4, Proposition 6.12], L is an injective R-module and therefore L is a direct summand of M. This completes the proof. The converse is obvious. □

The proof of the following which consider the relation of $\mathcal{I}$-supplemented modules with dual Rickart modules and amply $\mathcal{I}$-supplemented modules follows from [11, Proposition 4.39] and the definitions.

Theorem 2.15. *Let M be a projective module. Consider the following:*

(1) *M is dual Rickart;*

(2) *M is $\mathcal{I}$-supplemented;*

(3) *M is amply $\mathcal{I}$-supplemented.*

Then (1) ⇒ (2) ⇔ (3). *They are equivalent if, for each $f \in S$, Imf is a supplement submodule in M.*

Proof. (1) ⇒ (2) It is clear by definitions.

(2) ⇒ (1) With a similar strategy to the second part of the proof of [11, Proposition 4.39], it can be shown that for every $f \in S$, Imf is a direct summand of M.

(2) ⇒ (3) Follows from the proof of [17, 41.15].

(3) ⇒ (2) By definitions, every amply $\mathcal{I}$-supplemented module is $\mathcal{I}$-supplemented. □

Corollary 2.16. *Let M be a projective module such that Imf is a supplement in M for every $f \in S$. Then the following statements are equivalent:*

(1) *M is dual Rickart;*

(2) *M is $\mathcal{I}$-supplemented;*

(3) *M is E-⊕-supplemented.*

Proposition 2.17. *Let M be an indecomposable module. Then the following are equivalent:*

(1) *M is E-$\oplus$-supplemented;*

(2) *For every $f \in S$, $Imf \ll M$ or f is an epimorphism;*

(3) *M is E-H-supplemented.*

Proof. (1) $\Rightarrow$ (2) Let $f \in S$ be nonzero. By (1), there is a direct summand K of M, such that $M = Imf + K$ and $K \cap Imf \ll K$. M being indecomposable implies that $K = 0$ or $K = M$. If $K = M$, then Imf is a small submodule of M, otherwise f is epimorphism.

(2) $\Rightarrow$ (3) Suppose that f is an arbitrary endomorphism of M. If $Imf \ll M$, pick the summand $D := 0$ and if $Imf = M$ pick the summand $D := M$. Now for any $X \leq M$, $Imf + X = M$ if and only if $D + X = M$.

(3) $\Rightarrow$ (1) It follows from Proposition 2.4. □

Recall that a module M is *coHopfian* if every monomorphism $f \in S$ is an isomorphism. Following the last result, if M is indecomposable, E-$\oplus$-supplemented and $\mathcal{T}$-noncosingular, then M is coHopfian.

In [13], a module is called *duo* (resp. *weak duo*) if every submodule (resp. direct summand) of M is fully invariant in M.

Theorem 2.18. *For a module M consider the following:*

(1) *M is dual Rickart;*

(2) *M is E-$\oplus$-supplemented and $\mathcal{T}$-noncosingular.*

Then (1) $\Rightarrow$ (2). *The converse holds if M is a weak duo module.*

Proof. (1) $\Rightarrow$ (2) It is clear by definitions.
(2) $\Rightarrow$ (1) Let M be $\mathcal{T}$-noncosingular and E-$\oplus$-supplemented. Suppose that $f \in End(M)$. Now there is a direct summand K of M such that $Imf + K = M$ and $Imf \cap K \ll K$. Set $K \oplus K' = M$. Consider the R-homomorphism $\pi_K : M \to K$ such that $\pi_K(x + x') = x$ for every $x \in K$ and $x' \in K'$. Now $jo\pi_K of : M \to M$ is a homomorphism where $j : K \to M$ is the inclusion. So $jo\pi_K of(M) = Imf \cap K \ll M$ as M is a weak duo module. Since M is $\mathcal{T}$-noncosingular, $jo\pi_K of = 0$. It follows that $Imf \cap K = 0$. This completes the proof. □

Proposition 2.19. *Let M be an E-$\oplus$-supplemented module and N a direct summand of M. Suppose that for every direct summand D of M with $M = N + D$, $N \cap D$ is also a direct summand of M. Then N is E-$\oplus$-supplemented.*

Proof. Let $f\colon N \to N$ be an endomorphism of N. Set $N \oplus N' = M$ for a submodule N' of M. It is clear that $g = jofo\pi_N\colon M \to M$ is an endomorphism of M where $\pi_N\colon M \to N$ is projection of M onto N and $j\colon N \to M$ is the inclusion. It is not hard to see that $Img = Imf$. M being E-$\oplus$-supplemented implies that there is a direct summand D of M such that $Imf + D = M$ and $Imf \cap D \ll D$. By the assumption $N \cap D$ is a direct summand of M. Set $(N \cap D) \oplus K = M$, for a summand K of M. Consider an arbitrary d in D. Then $d = n + k$ for some $n \in (N \cap D)$ and $k \in K$. However, $k = d - n$ implies $k \in D$, so

$k \in K \cap D$. Hence $D = (N \cap D) \oplus (K \cap D)$. Now, $Imf + (N \cap D) + (K \cap D) = M$. The modular law implies that $Imf + (N \cap D) = N$. It also clear that $Imf \cap (N \cap D) \ll (N \cap D)$ since $N \cap D$ is a direct summand of D. □

A module M has D_3 if for any direct summands M_1 and M_2 of M with $M = M_1 + M_2$, $M_1 \cap M_2$ is a direct summand of M. A module M is said to have the SIP (*Summand Intersection Property*) if the intersection of two direct summands of M is again a direct summand of M.

Theorem 2.20. *Let M be an E-⊕-supplemented module with D_3 or having the SIP. Then every direct summand of M is E-⊕-supplemented. In particular, every direct summand of an E-⊕-supplemented weak duo module inherits the property.*

Proof. It follows immediately from Proposition 2.19. The last statement follows from [13, Corollary 2.2] and the first part. □

Condition D_3 is not necessary in Theorem 2.20, as the following example shows.

Example 2.21. ([14, Example 3.9]) Let I and J be two ideals of a commutative local ring R with maximal ideal m such that $I \subset J \subseteq m$ (e.g. R is a discrete valuation ring with maximal ideal m, $I = m^3$ and $J = m^2$). We consider the module $M = \frac{R}{I} \times \frac{R}{J}$ and its submodules $A = R(\bar{1}, \bar{0})$, $B = R(\bar{1}, \bar{1})$ and $C = R(\bar{0}, \bar{1})$. Note that $M = A + B = A \oplus C = B \oplus C$. On the other hand, we have $A \cap B = J/I \times 0$. Hence $A \cap B \subseteq Rad(M)$ and $A \cap B \ll M$. Therefore $0 \neq A \cap B$ is not a direct summand of M. So M does not satisfy D_3. Moreover, every direct summand of M is H-supplemented by [14, Proposition 2.1] and therefore is ⊕-supplemented. Hence every direct summand of M is E-⊕-supplemented.

We next show that under an assumption, a finite direct sum of E-⊕-supplemented modules is E-⊕-supplemented.

Proposition 2.22. *Let $M = M_1 \oplus M_2$ be a weak duo module. Then M_1 and M_2 are E-⊕-supplemented if and only if M is E-⊕-supplemented.*

Proof. Let $f \in End_R(M)$. Consider the R-homomorphisms $t_i : M_i \to M$ and $\pi_i : M \to M_i$ for $i = 1, 2$. Then $g_i = \pi_i f t_i \in End_R(M_i)$ for $i = 1, 2$. As M_i for $i = 1, 2$ is E-⊕-supplemented, there exists a direct summand K_i of M_i, for $i = 1, 2$, such that $Img_i + K_i = M_i$ and $Img_i \cap K_i \ll K_i$. Set $K_i \oplus K_i' = M_i$, where $K_i' \leq M_i$ and $K = K_1 \oplus K_2$. Since M_i is a fully invariant submodule of M for $i = 1, 2$, then $f(M_i) \subseteq M_i$, which implies that $Imf = Img_1 \oplus Img_2$. Now, $Imf + K = (Img_1 + K_1) + (Img_2 + K_2) = M_1 + M_2 = M$. Next, we show that $Imf \cap K = (Img_1 \cap K_1) + (Img_2 \cap k_2)$. Suppose that $x \in [Imf \cap K = (Img_1 \oplus Img_2) \cap (K_1 \oplus K_2)]$. Then $x = g(m_1) + g(m_2) = k_1 + k_2$ for $m_1 \in M_1$, $m_2 \in M_2$, $k_1 \in K_1$ and $k_2 \in K_2$. It follows that $g_1(m_1) = k_1$ and $g(m_2) = k_2$, which implies that $x = g_1(m_1) + g_2(m_2) \in (Img_1 \cap K_1) + (Img_2 \cap K_2)$. The other inclusion is obvious. Therefore, $Imf \cap K = (Img_1 \cap K_1) + (Img_2 \cap K_2) \ll$

$K_1 + K_2 = K$. Hence Imf has a supplement in M. This completes the proof. The converse holds by Theorem 2.20. □

In the next results, we try to present some assumptions under which every E-$\oplus$-supplemented module is $\mathcal{I}$-lifting.

Theorem 2.23. *Let M be an E-$\oplus$-supplemented module. In each of the following cases M is $\mathcal{I}$-lifting.*
(1) *M is projective.*
(2) *M is weak duo.*

Proof. (1) Let M be a projective E-$\oplus$-supplemented module and $f\colon M \to M$ be an endomorphism of M. Then there is a direct summand D of M such that $Imf + D = M$ and $Imf \cap D \ll D$. Set $D \oplus D' = M$. Since M is projective, there exists a decomposition $T \oplus D = M$ by [11, Lemma 4.47], where $T \subseteq Imf$. We show that $Imf/T \ll M/T$. To prove that, let $Imf/T + L/T = M/T$, where L is a submodule of M containing T. Then $Imf + L = M$. Since T is contained in Imf, modularity implies that $T \oplus (D \cap Imf) = Imf$. Now $T + (D \cap Imf) + L = M$ which, combined with $D \cap Imf \ll M$, implies that $T + L = M$. Hence $L = M$, as $T \subseteq L$.

(2) Let M be a weak duo E-$\oplus$-supplemented module and $f \in S$. Then there exists a direct summand D of M such that $Imf + D = M$ and $Imf \cap D \ll D$. Suppose that $D \oplus D' = M$ such that $D' \leq M$. Then $Imf = f(D) \oplus f(D')$. It follows that $f(D) + f(D') + D = M$. Since M is weak duo, $f(D) \subseteq D$. So, $f(D') \oplus D = M$. By the modular law we conclude that $f(D') = D'$, which implies that $D' \subseteq Imf$. Now, since $Imf \cap D \ll D$, we conclude that $Imf/D' \ll M/D'$. Therefore, M is $\mathcal{I}$-lifting. □

We end this manuscript by presenting a new characterization of f-semiperfect rings in terms of the E-$\oplus$-supplemented rings.

Recall from [17] that a ring R is f-semiperfect in the case when for every finitely generated right ideal I of R, the R-module R/I has a projective cover.

Corollary 2.24. *Let R be a ring. Then the following are equivalent:*
(1) *R_R is E-$\oplus$-supplemented;*
(2) *R_R is $\mathcal{I}$-lifting;*
(3) *Every cyclic right ideal of R lies above a direct summand of R_R;*
(4) *R is f-semiperfect.*

Proof. (1) ⇒ (2) It follows from Theorem 2.23.

(2) ⇒ (3) This follows from the fact that the image of every endomorphism of R_R is a cyclic right ideal of R.

(3) ⇒ (1) Let $g\colon R \to R$ be an endomorphism. Then Img is a cyclic right ideal of R which lies above a direct summand of R_R by assumption. The rest is clear.

(2) ⇔ (4) It is proved in [1, Theorem 2.7]. □

The following example introduces an E-$\oplus$-supplemented ring which is not semiperfect.

Example 2.25. Let $Q = \prod_{i=1}^{\infty} F_i$, where $F_i = \mathbb{Z}_2$ for all $i \in \mathbb{N}$ and R denotes the subring of Q generated by $\bigoplus_{i=1}^{\infty} F_i$ and 1_Q. It is well-known that R is a von Neumann regular V-ring and hence $J(R) = 0$. Therefore, R can not be semiperfect while R is $\mathcal{I}$-lifting and hence E-$\oplus$-supplemented.

We shall provide a condition under which the two concepts, E-$\oplus$-supplemented and semiperfect, are equivalent for rings.

Proposition 2.26. *Let R be a principal ideal domain. Then R_R is E-$\oplus$-supplemented if and only if R is semiperfect.*

Proof. Clear. □

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