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On a class of Humbert-Hermite polynomials

Waseem A. Khan[1] and M. A. Pathan[2,3]

Abstract. A unified presentation of a class of Humbert's polynomials in two variables which generalizes the well known class of Gegenbauer, Humbert, Legendre, Chebycheff, Pincherle, Horadam, Kinnsy, Horadam-Pethe, Djordjević, Gould, Milovanović and Djordjević, Pathan and Khan polynomials and many not so called 'named' polynomials has inspired the present paper and the authors define here generalized Humbert-Hermite polynomials of two variables. Several expansions of Humbert-Hermite polynomials, Hermite-Gegenbaurer (or ultraspherical) polynomials and Hermite-Chebyshev polynomials are proved.

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1. Introduction

The 2-variable Kampé de Fériet generalization of the Hermite polynomials[3] and [5] is defined as

$$H_n(x,y) = n! \sum_{r=0}^{[\frac{n}{2}]} \frac{y^r x^{n-2r}}{r!(n-2r)!}. \tag{1.1}$$

These polynomials are usually defined by the generating function

$$e^{xt+yt^2} = \sum_{n=0}^{\infty} H_n(x,y) \frac{t^n}{n!}, \tag{1.2}$$

and reduce to the ordinary Hermite polynomials $H_n(x)$ (see[1]) when $y = -1$ and x is replaced by $2x$.

Next, we recall the definition of N-variable generalized Hermite polynomials $H_n(\{x\}_1^N)$ defined by Dattoli et al. [6, p.602] :

$$\exp \sum_{s=1}^{N} x_s t^s = \sum_{n=0}^{\infty} H_n(\{x\}_1^N) \frac{t^n}{n!}, \tag{1.3}$$

[1]Department of Mathematics and Natural Sciences, Prince Mohammad Bin Fahd University, P.O Box 1664, Al Khobar 31952, Kingdom of Saudi Arabia,
e-mail: wkhan1@pmu.edu.sa, waseem08_khan@rediffmail.com
[2]Centre for Mathematical and Statistical Sciences (CMSS), Peechi P.O., Thrissur, Kerala-680653, India, e-mail: mapathan@gmail.com
[3]Corresponding author

where $\{x\}_1^N = x_1, x_2, ..., x_N$.

Generalized Hermite polynomials $H_n(\{x\}_1^N)$ for $N = 3$ also belong to the Bell type as shown in [7, p.403(26)]. The Gould-Hooper polynomials $g_n^m(x,y)$ (see [4] and [10]) are a special case of (1.3). The notation $H_n^m(x,y)$ or $g_n^m(x,y)$ was given by Dattoli et al. [4]. These are specified by

$$e^{xt+yt^m} = \sum_{n=0}^{\infty} H_n^m(x,y)\frac{t^n}{n!}. \tag{1.4}$$

Another generalization of Hermite polynomials which we wish to consider in this paper is given by $H_{n,m,\nu}(x,y)$ in the form of the generating function (see [16])

$$e^{\nu(x+y)t-(xy+1)t^m} = \sum_{n=0}^{\infty} H_{n,m,\nu}(x,y)\frac{t^n}{n!}, \tag{1.5}$$

which reduces to the ordinary Hermite polynomials $H_n(x)$ when $\nu = 2, x = 0$ or $\nu = 2, y = 0$.

We draw attention to familiar generating relations given by

$$(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x)t^n, \tag{1.6}$$

where $P_n(x)$ is Legendre's polynomial of the first kind.

$$(1-2xt+t^2)^{-1} = \sum_{n=0}^{\infty} U_n(x)t^n, \tag{1.7}$$

where $U_n(x)$ is the Chebychev polynomial of the second kind.

$$(1-2xt+t^2)^{-\nu} = \sum_{n=0}^{\infty} C_n^{\nu}(x)t^n, \tag{1.8}$$

where $C_n^{\nu}(x)$ is Gegenbauer's polynomial.

$$(1-mxt+t^m)^{-\nu} = \sum_{n=0}^{\infty} h_{n,m}^{\nu}(x)t^n, \tag{1.9}$$

$$h_{n,m}^{\nu}(x) = \sum_{k=0}^{[\frac{n}{m}]} \frac{(-1)^k(\nu)_{n+(1-m)k}(mx)^{n-mk}}{k!(n-mk)!},$$

where $h_{n,m}^{\nu}(x)$ is the Humbert polynomial and m is a positive integer. The Pochammer symbol $(a)_n$ is defined by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \left[\begin{array}{ll} 1 & \text{if } \ n=0 \\ a(a+1)(a+2)\cdots(a+n-1) & \text{if } \ n=1,2,3\cdots \end{array}\right.$$

In 1965, Gould [11] gave the following generating relation

$$(c - mxt + yt^m)^p = \sum_{n=0}^{\infty} P_n(m, x, y, p, c)t^n, \tag{1.10}$$

where m is a positive integer and other parameters are unrestricted in general. $P_n(m, x, y, p, c)$ is defined explicitly by [11, p.699]:
(1.11)

$$P_n(m, x, y, p, c) = \sum_{k=0}^{[\frac{n}{m}]} \binom{p}{k} \binom{p-k}{n-mk} c^{p-n+(m-1)k} y^k (-mx)^{n-mk}.$$

In 1989, Sinha [19] gave the following generating relation

$$\left[1 - 2xt + t^2(2x-1)\right]^{-\nu} = \sum_{n=0}^{\infty} S_n^{\nu}(x)t^n, \tag{1.12}$$

where

$$S_n^{\nu}(x) = \sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k (\nu)_{n-k} (2x)^{n-2k} (2x-1)^k}{k!(n-2k)!}, \tag{1.13}$$

$S_n^{\nu}(x)$ is the generalization of Shrestha polynomial $S_n(x)$ (see [16]).

In 1991, Milovanović and Djordjević [14] (see also [15]) gave the following generating relation

$$(1 - 2xt + t^m)^{-\lambda} = \sum_{n=0}^{\infty} p_{n,m}^{\lambda}(x)t^n, \tag{1.14}$$

where $m \in \mathbb{N}$ and $\lambda > -\frac{1}{2}$ and

$$p_{n,m}^{\lambda}(x) = \sum_{k=0}^{[\frac{n}{m}]} \frac{(-1)^k (\lambda)_{n-(m-1)_k} (2x)^{n-mk}}{k!(n-mk)!}. \tag{1.15}$$

It is to be noted that the polynomials represented by $p_{n,1}^{\lambda}(x)$, $p_{n,2}^{\lambda}(x)$ and $p_{n,3}^{\lambda}(x)$ are known as Horadam polynomials [12], Gegenbauer polynomials and Horadam-Pethe polynomials [13], respectively.

Many interesting generalizations of these polynomials appeared in the literature. In particular in 1997, Pathan and Khan [16, p.54] generalized these polynomials and gave the following generating relation

$$\begin{aligned} [c - axt + bt^m(2x-1)^d]^{-\nu} &= \sum_{n=0}^{\infty} p_{n,m,a,b,c,d}^{\nu}(x)t^n \\ &= \sum_{n=0}^{\infty} \Theta_n(x)t^n, \end{aligned} \tag{1.16}$$

where

$$(1.17)\quad \Theta_n(x)=\sum_{k=0}^{[\frac{n}{m}]}\frac{(-1)^k c^{-\nu-n+(m-1)k}(\nu)_{n+(1-m)k}(ax)^{n-mk}[b(2x-1)^d]^k}{k!(n-mk)!}.$$

Djordjević [9] provided a generalization of various polynomials of two variables in the form

$$(1.18)\qquad [1-2(x+y)t+t^m(2xy+1)]^{-\alpha}=\sum_{n=0}^{\infty}G_n^{\alpha,m}(x,y)t^n,$$

where

$$(1.19)\qquad G_n^{\alpha,m}(x,y)=\sum_{k=0}^{[\frac{n}{m}]}\frac{(-1)^k(\alpha)_{n-(m-1)k}(2x+2y)^{n-mk}(2xy+1)^k}{k!\ (n-mk)!}.$$

Note that $G_n^{1,m}(x,y)=C_n^m(x,y)$ and $G_n^{1/2,m}(x,y)=P_n^m(x,y)$ where $C_n^m(x,y)$ and $U_n^m(x,y)$ are Chebyshev and Legendre polynomials of two variables, respectively.

For $m=2$, $G_n^{\alpha,m}(x,y)$ reduces to a polynomial studied by Dave [8]. For $m=2$ and $y=0$, $G_n^{\alpha,m}(x,y)$ reduces to a Gegenbauer polynomial and for $m=3$ and $y=0$, $G_n^{\alpha,m}(x,y)$ are Horadam-Pethe polynomials [13]. Further, for $y=0$, $G_n^{\alpha,m}(x,y)$ reduces to a polynomial $p_{n,m}^{\alpha}(x)$ studied by Milovanović and Djordjević ([14] and [15]).

A generalization and unification of various polynomials mentioned above is provided by the definition of generalized Humbert polynomials in two variables given recently by Pathan and Khan [17] which has the generating function

$$(1.20)\quad [a-(bx+cy)t+dt^m(exy-1)^g]^{-h}=\sum_{n=0}^{\infty}Q_{n,m,g,h}^{a,b,c,d,e}(x,y)t^n=\sum_{n=0}^{\infty}Q_n(x,y)t^n,$$

where $m\in\mathbb{N}$, $h>0$ and the other parameters are unrestricted in general.

In (1.20), if we put $a=1$, $b=c=2$, $d=-1$, $e=-2$ and $g=1$, then we get a generating relation (1.18) studied by Djordjević [9]. For $y=1$, $e=2$ and $c=0$, we get a generating relation (1.16) studied by Pathan and Khan [16]. For $a=1$, $b=2$, $c=0$, $d=1$ and $g=0$, we get a generating relation (1.14) studied by Milovanović -Djordjević [15]. For $a=1,\ b=2,\ m=2,\ y=1,\ e=2$ and $g=1$, we get a polynomial defined by Sinha [19] and for $c=0,\ g=0,\ d=y$ and $h=-p$, we get a generating relation (1.4) given by Gould [11]. Some more interesting special cases which are recorded by G.B. Djordjević and G.V. Milovanović in [10] can be established similarly.

2. On a class of Humbert-Hermite polynomials

A generalization and unification of various polynomials mentioned above is provided by the definition of generalized Humbert-Hermite polynomials

${}_HG_n^{\nu,\alpha,m}(x,y)$ in two variables which has the generating function

$$[1-2(x+y)t+t^m(2xy+1)]^{-\nu}e^{\alpha(x+y)t-(xy+1)t^m}=\sum_{n=0}^{\infty}{}_HG_n^{\nu,\alpha,m}(x,y)t^n, \tag{2.1}$$

where $m\in\mathbb{N}$, $\alpha,\nu>0$ and the other parameters are unrestricted in general.

This is interesting since, as will be shown, the polynomials ${}_HG_n^{\nu,\alpha,m}(x,y)$ contain a number of known polynomials (see [4], [10], [9], [11], [12], [13], [14], [16], [17] and [18]).

Using the definitions of $H_{n,m,\nu}(x,y)$ and $G_n^{\alpha,m}(x,y)$ given by (1.5) and (1.1)) in (2.1), we find the representation

$$ {}_HG_n^{\nu,\alpha,m}(x,y)=\sum_{k=0}^{n}\frac{n!H_{k,m,\alpha}(x,y)G_{n-k}^{\nu,m}(x,y)}{k!}. \tag{2.2}$$

Some special cases of (2.2) are

$$ {}_HG_n^{\nu,1,m}(x,y)={}_HC_n^{\nu,m}(x,y)=\sum_{k=0}^{n}\frac{n!H_k^m(x,y)C_{n-k}^{\nu,m}(x,y)}{k!}.$$

Here ${}_HC_n^{\nu,m}(x,y)$ are Hermite-Gegenbaurer polynomials of two variables.

$$ {}_HC_n^{1,m}(x,y)={}_HU_n^{m}(x,y)=\sum_{k=0}^{n}\frac{n!H_k^m(x,y)U_{n-k}^{m}(x,y)}{k!},$$

where ${}_HU_n^m(x,y)$ are Hermite-Chebychev polynomials of two variables.

$$ {}_HC_n^{1/2,m}(x,y)={}_HP_n^{m}(x,y)=\sum_{k=0}^{n}\frac{n!H_k^m(x,y)P_{n-k}^{m}(x,y)}{k!},$$

where ${}_HP_n^m(x,y)$ are Hermite-Legendre polynomials of two variables.

As a special case, let $y=0$ and $\alpha=2$ be chosen in (2.1) so that generalized Humbert-Hermite polynomial ${}_HG_n^{\nu,\alpha,m}(x,y)$ of two variables reduces to Humbert-Hermite polynomial ${}_HG_n^{\nu,2,m}(x,0)={}_HG_n^{\nu,m}(x)$ of one variable. Then (2.1) yields the generating function

$$[1-2xt+t^m]^{-\nu}e^{2xt-t^m}=\sum_{n=0}^{\infty}{}_HG_n^{\nu,m}(x)t^n. \tag{2.3}$$

Furthermore, the Hermite-Gegenbaurer (or ultraspherical) polynomials ${}_HC_n^{\nu,2}(x)={}_HC_n^{\nu}(x)$ of one variable, for nonnegative integer ν are given by

$$e^{2xt-t^2}(1-2xt+t^2)^{-\nu}=\sum_{n=0}^{\infty}{}_HC_n^{\nu}(x)\frac{t^n}{n!}. \tag{2.4}$$

Letting $\nu=1/2$ and $\nu=1$ in (2.4) gives

$$e^{2xt-t^2}(1-2xt+t^2)^{-1/2}=\sum_{n=0}^{\infty}{}_HP_n(x)\frac{t^n}{n!}, \tag{2.5}$$

where ${}_HP_n(x)$ are Hermite-Legendre polynomials and

$$e^{2xt-t^2}(1-2xt+t^2)^{-1}=\sum_{n=0}^{\infty}{}_HU_n(x)\frac{t^n}{n!}, \tag{2.6}$$

where ${}_HU_n(x)$ are Hermite-Chebyshev polynomials.

3. On expansions of Hermite-Chebyshev and Hermite-Gegenbaurer polynomials

In this section, we prove several theorems on the expansions of Hermite-Gegenbaurer and Hermite-Chebyshev polynomials of two variables. We will start with (2.1), (2.3) and the special case of (2.1) for $\nu=1$,

$$[1-2(x+y)t+t^m(2xy+1)]^{-1}e^{\alpha(x+y)t-(xy+1)t^m}=\sum_{n=0}^{\infty}{}_HU_n^{\alpha,m}(x,y)\frac{t^n}{n!}, \tag{3.1}$$

which will be used in obtaining the corollaries of the following theorem.

Theorem 3.1. *For $k\in\mathbb{N}$ and $x,y\in\mathbb{C}$, we have*

$$\sum_{r=0}^{n}\frac{H_r^m(\alpha k(x+y),-k(xy+1))G_{n-r}^{\nu k,m}(x,y)}{r!}$$

$$=\sum_{n_1+n_2+\cdots+n_k=n}\frac{{}_HG_{n_1}^{\nu,\alpha,m}(x,y)\,{}_HG_{n_2}^{\nu,\alpha,m}(x,y)\cdots{}_HG_{n_k}^{\nu,\alpha,m}(x,y)}{n_1!n_2!\cdots n_k!}. \tag{3.2}$$

Proof. The definition of ${}_HG_n^{\nu,\alpha,m}(x,y)$ given in (2.1) can be written as

$$\left[[1-2(x+y)t+t^m(2xy+1)]^{-\nu}e^{\alpha(x+y)t-(xy+1)t^m}\right]^k$$
$$=[1-2(x+y)t+t^m(2xy+1)]^{-\nu k}e^{\alpha k(x+y)t-k(xy+1)t^m}$$
$$=\left[\sum_{n=0}^{\infty}{}_HG_n^{\nu,\alpha,m}(x,y)\frac{t^n}{n!}\right]^k.$$

Using (1.4), we can write

$$e^{\alpha k(x+y)t-k(xy+1)t^m}=\sum_{r=0}^{\infty}H_r^m(\alpha k(x+y),-k(xy+1))\frac{t^r}{r!}.$$

Thus it follows that the above result is essentially equivalent to

$$\sum_{n=0}^{\infty}G_n^{\nu k,m}(x,y)t^n\sum_{r=0}^{\infty}H_r^m(\alpha k(x+y),-k(xy+1))\frac{t^r}{r!}$$
$$=\sum_{n=0}^{\infty}\sum_{n_1+n_2+\cdots+n_k=n}\frac{{}_HG_{n_1}^{\nu,\alpha,m}(x,y)\,{}_HG_{n_2}^{\nu,\alpha,m}(x,y)\cdots{}_HG_{n_k}^{\nu,\alpha,m}(x,y)}{n_1!n_2!\cdots n_k!}t^n.$$

A manipulation of this series yields

$$\sum_{n=0}^{\infty}\sum_{r=0}^{n}\frac{H_r^m(\alpha k(x+y),-k(xy+1))G_{n-r}^{\nu k,m}(x,y)}{r!}t^n$$
$$=\sum_{n=0}^{\infty}\sum_{n_1+n_2+\cdots+n_k=n}\frac{{}_HG_{n_1}^{\nu,\alpha,m}(x,y)\,{}_HG_{n_2}^{\nu,\alpha,m}(x,y)\cdots{}_HG_{n_k}^{\nu,\alpha,m}(x,y)}{n_1!n_2!\cdots n_k!}t^n.$$

Now equating coefficients of t^n on both sides of the resulting equation will give the required result. □

Remark 3.2. Setting $\nu=1$ in Theorem 3.1, the result reduces to

Corollary 3.3. *For $k\in\mathbb{N}$ and $x,y\in\mathbb{C}$, we have*

$$\sum_{r=0}^{n}\frac{H_r^m(\alpha k(x+y),-k(xy+1))C_{n-r}^{k,m}(x,y)}{r!}$$
$$=\sum_{n_1+n_2+\cdots+n_k=n}\frac{{}_HU_{n_1}^{\alpha,m}(x,y)\,{}_HU_{n_2}^{\alpha,m}(x,y)\cdots{}_HU_{n_k}^{\alpha,m}(x,y)}{n_1!n_2!\cdots n_k!}. \tag{3.3}$$

Remark 3.4. Setting $\nu=0$ in Theorem 3.1, the result reduces to

Corollary 3.5. *For $k\in\mathbb{N}$ and $x,y\in\mathbb{C}$, we have*

$$\frac{H_n^m(\alpha k(x+y),-k(xy+1))}{n!}$$
$$=\sum_{n_1+n_2+\cdots+n_k=n}\frac{H_{n_1}^{\alpha,m}(x,y)H_{n_2}^{\alpha,m}(x,y)\cdots H_{n_k}^{\alpha,m}(x,y)}{n_1!n_2!\cdots n_k!}. \tag{3.4}$$

Remark 3.6. Setting $\alpha=m=2$, $\nu,y=0$ in Theorem 3.1, the result reduces to a known result of Batahan and Shehata [2, p.50.,Eq.(2.1)].

Corollary 3.7. *For $k\in\mathbb{N}$ and $x\in\mathbb{C}$, we have*

$$\sum_{r=0}^{[\frac{n}{2}]}\frac{(-k)^r(2kx)^{n-2r}}{(n-2r)r!}=\sum_{n_1+n_2+\cdots+n_k=n}\frac{H_{n_1}(x)H_{n_2}(x)\cdots H_{n_k}(x)}{n_1!n_2!\cdots n_k!}. \tag{3.5}$$

Theorem 3.8. *For $k\in\mathbb{N}$ and $X,Y\in\mathbb{C}$, we have*

$$\sum_{r=0}^{n}\frac{H_r^m(\alpha k(X+Y),-k(XY+1))G_{n-r}^{\nu k,m}(X,Y)}{r!}$$
$$=\sum_{n_1+n_2+\cdots+n_k=n}\frac{{}_HG_{n_1}^{\nu,\alpha,m}(X,Y)\,{}_HG_{n_2}^{\nu,\alpha,m}(X,Y)\cdots{}_HG_{n_k}^{\nu,\alpha,m}(X,Y)}{n_1!n_2!\cdots n_k!}, \tag{3.6}$$

where $X=\sum_{i=0}^{k}x_i$ and $Y=\sum_{j=0}^{k}y_j$.

Proof. The definition of ${}_HG_n^{\nu,\alpha,m}(x,y)$ can be written as

$$\Big[[1-2(X+Y)t+t^m(2XY+1)]^{-\nu}\,e^{\alpha(X+Y)t-(XY+1)t^m}\Big]^k$$

$$=[1-2(X+Y)t+t^m(2XY+1)]^{-\nu k}\,e^{\alpha k(X+Y)t-k(XY+1)t^m}$$

$$=\left[\sum_{n=0}^{\infty}{}_HG_n^{\nu,\alpha,m}(x_1+x_2+\cdots+x_k,y_1+y_2+\cdots+y_k)\frac{t^n}{n!}\right]^k.$$

Using (1.4), we can write

$$e^{\alpha k(X+Y)t-k(XY+1)t^m}=\sum_{r=0}^{\infty}H_n^m(\alpha k(X+Y),-k(XY+1))\frac{t^r}{r!}.$$

Thus it follows that the above result is essentially equivalent to

$$\sum_{n=0}^{\infty}G_n^{\nu k,m}(X,Y)t^n\sum_{r=0}^{\infty}H_n^m(\alpha k(X+Y),-k(XY+1))\frac{t^r}{r!}$$

$$=\sum_{n=0}^{\infty}\sum_{n_1+n_2+\cdots+n_k=n}\frac{{}_HG_{n_1}^{\nu,\alpha,m}(X,Y)\,{}_HG_{n_2}^{\nu,\alpha,m}(X,Y)\ldots{}_HG_{n_k}^{\nu,\alpha,m}(X,Y)}{n_1!n_2!\cdots n_k!}t^n.$$

A manipulation of this series yields

$$\sum_{n=0}^{\infty}\sum_{r=0}^{n}\frac{H_r^m(\alpha k(X+Y),-k(XY+1))G_{n-r}^{\nu k,m}(X,Y)}{r!}t^n$$

$$=\sum_{n=0}^{\infty}\sum_{n_1+n_2+\cdots+n_k=n}\frac{{}_HG_{n_1}^{\nu,\alpha,m}(X,Y)\,{}_HG_{n_2}^{\nu,\alpha,m}(X,Y\cdots{}_HG_{n_k}^{\nu,\alpha,m}(X,Y)}{n_1!n_2!\cdots n_k!}t^n.$$

Now equating coefficients of t^n on both sides of the resulting equation will give the required result. □

Remark 3.9. Setting $\nu=1$ in Theorem 3.8, the result reduces to

Corollary 3.10. *For $k\in\mathbb{N}$ and $x,y\in\mathbb{C}$, we have*

$$\sum_{r=0}^{n}\frac{H_r^m(\alpha k(X+Y),-k(XY+1))C_{n-r}^{k,m}(X,Y)}{r!}$$

$$=\sum_{n_1+n_2+\cdots+n_k=n}\frac{{}_HU_{n_1}^{\alpha,m}(X,Y)\,{}_HU_{n_2}^{\alpha,m}(X,Y)\cdots{}_HU_{n_k}^{\alpha,m}(X,Y)}{n_1!n_2!\cdots n_k!}. \tag{3.7}$$

Remark 3.11. Setting $\nu=0$ in Theorem 3.8, the result reduces to

Corollary 3.12. *For $k \in \mathbb{N}$ and $X, Y \in \mathbb{C}$, we have*

$$\frac{H_n^m(\alpha k(X+Y), -k(XY+1))}{n!}$$

$$= \sum_{n_1+n_2+\cdots+n_k=n} \frac{H_{n_1}^{\alpha,m}(X,Y) H_{n_2}^{\alpha,m}(X,Y) \cdots H_{n_k}^{\alpha,m}(X,Y)}{n_1! n_2! \cdots n_k!}. \tag{3.8}$$

Remark 3.13. Setting $\alpha = m = 2$, $\nu = 0$, $x_2 = \cdots x_k = 0$, $y_1 = \cdots y_k = 0$ and replacing x_1 by x in Theorem 3.8, the result reduces to a known result of Batahan and Shehata [2, p.51., Eq.(2.4)].

Corollary 3.14. *For $k \in \mathbb{N}$ and $x \in \mathbb{C}$, we have*

$$\sum_{r=0}^{[\frac{n}{2}]} \frac{(-k)^r (2kx)^{n-2r}}{(n-2r)r!} = \sum_{n_1+n_2+\cdots+n_k=n} \frac{H_{n_1}(x) H_{n_2}(x) \cdots H_{n_k}(x)}{n_1! n_2! \cdots n_k!}. \tag{3.9}$$

Theorem 3.15. *For $k \in \mathbb{N}$ and $x, y \in \mathbb{C}$, we have*

$$\sum_{s=0}^{[\frac{n}{m}]} \frac{(-1)^s (\nu k)_{n-(m-1)s} (2x+2y)^{n-ms} (2xy+1)^s}{s!\ (n-ms)!}$$

$$= \sum_{n_1+n_2+\cdots+n_k=n} G_{n_1}^{\nu,m}(x,y) G_{n_2}^{\nu,m}(x,y) \cdots G_{n_k}^{\nu,m}(x,y). \tag{3.10}$$

Proof. Using the power series of $[1-2(x+y)t+t^m(2xy+1)]^{-k}$ and making the necessary series arrangements gives

$$[1-2(x+y)t+t^m(2xy+1)]^{-\nu k}$$

$$= \sum_{n=0}^{\infty} \sum_{s=0}^{[\frac{n}{m}]} \frac{(-1)^s (\nu k)_{n-(m-1)s} (2x+2y)^{n-ms} (2xy+1)^s}{s!\ (n-ms)!} t^n.$$

In addition to this, we can write

$$[1-2(x+y)t+t^m(2xy+1)]^{-k} = \left[[1-2(x+y)t+t^m(2xy+1)]^{-\nu}\right]^k$$

$$= \left[\sum_{n=0}^{\infty} G_n^{\nu,m}(x,y) t^n\right]^k$$

$$= \sum_{n=0}^{\infty} \sum_{n_1+n_2+\cdots+n_k=n} G_{n_1}^{\nu,m}(x,y) G_{n_2}^{\nu,m}(x,y) \cdots G_{n_k}^{\nu,m}(x,y) t^n.$$

Now equating coefficients of t^n on both sides of the resulting equation will give the required result. □

Remark 3.16. For $\nu = 1$ in Theorem 3.15, the result reduces to

Corollary 3.17. *For $k \in \mathbb{N}$ and $x, y \in \mathbb{C}$, we have*

$$\sum_{s=0}^{[\frac{n}{m}]} \frac{(-1)^s (k)_{n-(m-1)s} (2x+2y)^{n-ms} (2xy+1)^s}{s!\ (n-ms)!}$$

$$= \sum_{n_1+n_2+\cdots+n_k=n} U_{n_1}^m(x,y) U_{n_2}^m(x,y) \cdots U_{n_k}^m(x,y). \tag{3.11}$$

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