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Complete topology on BCK-algebras

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Abstract. In this paper, we introduce the concept of a complete BCK-algebra by using of a system of ideals on a BCK-algebra.

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1. Introduction

In 1966, Y. Imai and K. Iséki introduced in [2] a new notion, called a BCK-algebra. This notion originated from two different sources: one of them is based on set theory; another is from classical and nonclassical propositional calculi. The BCK-operator $*$ is an analogue of the set theoretical difference. As is well known, there is a close relationship between the notions of the set difference in set theory and the implication functor in logical systems. Then the following problems arise from this relationship: What are the most essential and fundamental common properties? Can we formulate a new general algebra from this wiewpoint? How can we find an axiom system to establish a good theory of general algebras? To give an answer these problems, Y. Y. Imai and K. Iséki introduced a new class of general algebras which are called BCK-algebras. This name is taken from the BCK-system of C. A. Meredith. Since then many researchers studied several notions and properties of BCK-algebras. Today BCK-algebras have been studied by many authors and they have been applied to many branches of mathematics, such as group theory, functional analysis, probability theory, topology, fuzzy set theory, and so on. In this paper, we work on a special type of topology induced by a system of ideals on BCK-algebras and study some general properties of this topology. We also define the inverse system and inverse limit in the category of BCK-algebras. Finally, we introduce the concept of a Cauchy sequence and complete BCK-algebras and study their properties.

2. Definitions

An algebra $(X, *, 0)$ of type $(2, 0)$ is called a *BCK-algebra* if it satisfies the following axioms: for any $x, y, z \in X$,
(1) $((x * y) * (x * z)) * (z * y) = 0$;
(2) $(x * (x * y)) * y = 0$;

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(3) $x * x = 0$;
(4) $x * y = y * x = 0 \Rightarrow x = y$;
(5) $0 * x = 0$.
In BCK-algebra X, if we define $\leqslant$ by $x \leqslant y$ if and only if $x * y = 0$, then $\leqslant$ is a partial order and the following conclusions hold:
(6) $(y * x) * (z * x) \leqslant (y * z)$,
(7) $x * (x * (x * y)) = x * y$,
(8) $(x * y) * z = (x * z) * y$,
(9) $x * 0 = x$,
(10) $x * y \leqslant x$,
(11) $x \leqslant y$ implies $x * z \leqslant y * z$.(See,[4])

Definition 2.1. [4] Let X be a BCK-algebra. An *ideal* is a nonempty set $I \subseteq X$ such that for each $x, y \in X$,
(a) $0 \in I$,
(b) $x * y \in I$, $y \in I \Rightarrow x \in I$.

Proposition 2.2. *[4] Let I be an ideal in a BCK-algebra $(X, *, 0)$. Then*
(i) if $x \leqslant y$ and $y \in I$, then $x \in I$,
(ii) the relation

$$x \equiv^I y \;\Leftrightarrow\; x * y, \quad y * x \in I$$

is a congruence relation on X.
(iii) if $\frac{x}{I} = \{y \in A : x \equiv^I y\}$ and $\frac{X}{I} = \{\frac{x}{I} : x \in X\}$, then $\frac{X}{I}$ is a BCK-algebra under the binary operation

$$\frac{x}{I} * \frac{y}{I} = \frac{x * y}{I}.$$

In this case $\frac{X}{I}$ is said to be the quotient BCK-algebra.
(iv) the mapping $\pi_I : X \hookrightarrow \frac{X}{I}$ by $\pi_I(x) = x/I$ is an epimorphism and for each $S \subseteq X$,

$$(\pi_I^{-1} \circ \pi_I)(S) = \bigcup_{x \in S} \frac{x}{I}.$$

π_I is also called the canonical epimorphism*.*

Recall that an *inverse system* in a category D is a family $\{A_i\}_{i \in \Lambda}$ of objects, indexed by an upward directed set Λ, together with a family of morphisms $\pi_{ij} : A_j \to A_i$ for each $i \leq j$, satisfying the conditions:
(1) $\pi_{ik} = \pi_{ij} \circ \pi_{jk}$ for all $i \leq j \leq k$,
(2) $\pi_{ii} = Id_{A_i}$ for all $i \in \Lambda$.
Given an inverse system $\{A_i, \pi_{ij}\}_{i \leq j \in \Lambda}$ in D, an *inverse limit* of this system is an object A of D together with a family of morphisms $\phi_i : A \to A_i$ satisfying the condition $\pi_{ij} \circ \phi_j = \phi_i$ when $i \leq j$ and having the following universal property: for every object B of D together with family of morphisms $\psi_i : B \to A_i$, if $\pi_{ij} \circ \psi_j = \psi_i$ whenever $i \leq j$, then there exists a unique morphism $\psi : B \to A$ such that $\phi_i \circ \psi = \psi_i$ for all $i \in \Lambda$. The inverse limit of an inverse system $\{A_i, \pi_{ij}\}$ in D often denoted by $\varprojlim A_i$.
The inverse limits of family of algebras are constructed in the following way.

Let $\{A_i, \pi_{ij}\}$ be an inverse system of algebras, $\prod_{i\in\Lambda} A_i$ be its product and $\pi_i : \prod_{i\in\Lambda} A_i \to A_i$ is defined by $\pi_i((x_i)_{i\in\Lambda}) = x_i$ for any $i \in \Lambda$. Let $A = \{(a_i)_{i\in\Lambda} \in \prod_{i\in\Lambda} A_i : \pi_{ij}(a_j) = a_i, i \le j\}$. Then A is a subalgebra of $\prod_{i\in\Lambda} A_i$ and $\{A, \phi_i\}_{i\in\Lambda}$ is inverse limit of $\{A_i, \pi_{ij}\}_{i,j\in\Lambda}$ where $\phi_i = \pi_i|_A$ for any $i \in \Lambda$.[See,[1]]

3. Main results

Recall, a poset $(D, \le)$ is called an *upward directed* set if for any $x, y \in D$ there exists $z \in D$ such that $x \le z$ and $y \le z$.

Definition 3.1. Topology τ on BCK-algebra $(X, *, 0)$ is called *linear topology* if there exists a base β for τ such that for any element B of β containing 0, B is an ideal of X.

Definition 3.2. Let Λ be an upward directed set and $\{I_i : i \in \Lambda\}$ be a family of ideals in a BCK-algebra $(X, *, 0)$. Then $\{I_i : i \in \Lambda\}$ is called a *system of ideals* of X if $i \le j$ implies $I_j \subseteq I_i$, for any $i, j \in \Lambda$.

Proposition 3.3. *[3] Let $\{I_i : i \in \Lambda\}$ be a system of ideals of BCK-algebra $(X, *, 0)$. Then the set $\beta = \{x/I_i : x \in X,\ i \in \Lambda\}$ is a base for a linear topology τ_Λ on X. Moreover, (X, τ_Λ) is a topological BCK-algebra.*

Proposition 3.4. *If $U \in \tau_\Lambda$, then $U * X$ is open.*

Proof. Let $a \in U * X$. Then $a = u * x$ for some $u \in U$ and $x \in X$. There exists $i \in \Lambda$ such that $\frac{u}{I_i} \subseteq U$, because $u \in U \in \tau_\Lambda$. Hence

$$\frac{a}{I_i} = \frac{u * x}{I_i} = \frac{u}{I_i} * \frac{x}{I_i} \subseteq \frac{u}{I_i} * X \subseteq U * X.$$

Thus $U * X \in \tau_\Lambda$. □

Proposition 3.5. *The space (X, τ_Λ) is locally convex.*

Proof. Let $x \in U \subseteq \tau_\Lambda$. Then $\frac{x}{I_i} \subseteq U$ for some $i \in \Lambda$. We claim that $\frac{x}{I_i}$ is convex. Let $a, b \in \frac{x}{I_i}$ and $z \in X$ such that $a \le z \le b$. Since $a \le z \le b$, $a * z = z * b = 0 \in I_i$. Since $a, b \in \frac{x}{I_i}$, $a * x, x * a, b * x, x * b \in I_i$. By (1) and (6) we have $((z * b) * (x * b)) * (z * x) = 0 \in I_i$ and $((a * z) * (a * x)) * (x * z) = 0 \in I_i$. Hence $z * x, x * z \in I_i$, and so $z \in \frac{x}{I_i}$. □

Proposition 3.6. *[3] Let $(X, *, 0, \tau)$ be a (semi)topological BCK-algebra. An ideal I of X is an open(closed) subset of X iff, for each $x \in X$, x/I is an open(closed) subset of X.*

A topological space (A, τ) is called zero-dimensional if τ has a clopen base.

Proposition 3.7. *The topological space (X, τ_Λ) is a zero-dimensional space.*

Proof. By Proposition 3.3, the set $\{\frac{x}{I_i} : x \in X, i \in \Lambda\}$ is a base for τ_Λ. We claim that $\frac{x}{I_i}$ is closed for any $x \in X$ and $i \in \Lambda$. Cleary, $I_i = X \setminus \bigcup_{x\notin I_i} \frac{x}{I_i}$ and hence I_i is closed in (X, τ_Λ). Thus by Proposition 3.6, $\frac{x}{I_i}$ is closed in (X, τ_Λ). □

Proposition 3.8. *[3] Let $\{I_i : i \in \Lambda\}$ be a system of ideals of the topological BCK-algebra $(X, *, 0, \tau_\Lambda)$. If $S \subseteq X$ and $S/I_i = \bigcup_{x\in S} x/I_i$, then $\overline{S} = \bigcap_{i\in\Lambda} S/I_i$,*

Proposition 3.9. *Let I be an ideal of X. Then*
(i) If $I_i \subseteq I$ for some $i \in \Lambda$, then I is closed.
(ii) If $I \subseteq \bigcap_{i\in I} I_i$, then $\overline{I} \subseteq \bigcap_{i\in I} I_i$.
(iii) If $I \subseteq \bigcap_{i\in I} I_i$, then $\bigcap_{i\in I} I_i$ is closed.
(iv) If I is closed, then $\bigcap_{i\in I} I_i \subseteq I$.

Proof. (i) By Proposition 3.8, we have $\overline{I} = \bigcap_{i\in\Lambda}(\bigcup_{x\in I} \frac{x}{I_i})$. If $z \in \overline{I}$, then $z \in \frac{x}{I_i}$ for any $i \in \Lambda$ and some $x \in I$. Since $x \in I$ and $z * x \in I_i \subseteq I$, $x \in I$ and so $\overline{I} \subseteq I$.
(ii) If $z \in \overline{I}$, then $z \in \frac{x}{I_i}$ for any $i \in \Lambda$ and some $x \in I$. Since for each $i \in \Lambda$, $z * x \in I_i$ and $x \in I \subseteq I$, we get $z \in I_i$ and hence $z \in \bigcap_{i\in I} I_i$.
(iii) By (ii) the proof is clear.
(iv) If I is closed, then $\overline{I} = I$. If $z \in \bigcap_{i\in I} I_i$, then $z \in I_i$ for some $i \in \Lambda$. Since $z * 0, 0 * z \in I_i$, $z \in \frac{0}{I_i} \subseteq \frac{I}{I_i}$ for any $i \in \Lambda$. Hence by Proposition 3.8, we have $z \in \bigcap_{i\in\Lambda} \frac{I}{I_i} = \overline{I}$. □

Proposition 3.10. *Let τ_Λ be the linear topology on X and $\pi_{ij} : \frac{X}{I_i} \to \frac{X}{I_j}$ be the mapping defined by $\pi_{ji}(\frac{x}{I_i}) = \frac{x}{I_j}$ for any $x \in X$, $i, j \in \Lambda$ and $j \leq i$. Then*
(i) $\{\frac{X}{I_i}, \pi_{ij}\}_{i\in\Lambda}$ is an inverse system, whose limit shall be denoted by $\hat{X} = \varprojlim \frac{X}{I_i}$.
(ii) Let $\phi_i : \hat{X} \to \frac{X}{I_i}$ be the natural projection and $K_i = ker(\phi_i)$ for any $i \in \Lambda$. Then $\mathcal{I} = \{K_i : i \in \Lambda\}$ is a system of ideals of $\hat{X}$.

Proof. (i) It is clear that $\pi_{ji} : \frac{X}{I_i} \to \frac{X}{I_j}$ is a homomorphism and $\pi_{ii} : \frac{X}{I_i} \to \frac{X}{I_i}$ is identity map. Let $i, j, k \in \Lambda$ and $i \leq j \leq k$. Then

$$(\pi_{ij} \circ \pi_{jk})(\frac{x}{I_k}) = \pi_{ij}(\pi_{jk}(\frac{x}{I_k})) = \pi_{ij}(\frac{x}{I_j}) = \frac{x}{I_i} = \pi_{ik}(\frac{x}{I_k}).$$

Hence $\{\frac{X}{I_i}, \pi_{ij}\}_{i\in\Lambda}$ is an inverse system.
(ii) Clearly, K_i is an ideal of $\hat{X}$ for any $i \in \Lambda$. Let $i, j \in \Lambda$ and $i \leq j$. Since $\phi_i = \pi_{ij} \circ \phi_j$, we get that $K_j \subseteq K_i$. Hence $\{K_i : i \in \Lambda\}$ is a system of ideals of $\hat{X}$. □

Proposition 3.11. *If $\psi : X \to \hat{X}$ is a natural homomorphism defined by $\psi(x) = (\frac{x}{I_i})_{i\in\Lambda}$, then the following are equivalent*
(i) ψ is one to one,
(ii) (X, τ_Λ) is Hausdorff space,
(iii) $\bigcap\{I_i : i \in \Lambda\} = \{0\}$.

Proof. $(i) \Rightarrow (ii)$ Let $x, y \in X$ such that for each $U \in \tau_\Lambda$, $x \in U$ if and only if $y \in U$. Since $\frac{x}{I_i} \in \tau_\Lambda$ for any $i \in \Lambda$ and $x \in \frac{x}{I_i}$, we get $y \in \frac{x}{I_i}$ for any $i \in \Lambda$. Hence $x * y, y * x \in I_i$ for any $i \in \Lambda$ and so $\frac{x}{I_i} = \frac{y}{I_i}$ for any $i \in \Lambda$. Thus

$(\frac{x}{I_i})_{i\in\Lambda} = (\frac{y}{I_i})_{i\in\Lambda}$ and hence $\psi(x) = \psi(y)$. Since ψ is one to one, $x = y$.
$(ii) \Rightarrow (iii)$ Let $x \in \bigcap\{I_i : i \in \Lambda\}$ and $x \neq 0$. By assumption, there exist $U, V \in \tau_\Lambda$ such that $0 \in U$, $x \in V$ and $U \cap V = \emptyset$. Hence $\frac{0}{I_j} = I_j \subseteq U$ for some $j \in \Lambda$. Since $x \in \bigcap\{I_i : i \in \Lambda\}$, $x \in I_j$ and so $x \in U$. Thus $U \cap V \neq \emptyset$, which is a contradiction.
$(iii) \Rightarrow (i)$ Let $x, y \in X$ and $\psi(x) = \psi(y)$. Then $(\frac{x}{I_i})_{i\in\Lambda} = (\frac{y}{I_i})_{i\in\Lambda}$. Hence $x * y, y * x \in I_i$ for any $i \in \Lambda$. Therefore $x * y, y * x \in \bigcap\{I_i : i \in \Lambda\} = \{0\}$. Thus $x * y = y * x = 0$ and so $x = y$. Hence ψ is one to one. □

Proposition 3.12. *The mapping $\psi : X \to \hat{X}$ is onto if and only if whenever $\{x_i\}_{i\in\Lambda}$ is a family of elements of X with the property that for $i \leq j$ we have $x_i \equiv^{I_i} x_j$, then there exists $x \in X$ such that $x \equiv^{I_i} x_i$ for any $i \in \Lambda$.*

Proof. Let ψ be onto. If $\{x_i\}_{i\in\Lambda}$ is a family of elements of X such that $x_i \equiv^{I_i} x_j$ for $j \leq i$, then $\frac{x_i}{I_i} = \frac{x_j}{I_i}$. Hence $\pi_{ji}(\frac{x_i}{I_i}) = \pi_{ji}(\frac{x_j}{I_i}) = \frac{x_j}{I_j}$. Thus $(\frac{x_i}{I_i})_{i\in\Lambda} \in \hat{X}$. Since ψ is onto, there exists $x \in X$ such that $\psi(x) = (\frac{x_i}{I_i})_{i\in\Lambda}$. We claim that $x \equiv^{I_i} x_i$ for any $i \in \Lambda$. Since $\psi(x) = (\frac{x}{I_i})_{i\in\Lambda}$, we get that $(\frac{x}{I_i})_{i\in\Lambda} = (\frac{x_i}{I_i})_{i\in\Lambda}$. Hence $\frac{x}{I_i} = \frac{x_i}{I_i}$ for any $i \in \Lambda$. Therefore, $x \equiv^{I_i} x_i$ for any $i \in \Lambda$. Conversely, let $\{x_i\}_{i\in\Lambda}$ be the family of elements of X with the property that for $i \leq j$ we have $x_i \equiv^{I_i} x_j$, then there exists $x \in X$ such that $x \equiv^{I_i} x_i$ for any $i \in \Lambda$. Let $(\frac{x_i}{I_i})_{i\in\Lambda} \in \hat{X}$. If $i, j \in \Lambda$ and $j \leq i$, then there exists $k \in \Lambda$ such that $i, j \leq k$ and so $I_k \subseteq I_i \cap I_j$. Hence $\frac{x_k}{I_i} = \frac{x_i}{I_i}$ and $\frac{x_k}{I_j} = \frac{x_j}{I_j}$. Thus $x_k \equiv^{I_i} x_i$ and $x_k \equiv^{I_j} x_j$. Since $I_j \subseteq I_i$, we get that $x_i \equiv^{I_i} x_j$. By the assumption, there exists $x \in X$ such that $x \equiv^{I_i} x_i$ for any $i \in \Lambda$. We claim that $\psi(x) = (\frac{x_i}{I_i})_{i\in\Lambda}$. Since $x \equiv^{I_i} x_i$ for any $i \in \Lambda$, we get that $\frac{x}{I_i} = \frac{x_i}{I_i}$ for any $i \in \Lambda$ and so $(\frac{x}{I_i})_{i\in\Lambda} = (\frac{x_i}{I_i})_{i\in\Lambda}$. Thus $\psi(x) = (\frac{x_i}{I_i})_{i\in\Lambda}$, and hence ψ is onto. □

Proposition 3.13. (i) *If τ is the topology on $\hat{X}$ and $\hat{\tau}_\Lambda$ is the linear topology induced by $\mathcal{I}$ on $\hat{X}$, then $\tau = \hat{\tau}_\Lambda$.*
(ii) *The mapping $\hat{\psi} : \hat{X} \to \varprojlim \frac{\hat{X}}{K_i}$ defined by $\hat{\psi}(x) = (\frac{x}{K_i})_{i\in\Lambda}$ for any $x \in \hat{X}$ is an isomorphism.*

Proof. (i) Let $(\frac{x_i}{I_i})_{i\in\Lambda} \in \hat{X}$, $j \in \Lambda$ and

$$\frac{(\frac{x_i}{I_i})_{i\in\Lambda}}{K_j} = \{(\frac{y_i}{I_i})_{i\in\Lambda} \in \hat{X} : (\frac{y_i}{I_i})_{i\in\Lambda} \equiv^{K_j} (\frac{x_i}{I_i})_{i\in\Lambda}\}.$$

Then by Proposition 3.3, the set $\{\frac{(\frac{x_i}{I_i})_{i\in\Lambda}}{K_j} : (\frac{x_i}{I_i})_{i\in\Lambda} \in \hat{X}, j \in \Lambda\}$ is a base for $\hat{\tau}_\Lambda$. Moreover, it is clear that $\frac{(\frac{x_i}{I_i})_{i\in\Lambda}}{K_j} = \prod_{i\in\Lambda} U_i$, where

$$U_i = \begin{cases} \phi_i(\hat{X}) & \text{if } i \neq j \\ \{\frac{x_j}{I_j}\} & \text{if } i = j. \end{cases}$$

Hence $\frac{(\frac{x_i}{I_i})_{i\in\Lambda}}{K_j} \in \tau$ for any $j \in \Lambda$ and $(\frac{x_i}{I_i})_{i\in\Lambda} \in \hat{X}$. Therefore $\hat{\tau}_\Lambda \subseteq \tau$. It is easily verified that $\tau \subseteq \hat{\tau}_\Lambda$.

(ii) By Proposition 2.2, the mapping $\pi_{I_i} : X \to \frac{X}{I_i}$ given by $\pi_{I_i}(x) = \frac{x}{I_i}$ is an epimorphism. Hence the mapping $\phi_i : \hat{X} \to \frac{X}{I_i}$ is onto, because $\phi_i(\psi(X)) = \pi_{I_i}(X) = \frac{X}{I_i}$. Thus ϕ_i is an epimorphism for any $i \in \Lambda$. Therefore by the first isomorphism theorem in BCK-algebras, we have $\frac{X}{I_i} \cong \frac{\hat{X}}{K_i}$ for any $i \in \Lambda$. Thus $\hat{X} = \varprojlim \frac{X}{I_i} \cong \varprojlim \frac{\hat{X}}{K_i}$. □

Definition 3.14. A sequence $\{x_i : i \in \Lambda\}$ of elements of X
(i) *Converges* to $x \in X$ if for each $i \in \Lambda$ there exists $N_i \in \Lambda$ such that $x_n \in \frac{x}{I_i}$ for any $n \geq N_i$ and $n \in \Lambda$. In this case we write $x_n \to x$.
(ii) is called *Cauchy* if for each $i \in \Lambda$ there exists $N_i \in \Lambda$ such that $\frac{x_n}{I_i} = \frac{x_m}{I_i}$ for any $m, n \geq N_i$ and $m, n \in \Lambda$.

Theorem 3.15. (i) *If (X, τ_Λ) is Hausdorff space, then the limit of any sequence in X is unique, if it exists.*
(ii) *If $x_i \to x$ and $y_i \to y$, then $x_i * y_i \to x * y$.*
(iii) *Each convergent sequence in X is a Cauchy sequence.*

Proof. (i) Let $\{x_i\}_{i\in\Lambda}$ be a sequence in X and $x, y \in X$. If $x_i \to x$ and $x_i \to y$, then for each $i \in \Lambda$ there exists $N_i, M_i \in \Lambda$ such that $x_n \in \frac{x}{I_i}$ for any $n \geq N_i$ and $x_n \in \frac{y}{I_i}$ for any $n \geq M_i$. Since Λ is an upward directed set, there exists $N \in \Lambda$ such that $N_i, M_i \leq N$. Hence $x_n \in \frac{x}{I_i}$ and $x_n \in \frac{y}{I_i}$ for any $n \geq N$ and so $x \equiv^{I_i} y$ for any $i \in \Lambda$. Thus $x * y, y * x \in I_i$ for any $i \in \Lambda$. Since (X, τ_Λ) is a Hausdorff space, by Proposition 3.11, we have $x * y = y * x \in \bigcap\{I_i : i \in \Lambda\} = \{0\}$. Hence $x = y$.
(ii) Let $j \in \Lambda$. Then there exist $N_j, M_j \in \Lambda$ such that $x_n \equiv^{I_j} x$ for any $n \geq N_j$ and $y_n \equiv^{I_j} y$ for any $n \geq M_j$. There exists $N \in \Lambda$ such that $N_j, M_j \leq N$ and so $x_n \equiv^{I_j} x$ and $y_n \equiv^{I_j} y$ for any $n \geq N$. Thus $x_n * y_n \equiv^{I_j} x * y$ for any $n \geq N$ and hence $x_n * y_n \in \frac{x*y}{I_j}$ for any $n \geq N$. Therefore $x_i * y_i \to x * y$.
(iii) Let $\{x_i\}_{i\in\Lambda}$ be a sequence in X and $x_i \to x \in X$. Then for each $j \in \Lambda$ there exists $N_j \in \Lambda$ such that $x_n \in \frac{x}{I_j}$ for any $n \geq N_j$. If $m, n \geq N_j$, then $x_n \in \frac{x}{I_j}$ and $x_m \in \frac{x}{I_j}$ and so $\frac{x_n}{I_j} = \frac{x}{I_j} = \frac{x_m}{I_j}$ for any $m, n \geq N_j$. Thus $\{x_i\}_{i\in\Lambda}$ is a Cauchy sequence in X. □

Let $\mathcal{C}$ be the set of all Cauchy sequences in X. define a binary operation on $\mathcal{C}$ as follow

$$* : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \quad \{x_i\}_{i\in\Lambda} * \{y_i\}_{i\in\Lambda} \to \{x_i * y_i\}_{i\in\Lambda}.$$

Theorem 3.16. (i) *$(\mathcal{C}, *, \{0\}_{i\in\Lambda})$ is a BCK-algebra.*
(ii) *If C_0 is the set of all sequences in X which converge to 0, then C_0 is an ideal of $\mathcal{C}$.*

Proof. (i) Let $\{x_i\}_{i\in\Lambda}, \{y_i\}_{i\in\Lambda} \in \mathcal{C}$ and $j \in \Lambda$. There exist $N_j, M_j \in \Lambda$ such that $\frac{x_m}{I_j} = \frac{x_n}{I_j}$ for any $n, m \geq N_j$ and $\frac{y_s}{I_j} = \frac{y_t}{I_j}$ for any $s, t \geq M_j$. Since Λ is an upward directed set, there exists $N \in \Lambda$ such that $N_j, M_j \leq N$. Then $\frac{x_m}{I_j} = \frac{x_n}{I_j}$ and $\frac{y_m}{I_j} = \frac{y_n}{I_j}$ for any $m, n \geq N$. Hence $x_m \equiv^{I_j} x_n$ and $y_m \equiv^{I_j} y_n$ for any $m, n \geq N$ and so $x_m * y_m \equiv^{I_j} x_n * y_m$ for any $m, n \geq N$. Hence $\{x_i * y_i\}_{i\in\Lambda} \in \mathcal{C}$.

It is easily verified that $(\mathcal{C}, *, \{0\}_{i\in\Lambda})$ is a BCK-algebra.
(ii) Clearly, $\{0\}_{i\in\Lambda} \in C_0$. Let $\{x_i\}_{i\in\Lambda} * \{y_i\}_{i\in\Lambda} = \{x_i * y_i\}_{i\in\Lambda} \in C_0$ and $\{y_i\}_{i\in\Lambda} \in C_0$. Then $x_i * y_i \to 0$ and $y_i \to 0$. Let $j \in \Lambda$. There exist $N_j, M_j \in \Lambda$ such that $x_n * y_n \in \frac{0}{I_j} = I_j$ for any $n \geq N_j$ and $y_n \in \frac{0}{I_j} = I_j$ for any $n \geq M_j$. There exists $N \in \Lambda$ such that $N_i, N_j \geq N$. Hence $x_n * y_n \in I_j$ and $y_n \in I_j$ for any $n \geq N$. Since I_j is an ideal, $x_n \in I_j$ for any $n \geq N$ and so $x_n \in \frac{0}{I_i}$ for any $n \geq N$. Thus $x_i \to 0$ and so $\{x_i\}_{i\in\Lambda} \in C_0$. Therefore C_0 is an ideal of $\mathcal{C}$. □

Lemma 3.17. *If $(\frac{x_i}{I_i})_{i\in\Lambda} \in \hat{X}$, then $\{x_i\}_{i\in\Lambda}$ is a Cauchy sequence in X.*

Proof. Let $j \in \Lambda$. For any $m, n \geq j$ we have

$$\frac{x_m}{I_j} = \pi_{jm}(\frac{x_m}{I_m}) = \pi_{jm}(\phi_m((\frac{x_i}{I_i})_{i\in\Lambda})) = \phi_j((\frac{x_i}{I_i})_{i\in\Lambda}) = \frac{x_j}{I_j}.$$

Hence $x_m \in \frac{x_j}{I_j}$. Similarly, we have $x_n \in \frac{x_j}{I_j}$. Thus $\frac{x_m}{I_j} = \frac{x_j}{I_n}$ for any $m, n \geq j$ and so $\{x_i\}_{i\in\Lambda}$ is a Cauchy sequence in X. □

Definition 3.18. The topological space (X, τ_Λ) is called *complete* if (X, τ_Λ) is a Hausdorff space and each Cauchy sequence in X is convergent.

Theorem 3.19. *The topological space (X, τ_Λ) is complete if and only if the mapping $\psi : X \to \hat{X}$ is an isomorphism.*

Proof. If (X, τ_Λ) is complete, then (X, τ_Λ) is Hausdorff space and by Proposition 3.11, ψ is one to one. Let $(\frac{x_i}{I_i})_{i\in\Lambda} \in \hat{X}$. By Lemma 3.17, $\{x_i\}_{i\in\Lambda}$ is a Cauchy sequence in X. Since (X, τ_Λ) is complete, there exists $x \in X$ such that $x_i \to x$. We claim that $\psi(x) = (\frac{x_i}{I_i})_{i\in\Lambda}$. Let $j \in \Lambda$. There exists $N_j \in \Lambda$ such that $x_i \in \frac{x}{I_i}$ for any $i \geq N_j$. Since Λ ia an upward directed set, there exists $k \in \Lambda$ such that $j, N_j \leq k$ and so $x_k \in \frac{x}{I_i}$. Since $j \leq k$, $\frac{x_k}{I_j} = \frac{x_j}{I_j}$ and hence $\frac{x}{I_j} = \frac{x_j}{I_j}$. Thus $(\frac{x}{I_i})_{i\in\Lambda} = (\frac{x_i}{I_i})_{i\in\Lambda}$. Therefore $\psi(x) = (\frac{x_i}{I_i})_{i\in\Lambda}$ and hence ψ is onto. Now, Since ψ is a homomorphism, ψ is an isomorphism.

Conversely, Let ψ be an isomorphism. Then ψ is one to one and by Proposition 3.11, (X, τ_Λ) is a Hausdorff space. Let $\{x_i\}_{i\in\Lambda}$ be a Cauchy sequence in X. Then for each $j \in \Lambda$ there exists $N_j \in \Lambda$ such that $\frac{x_m}{I_j} = \frac{x_n}{I_j}$ for any $m, n \geq N_j$ and $m, n \in \Lambda$. If $j \leq k \in \Lambda$, then $N_j \leq N_k$. Let $a_i = x_{N_i}$ for any $i \in \Lambda$. If $i \leq j$, then $a_i \equiv^{I_i} a_j$ because there exist $N_i, N_j \in \Lambda$ such that $\frac{x_m}{I_i} = \frac{x_n}{I_i}$ for any $m, n \geq N_i$ and $\frac{x_m}{I_j} = \frac{x_n}{I_j}$ for any $m, n \geq N_j$. Since Λ is an upward directed set, there exists $\lambda \in \Lambda$ such that $N_i, N_j \leq \lambda$. Hence $x_\lambda \equiv^{I_i} x_{N_i}$ and $x_\lambda \equiv^{I_j} x_{N_j}$. Since $i \leq j$, $I_j \subseteq I_i$ and so $x_{N_i} \equiv^{I_i} x_{N_j}$. Hence we have a family $(a_i)_{i\in\Lambda}$ of elements of X such that if $i \leq j$, then $a_i \equiv^{I_i} a_j$. By Proposition 3.12, there exists $x \in X$ such that $x \equiv^{I_i} a_i$ for any $i \in \Lambda$. Hence $x \equiv^{I_j} a_j$. For each $n \geq N_j$ we have $a_n \equiv^{I_j} x_{N_j} (= a_j)$. Thus $x \equiv^{I_j} a_n$ for any $n \geq N_j$ and so $x_n \in \frac{x}{I_j}$ for any $n \geq N_j$. Therefore $x_i \to x$ and hence (X, τ_Λ) is complete. □

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