Relation theoretic fixed point results for multivalued mappings in rectangular *b*-metrics spaces

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Abstract. In this paper, we introduce the concepts of \mathcal{R} -completeness and weak \mathcal{R}^{\neq} -preserving mapping and employ them to prove some fixed point results for multi-valued mappings satisfying an implicit relation in rectangular *b*-metric spaces. We also deduce a fixed point result for the same in rectangular *b*-metric spaces endowed with graph \mathcal{G} . Furthermore, we adopt some examples to exhibit the utility of our definitions and results.

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1. Introduction

Fixed point theory is one of rapidly growing and flourishing fields in analysis due to its capacious applications in game theory, mathematical economics and engineering etc. The concept of metric space has been investigated by many researchers in different directions by varying the metric conditions leading to the evolution of various types of generalized metric spaces, e.g., partial metric spaces by Matthews [19], quasi metric spaces by Wilson [29], *b*-metric spaces by Bakhtin [8] and Czerwik [11], rectangular metric spaces by Branciari [9], rectangular *b*-metric spaces by George et al. [12], *G*-metric space by Mustafa and Sims [24] and JS-metric spaces by Jleli and Samet [14] etc.

Specifically, Bakhtin [8] defined the b-metric spaces by changing the triangular inequality as follows:

Definition 1.1. [8] Let M be a non-empty set. Then a mapping $\rho : M \times M \rightarrow [0, \infty)$ is called a *b*-metric if for all $x, y, z \in M$ and $b \ge 1$ it satisfies the following conditions:

- (a) $\rho(x, y) = 0$ if and only if x = y;
- (b) $\rho(x, y) = \rho(y, x);$
- (c) $\rho(x,y) \leq b[\rho(x,z) + \rho(z,y)].$

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The space (M, ρ) is known as *b*-metric space.

It can be noticed that each metric space is a *b*-metric space with b = 1. On the other hand, the notion of rectangular metric spaces was introduced by Branciari [9].

Definition 1.2. [9] Let M be a non-empty set. Then a mapping $\rho : M \times M \rightarrow [0, \infty)$ is called a rectangular metric if for all $x, y \in M$, it satisfies the following conditions:

- (i) $\rho(x, y) = 0$ if and only if x = y;
- (ii) $\rho(x, y) = \rho(y, x);$
- (iii) $\rho(x,y) \le \rho(x,z) + \rho(z,w) + \rho(w,y)$, for all distinct $z, w \in M \setminus \{x,y\}$.

The space (M, ρ) is known as rectangular metric space.

Thereafter, inspired by these, R. George et. al. [12] came out with rectangular b-metric spaces and proved an analogue of Banach contraction principle which was later modified by Z. D. Mitrovic [20].

Definition 1.3. [12] Let M be a non-empty set. Then a mapping $\rho : M \times M \rightarrow [0, \infty)$ is called a rectangular *b*-metric if, for all $x, y \in M$ and $b \ge 1$, it satisfies the following conditions:

(a) $\rho_b(x, y) = 0$ if and only if x = y;

(b)
$$\rho_b(x, y) = d(y, x);$$

(c) $\rho_b(x,y) \le b[\rho_b(x,z) + \rho_b(z,w) + \rho_b(w,y)]$, for all distinct $z, w \in M \setminus \{x,y\}$.

The space (M, ρ_b) is known as a rectangular *b*-metric space.

Remark 1.1. Every metric space is a rectangular metric space and every rectangular metric space is a rectangular *b*-metric space with coefficient b = 1.

In a rectangular *b*-metric space, the open ball with center $z \in M$ and radius r is defined as:

$$\mathcal{B}_r(z) = \{ w \in M : \rho_b(z, w) < r \}.$$

The open ball in rectangular *b*-metric space is not necessarily open. The collection \mathfrak{T}_{ρ_b} of all subsets $A \subseteq M$ with the condition that for every $z \in A$ there exists r > 0 such that $\mathcal{B}_r(z) \subseteq A$ forms a topology for (M, ρ_b) . For more details, we refer the reader to [12].

Let 2^M denote the set of all non-empty subsets of M. Suppose that the closure of $A \in 2^M$ is denoted by \overline{A} . Let us take an element $z \in M$, then $z \in \overline{A}$ if and only if $\rho_b(z, A) = 0$. Moreover, A is said to be closed if and only if $A = \overline{A}$.

Definition 1.4. [12] Let (M, ρ_b) be a rectangular *b*-metric space and $\{z_n\}$ a sequence in M. Then

- (a) $\{z_n\}$ is said to be convergent to $z \in M$ if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\rho_b(z_n, z) < \epsilon$, for all n > N.
- (b) $\{z_n\}$ is said to be Cauchy in (M, ρ_b) if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\rho_b(z_n, z_m) < \epsilon$, for all distinct m, n > N.
- (c) (M, ρ_b) is said to be complete rectangular *b*-metric space if every Cauchy sequence in *M* converges to some $z \in M$.

For more details, we refer readers to [12, 26, 20]. Many mathematicians did excellent works in the area of these generalized metric spaces in the recent past (see [16, 2, 22, 23, 6, 21] and others). For a non-empty set M, a point $z \in M$ is said to be a fixed point of a multi-valued mapping $S : M \to 2^M$ if $z \in Sz$. Nadler [25] was the first to introduce fixed point theory for multi-valued mappings.

Recently, several authors extended the branch of fixed point theory by introducing different type of binary relations and proved relation theoretic fixed point results for single-valued and multi-valued mappings in metric and generalized metric spaces. In this direction, Sintunavarat et. al. [27] gave the idea of a multi-valued mapping to be preserved under a binary relation.

In this paper, employing the above idea, we introduce the concept of weak \mathcal{R}^{\neq} -preserving mappings and present a fixed point result for such mappings in rectangular *b*-metric spaces. We furnish an example in support of our result and some consequences. As an application of our main result, we derive fixed point result for multi-valued mappings on a rectangular *b*-metric space endowed with a graph.

Throughout the paper, all notations are used in their natural meaning.

2. Preliminaries

In order to prove our results, the following definitions, notions and results are used. In the sequel, M is a non-empty set and $S: M \to 2^M$ a multi-valued mapping.

Let (M, ρ) be a metric space, CL(M) and CB(M) denote the family of all non-empty closed and closed and bounded subsets of (M, ρ) , respectively. Then, for $z \in M$ and $A, B \in CB(M)$, we have

$$H(A, B) = \max\{\delta(A, B), \delta(B, A)\},\$$

where $\rho(z, A) = \inf\{\rho(z, a) : a \in A\}$ and $\delta(A, B) = \sup\{\rho(a, B) : a \in A\}.$

Lemma 2.1. [25] Let (M, ρ) be a metric space and $A, B \in CL(M)$. Then for each $\epsilon > 0$ and $a \in A$ there exists $b \in B$ such that $\rho(a, b) \leq H(A, B) + \epsilon$.

Definition 2.1. [10] Let (M, ρ) be a *b*-metric space with $b \ge 1$. Then

(a) a multi-valued mapping $S: M \to CL(M)$ is continuous iff for each $z \in M$ and $\{z_n\} \subseteq M$ with $\{z_n\} \to z$, we have $\lim_{n\to\infty} H(Sz_n, Sz) = 0$. (b) a multi-valued mapping $S: M \to CL(M)$ is h-upper semi-continuous iff for each $z \in M$ and $\{z_n\} \subseteq M$ with $\{z_n\} \to z$, we have

$$\lim_{n \to \infty} \delta(Sz_n, Sz) = 0.$$

(c) a function $f: M \to [0, \infty)$ is said to be upper semi-continuous iff for each $z \in M$ and $\{z_n\} \subseteq M$ with $\{z_n\} \to z$, we have

$$\limsup_{n \to \infty} f z_n \le f z.$$

(d) a function $f: M \to [0, \infty)$ is said to be lower semi-continuous iff for each $z \in M$ and $\{z_n\} \subseteq M$ with $\{z_n\} \to z$, we have

$$\liminf_{n \to \infty} f z_n \ge f z.$$

Remark 2.1. Let (M, ρ) be a complete *b*-metric space and $S: M \to CL(M)$ a multi-valued map. If S is continuous, then it is *h*-upper semi-continuous.

Let M be a non-empty set. A binary relation \mathcal{R} on M is a non-empty subset of $M \times M$. We write $(z_1, z_2) \in \mathcal{R}$ (sometimes $z_1\mathcal{R}z_2$), if z_1 is related to z_2 under \mathcal{R} and $(z_1, z_2) \in \mathcal{R}^{\neq}$, whenever $(z_1, z_2) \in \mathcal{R}$ with $z_1 \neq z_2$. It can be very easily seen that \mathcal{R}^{\neq} is also a binary relation on M. \mathcal{R} is the inverse/transpose/dual relation of \mathcal{R} defined by $\mathcal{R}^{-1} = \{(z_1, z_2) \in M \times M : (z_2, z_1) \in \mathcal{R}\}$ and \mathcal{R}^s is the symmetric closure of \mathcal{R} defined by $\mathcal{R}^s = \mathcal{R} \cup \mathcal{R}^{-1}$.

Definition 2.2. [5] Let M be a non-empty set and \mathcal{R} a binary relation on M. A sequence $\{z_n\} \subseteq M$ is said to be \mathcal{R} -preserving if $(z_n, z_{n+1}) \in \mathcal{R}$, for all $n \in \mathbb{N}$.

Definition 2.3. [18] For $z_1, z_2 \in M$, a path of length $l \in \mathbb{N}$ in \mathcal{R} from z_1 to z_2 is a finite sequence $\{p_0, p_1, ..., p_l\} \subseteq M$ such that $p_0 = z_1, p_l = z_2$ and $(p_i, p_{i+1}) \in \mathcal{R}$, for each $0 \leq i \leq l-1$.

Definition 2.4. [4] A subset A of M is said to be \mathcal{R} -connected if for each $z_1, z_2 \in A$, there exists a path in \mathcal{R} from z_1 to z_2 .

Definition 2.5. [3] Let M be a non-empty set and $S: M \to M$. A binary relation \mathcal{R} is said to be S-closed if $(z_1, z_2) \in \mathcal{R}$ implies that $(Sz_1, Sz_2) \in \mathcal{R}$, for all $z_1, z_2 \in M$.

Definition 2.6. [27] Let M be a non-empty set, \mathcal{R} a binary relation on M and $S: M \to 2^M$ a multi-valued mapping. S is said to be a preserving mapping if for each $x \in M$ and $y \in Sx$ with $(x, y) \in \mathcal{R}$, we have $(y, z) \in \mathcal{R}$, for all $z \in Sy$.

3. Implicit Relation

To depict Implicit relation, we present the following class of mappings.

Let Φ denote the class of all continuous mappings $\phi : [0, \infty)^4 \to [0, \infty)$ satisfying the following properties:

- (ϕ 1) ϕ is non-decreasing in first variable and non-increasing in third and fourth variable;
- $(\phi 2) \ \phi(u, v, v, u) \leq 0 \text{ implies } u \leq \lambda v, \ \lambda \in [0, 1).$

Example 3.1. Let $\phi : [0, \infty)^4 \to [0, \infty)$ be defined by the following:

(i)
$$\phi(t_1, t_2, t_3, t_4) = t_1 - \lambda \max\{t_2, t_3, t_4\}$$
, where $\lambda \in [0, 1)$;

- (ii) $\phi(t_1, t_2, t_3, t_4) = t_1 \lambda \max\{t_3, t_4\}$, where $\lambda \in [0, 1)$;
- (iii) $\phi(t_1, t_2, t_3, t_4) = t_1 \lambda \max\{t_2, \frac{t_3 + t_4}{2}\}$, where $\lambda \in [0, 1)$;
- (iv) $\phi(t_1, t_2, t_3, t_4) = t_1 (a_1t_2 + a_2t_3 + a_3t_4)$, where $a_i \ge 0$ for $i \in \{1, 2, 3\}$ and $a_1 + a_2 + a_3 < 1$;
- (v) $\phi(t_1, t_2, t_3, t_4) = t_1 a(t_3 + t_4)$, where $a \in [0, \frac{1}{2})$ and

(vi)
$$\phi(t_1, t_2, t_3, t_4) = t_1 - at_2$$
, where $a \in [0, 1)$.

Then $\phi \in \Phi$.

4. Main Results

Before presenting our main result, we need the following:

Let (M, ρ_b) be a rectangular b-metric space, $CL^{\rho_b}(M)$ and $CB^{\rho_b}(M)$ the family of all non-empty closed and closed and bounded subsets of (M, ρ_b) , respectively. Then for $z \in M$ and $A, B \in CL^{\rho_b}(M)$, we write

$$\rho_b(z, A) = \inf\{\rho_b(z, a) : a \in A\}, \ \delta_{\rho_b}(A, B) = \sup\{\rho_b(a, B) : a \in A\}$$

and

$$H_{\rho_b}(A,B) = \begin{cases} \max\{\delta_{\rho_b}(A,B), \delta_{\rho_b}(B,A)\}, & \text{if maximum exists,} \\ \infty, & \text{otherwise.} \end{cases}$$

Remark 4.1. [15] If (M, ρ_b) is a rectangular *b*-metric space, then $(CB^{\rho_b}(M), H_{\rho_b})$ need not be a rectangular *b*-metric space.

Lemma 4.1. Let (M, ρ_b) be a rectangular b-metric space and $A, B \in CL^{\rho_b}(M)$. Then for each $\epsilon > 0$ and $a \in A$ there exists $b \in B$ such that $\rho_b(a, b) \leq H_{\rho_b}(A, B) + \epsilon$.

Proof. Suppose, on the contrary, that there exists $\epsilon > 0$ and $a \in A$ such that, for all $b \in B$, we have $\rho_b(a, b) > H_{\rho_b}(A, B) + \epsilon$. Then taking $\inf_{b \in B}$, we get $\rho_b(a, B) \ge H_{\rho_b}(A, B) + \epsilon$, but we have $H_{\rho_b}(A, B) \ge \rho_b(a, B)$. Hence, we get $\epsilon \le 0$, which is a contradiction.

Similarly, we can prove the following lemma.

Lemma 4.2. Let (M, ρ_b) be a rectangular b-metric space, $A, B \in CL^{\rho_b}(M)$ and r > 1. Then for each $z \in M$ and $A \in 2^M$ there exists $a \in A$ such that $\rho_b(z, a) \leq r\rho_b(z, A)$.

Now, we define relation theoretic versions of some well known notions.

Definition 4.1. Let (M, ρ_b, \mathcal{R}) be a rectangular *b*-metric space and \mathcal{R} a binary relation on M. Then

- (a) (M, ρ_b, \mathcal{R}) is \mathcal{R} -complete if every \mathcal{R} -preserving Cauchy sequence in M converges (with respect to \mathfrak{T}_{ρ_b}) to a point in M.
- (b) (M, ρ_b, \mathcal{R}) is \mathcal{R} -regular if for a sequence $\{z_n\}$ in M such that $(z_n, z_{n+1}) \in \mathcal{R}$, for all $n \in \mathbb{N}$, and $\{z_n\} \to z$, for some $z \in M$, then $(z_n, z) \in \mathcal{R}$, for all $n \in \mathbb{N}$.

Definition 4.2. Let (M, ρ_b, \mathcal{R}) be a rectangular *b*-metric space with $b \ge 1$ and binary relation \mathcal{R} . Then

(a) a multi-valued mapping $S: M \to 2^M$ is \mathcal{R} -continuous iff for each $z \in M$ and \mathcal{R} -preserving sequence $\{z_n\}$ such that $\{z_n\} \to z$ we have

$$\lim_{n \to \infty} H_{\rho_b}(Sz_n, Sz) = 0.$$

(b) a multi-valued mapping $S: M \to 2^M$ is \mathcal{R} -h-upper semi-continuous iff for each $z \in M$ and \mathcal{R} -preserving sequence $\{z_n\}$ such that $\{z_n\} \to z$ we have

$$\lim_{n \to \infty} \delta_{\rho_b}(Sz_n, Sz) = 0.$$

Remark 4.2. \mathcal{R} -continuity of S implies its \mathcal{R} -h-upper semi-continuity.

Definition 4.3. Let M be a non-empty set, \mathcal{R} a binary relation on M and $S: M \to 2^M$ a multi-valued mapping. We say S to be a weak \mathcal{R}^{\neq} -preserving mapping if for each $x \in M$ and $y \in Sx$ with $(x, y) \in \mathcal{R}^{\neq}$, we have $(y, z) \in \mathcal{R}^{\neq}$, for all $z \neq y \in Sy$.

Remark 4.3. Every preserving mapping is weak \mathcal{R}^{\neq} -preserving but the converse need not be true in general.

Example 4.1. Let $M = \{0, 1, 2\}$ with $\mathcal{R} = \{(0, 0), (0, 1), (1, 1), (1, 2), (2, 1)\}$. Let $S : M \to 2^M$ be defined by:

$$S(0) = \{0\}, S(1) = \{2\} \text{ and } S(2) = \{1, 2\}.$$

Then S is a weak \mathcal{R}^{\neq} -preserving mapping but it is not preserving. Indeed, for z = 1 and $2 \in S1$, we have $(1, 2) \in \mathcal{R}$, but there exists $2 \in S(2)$ such that $(2, 2) \notin \mathcal{R}$.

Now, we are ready to state our main result.

Theorem 4.1. Let (M, ρ_b, \mathcal{R}) be a rectangular b-metric space with coefficient $b \geq 1$ equipped with a binary relation \mathcal{R} such that ρ_b is continuous on $M \times M$ and $S : M \to CL^{\rho_b}(M)$ a multi-valued mapping satisfying the following assertions:

- (i) there exists $z_0 \in M$ and $z_1 \in Sz_0$ such that $(z_0, z_1) \in \mathbb{R}^{\neq}$;
- (ii) S is weak \mathcal{R}^{\neq} -preserving;
- (iii) (M, ρ_b) is \mathcal{R}^{\neq} -complete;
- (iv) S satisfies

(4.1)
$$\phi(H_{\rho_b}(Sz, Sw), \rho_b(z, w), \rho_b(z, Sz), \rho_b(w, Sw)) \le 0,$$

for all $z, w \in M$ with $(z, w) \in \mathcal{R}^{\neq}$;

- (v) (M, ρ_b) is \mathcal{R}^{\neq} -regular.
- Then S has a fixed point.

Proof. Owing to condition (i), choose $z_0 \in M$ and $z_1 \in Sz_0$ such that $(z_0, z_1) \in \mathbb{R}^{\neq}$. If $z_1 \in Sz_1$, then z_1 is a fixed point of S and we are done. Otherwise, by (4.1), we obtain

$$\phi(H_{\rho_b}(Sz_0, Sz_1), \rho_b(z_0, z_1), \rho_b(z_0, Sz_0), \rho_b(z_1, Sz_1)) \le 0,$$

which by $(\phi 1)$ implies that

$$\phi(H_{\rho_b}(Sz_0, Sz_1), \rho_b(z_0, z_1), \rho_b(z_0, z_1), H_{\rho_b}(Sz_0, Sz_1)) \le 0$$

Hence, by $(\phi 2)$, we have

$$H_{\rho_b}(Sz_0, Sz_1) \le \lambda \rho_b(z_0, z_1).$$

Now, choose $\epsilon = \frac{1}{2}(1-\lambda)$ and, by Lemma 4.1, there exists $z_2 \in Sz_1$ such that

$$\rho_b(z_1, z_2) \le H_{\rho_b}(Sz_0, Sz_1) + \epsilon \rho_b(z_0, z_1) \\ \le \lambda \rho_b(z_0, z_1) + \epsilon \rho_b(z_0, z_1) \\ = q \rho_b(z_0, z_1),$$

where $q = \lambda + \epsilon$.

Since $z_0 \in M$ and $z_1 \in Sz_0$ such that $(z_0, z_1) \in \mathcal{R}^{\neq}$, so from condition (ii), we have $(z_1, z_2) \in \mathcal{R}^{\neq}$. Similarly, if $z_2 \in Sz_2$, we are done. Otherwise, we choose $z_3 \in Sz_2$ ($(z_2, z_3) \in \mathcal{R}^{\neq}$, by condition (ii)) such that

$$\rho_b(z_2, z_3) \le H_{\rho_b}(Sz_1, Sz_2) + \epsilon \rho_b(z_1, z_2) \\ \le \lambda \rho_b(z_1, z_2) + \epsilon \rho_b(z_1, z_2) \\ = q \rho_b(z_1, z_2).$$

Inductively, we construct a sequence $\{z_n\} \subseteq M$ satisfying the following:

$$z_{n-1} \notin Sz_{n-1}, \ z_n \in Sz_{n-1}, \ (z_{n-1}, z_n) \in \mathcal{R}^{\neq}$$

and

(4.2)
$$\rho_b(z_n, z_{n+1}) \le q \rho_b(z_{n-1}, z_n),$$

for each $n \in \mathbb{N}$. As $\epsilon = \frac{1}{2}(1-\lambda)$, so we get

$$q = \frac{1}{2}(1+\lambda) < 1.$$

Since q < 1, we have $\lim_{n\to\infty} q^n = 0$, and hence there exists $N_1 \in \mathbb{N}$ such that (for all $k \ge N_1$)

$$(4.3) 0 \le bq^k < 1.$$

We also have $z_n \neq z_{n+k}$, for all $k \ge 1$ and $n \in \mathbb{N}_0$. By using the condition that if $z_n = z_{n+k}$, for some $k \ge 1$ and $n \in \mathbb{N}_0$, then from $Sz_n = Sz_{n+k}$ and (4.2), we get

$$\rho_b(z_n, Sz_n) = \rho_b(z_{n+k}, Sz_{n+k})$$

$$\leq \rho_b(z_{n+k}, z_{n+k+1})$$

$$\leq q^k \rho_b(z_n, z_{n+1})$$

$$< q^k r \rho_b(z_n, Sz_n)), \text{ for some } r > 1 \text{ (by lemma 4.2)}$$

As in (4.3), there exists some $N_2 \in \mathbb{N}$ such that

$$0 \le rq^k < 1$$
, for all $k \ge N_2$.

So, for $k \geq N_2$, we get

$$\rho_b(z_n, Sz_n) < \rho_b(z_n, Sz_n),$$

a contradiction. Thus, we have $z_n \neq z_m$, for all distinct $m, n \in \mathbb{N}_0$. Now, for all distinct $m, n \in \mathbb{N}_0$ and $k \geq \max\{N_1, N_2\}$, using condition (iii) of rectangular *b*-metric spaces and (4.2), we obtain

$$\rho_b(z_m, z_n) \le b[\rho_b(z_m, z_{m+k}) + \rho_b(z_{m+k}, z_{n+k}) + \rho_b(z_{n+k}, z_n)]$$

$$\le b[q^m \rho_b(z_0, z_k) + q^k \rho_b(z_m, z_n) + q^n \rho_b(z_0, z_k)].$$

This implies

$$(1 - bq^k)\rho_b(z_m, z_n) \le b(q^m + q^n)\rho_b(z_0, z_k),$$

yielding thereby

$$\rho_b(z_m, z_n) \le \frac{b(q^m + q^n)}{1 - bq^k} \rho_b(z_0, z_k)$$

$$\longrightarrow 0, \qquad \text{as } m, n \to \infty.$$

Therefore, $\{z_n\}$ is a Cauchy sequence which is \mathcal{R}^{\neq} -preserving. By \mathcal{R}^{\neq} -completeness of M, there exists $z^* \in M$ such that

(4.4)
$$\lim_{n \to \infty} z_n = z^*.$$

Next, we prove that z^* is the fixed point of S. Firstly, employing condition (v), we obtain $(z_n, z) \in \mathcal{R}$, for all $n \in \mathbb{N}$. Therefore (4.1) gives

$$\phi(H_{\rho_b}(Sz_n, Sz^*), \rho_b(z_n, z^*), \rho_b(z_n, Sz_n), \rho_b(z^*, Sz^*)) \le 0.$$

Now, by condition $(\phi 1)$, we get

(4.5)
$$\phi(\rho_b(z_{n+1}, Sz^*), \rho_b(z_n, z^*), \rho_b(z_n, z_{n+1}), \rho_b(z^*, Sz^*)) \le 0.$$

Taking limit in (4.5) and using continuity of ρ_b , we obtain

$$\phi(\rho_b(z^*, Sz^*), 0, 0, \rho_b(z^*, Sz^*)) \le 0,$$

which on applying $(\phi 2)$ yields

$$\rho_b(z^*, Sz^*) \le 0.$$

Therefore, $z^* \in Sz^*$ (as Sz^* is closed) and hence, S has a fixed point.

We give the following example to support our result.

Example 4.2. Let $M = \{1, 2, 3, 4, 5\}$ equipped with the binary relation

$$\mathcal{R} = \{(2,3), (3,4), (4,5), (5,1), (5,2), (5,4)\}$$

and define $\rho_b: M \times M \to [0, \infty)$ by:

$$\begin{split} \rho_b(z,z) &= 0, \text{ for all } z \in M, \\ \rho_b(z,w) &= \rho_b(w,z), \text{ for all } z, w \in M, \\ \rho_b(1,2) &= \rho_b(2,4) = \alpha, \ \rho_b(1,3) = \rho_b(3,5) = 2\alpha, \\ \rho_b(1,4) &= \rho_b(2,5) = 3\alpha, \ \rho_b(1,5) = \rho_b(2,3) = \rho_b(3,4) = 20\alpha, \end{split}$$

where $\alpha > 0$. Then (M, ρ_b) is a rectangular *b*-metric space with b = 4. Let us consider a multi-valued mapping $S: M \to CL^{\rho_b}(M)$ defined by:

$$Sz = \begin{cases} \{z, z+1\}, & \text{if } z \in \{1, 2, 3, 4\}; \\ \{1, 2, 4\}, & \text{if } z = 5. \end{cases}$$

Then S is a weak \mathcal{R}^{\neq} -preserving mapping. We show that the contraction condition of Theorem 4.1 is also satisfied. We consider all the cases when $(z, y) \in \mathcal{R}^{\neq}$ and see that

$$H_{\rho_b}(Sz, Sw) \le \lambda \max\{\rho_b(z, w), \rho_b(z, Sz), \rho_b(w, Sw)\}, \text{for all } z, w \in M \text{ with}$$
$$(z, w) \in \mathcal{R}^{\neq},$$

for all $\lambda \in [\frac{3}{20}, 1)$. Thus, all conditions of Theorem 4.1 with $\phi(t_1, t_2, t_3, t_4) = t_1 - \lambda \max\{t_2, t_3, t_4\}$, where $\lambda \in [0, 1)$, are satisfied and hence S has a fixed point.

For a mapping $S: M \to 2^M$, let $\rho_b^S: M \to [0, \infty)$ be the mapping defined by: $\rho_b^S(z) = \rho_b(z, Sz)$. We give another variant of Theorem 4.1 by replacing condition (v) with (v)^{*} as follows:

Theorem 4.2. Let (M, ρ_b, \mathcal{R}) be a rectangular b-metric space with coefficient $b \geq 1$ and binary relation \mathcal{R} and $S : M \to CL^{\rho_b}(M)$ a multi-valued mapping. If we replace condition (v) of Theorem 4.1 (with all others remaining the same) by the following:

 $(v)^*$ either S is \mathcal{R}^{\neq} -h-upper semi-continuous or ρ_b^S is lower semi-continuous.

Then S has a fixed point.

Proof. The proof runs along the same lines as that of Theorem 4.1 upto (4.4). Next, to prove that z^* is a fixed point, we use condition $(v)^*$ and have

$$\rho_b(z^*, Sz^*) \le b[\rho_b(z^*, z_n) + \rho_b(z_n, z_{n+1}) + \rho_b(z_{n+1}, Sz^*)] \\ \le b[\rho_b(z^*, z_n) + \rho_b(z_n, z_{n+1}) + \delta(Sz_n, Sz^*)].$$

Letting $n \to \infty$ and using \mathcal{R}^{\neq} -h-upper semi-continuity of S, we obtain $\rho_b(z^*, Sz^*) = 0$, i.e., $z^* \in Sz^*$.

Further, if ρ_b^S is lower semi-continuous, then we have $\rho_b^S(z^*) \leq \liminf_{n \to \infty} \rho_b^S(z_n)$, i.e.,

$$\rho_b(z^*, Sz^*) \le \liminf_{n \to \infty} \rho_b(z_n, Sz_n)$$
$$\le \lim_{n \to \infty} \rho_b(z_n, z_{n+1}) = 0.$$

Hence, $\rho_b(z^*, Sz^*) = 0$ and z^* is a fixed point of S.

Now, we deduce the following corollaries in order to present the utility and unifiedness of our result.

Corollary 4.1. Let (M, ρ_b, \mathcal{R}) be a rectangular b-metric space with coefficient $b \geq 1$ equipped with binary relation \mathcal{R} such that ρ_b is continuous on $M \times M$ and $S: M \to CL^{\rho_b}(M)$ a multi-valued mapping satisfying the following assertions:

- (i) there exists $z_0 \in M$ and $z_1 \in Sz_0$ such that $(z_0, z_1) \in \mathbb{R}^{\neq}$;
- (ii) S is weak \mathcal{R}^{\neq} -preserving;
- (iii) (M, ρ_b) is \mathcal{R}^{\neq} -complete;
- (iv) for all $z, w \in M$ with $(z, w) \in \mathbb{R}^{\neq}$, S satisfies anyone of the following:
 - (A) $\begin{aligned} H_{\rho_b}(Sz, Sw) &\leq \lambda \max\{\rho_b(z, w), \rho_b(z, Sz), \rho_b(w, Sw)\}, \\ where \ \lambda \in [0, 1); \end{aligned}$
 - (B) $H_{\rho_b}(Sz, Sw) \le \lambda \max\{\rho_b(z, Sz), \rho_b(w, Sw)\},$ where $\lambda \in [0, 1);$

- (C) $H_{\rho_b}(Sz, Sw) \le \lambda \max\{\rho_b(z, w), \frac{\rho_b(z, Sz) + \rho_b(w, Sw)}{2}\},$ where $\lambda \in [0, 1);$
- (D) $\begin{aligned} H_{\rho_b}(Sz,Sw) &\leq a_1\rho_b(z,w) + a_2\rho_b(z,Sz) + a_3\rho_b(w,Sw), \\ where \ a_i &\geq 0, \ for \ all \ i = 1,2,3, \ where \ a_1 + a_2 + a_3 < 1; \end{aligned}$
- (E) $\begin{aligned} H_{\rho_b}(Sz,Sw) &\leq \lambda [\rho_b(z,Sz) + \rho_b(w,Sw)], \\ where \ \lambda \in [0,\frac{1}{2}); \end{aligned}$

(F)
$$H_{\rho_b}(Sz, Sw) \leq \lambda \rho_b(z, w),$$

where $\lambda \in [0, 1);$

(v) (M,d) is \mathcal{R}^{\neq} -regular or

 $(v)^*$ either S is \mathcal{R}^{\neq} -h-upper semi-continuous or ρ_b^S is lower semi-continuous.

Then S has a fixed point.

Proof. The result follows immediately from Theorem 4.1, 4.2 and Example 3.1. \Box

Next, we deduce the following result for single-valued mappings from Corollary 4.1.

Corollary 4.2. Let (M, ρ_b, \mathcal{R}) be a rectangular b-metric space with coefficient $b \geq 1$ equipped with binary relation \mathcal{R} such that ρ_b is continuous on $M \times M$ and $S: M \to M$ a mapping satisfying the following assertions:

- (i) there exists $z_0 \in M$ such that $(z_0, Sz_0) \in \mathcal{R}^{\neq}$;
- (ii) \mathcal{R} is S-closed;
- (iii) (M, ρ_b) is \mathcal{R}^{\neq} -complete;
- (iv) S satisfies (4.6) $\rho_b(Sz, Sw) \leq \lambda \rho_b(z, w), \text{ for all } z, w \in M \text{ with } (z, w) \in \mathbb{R}^{\neq} \text{ and } \lambda \in [0, 1);$
- (v) (M, ρ_b) is \mathcal{R}^{\neq} -regular.

Then S has a fixed point. Moreover, the fixed point is unique if the following condition is satisfied:

(vi) Fix(S) is \mathcal{R}^s -connected.

Proof. The existence part follows directly from Corollary 4.1 by taking into consideration the contraction condition (F) in assertion (iv). For the uniqueness of fixed point, suppose that $z^*, \bar{z} \in Fix(S)$ such that $z^* \neq \bar{z}$. Then by condition (vi), there exists a path in \mathcal{R}^s , say $\{u_0, u_1, u_2, ..., u_k\} \subseteq Fix(S)$ of some finite length k from z^* to \bar{z} satisfying

$$u_0 = z^*, \ u_k = \overline{z} \text{ and } [u_i, u_{i+1}] \in \mathcal{R}, \text{ for all } 0 \le i \le k-1.$$

If $u_i = u_{i+1}$, for each *i*, then $z^* = \overline{z}$, a contradiction. Thus, if $u_p \neq u_{p+1}$, for some $p \in \{0, 1, 2, ..., k-1\}$, then $Su_p = u_p$ and $Su_{p+1} = u_{p+1}$, so (4.6) yields (with $z = u_p$ and $w = u_{p+1}$)

$$\rho_b(Su_p, Su_{p+1}) \le \lambda \rho_b(u_p, u_{p+1}), \ \lambda \in [0, 1),$$

which implies

$$\rho_b(u_p, u_{p+1}) \le \lambda \rho_b(u_p, u_{p+1}), \ \lambda \in [0, 1),$$

a contradiction. Hence, we arrive at the conclusion.

In the below example, we demonstrate that Corollary 4.2 is a more generalized version of Theorem 2.1 of [20].

Example 4.3. Let $M = A \cup B$, where $A = \left\{0, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}\right\}$ and B = [3, 4]. Define a mapping $\rho_b : M \times M \to [0, \infty)$ as: $\rho_b(0, \frac{1}{3}) = \rho_b(\frac{1}{3}, \frac{1}{5}) = \rho_b(0, \frac{1}{7}) = 1;$ $\rho_b(0, \frac{1}{5}) = \rho_b(\frac{1}{3}, \frac{1}{7}) = 4;$ $\rho_b(\frac{1}{5}, \frac{1}{7}) = 3;$ $\rho_b(z, w) = |z - w|$, for all $z, w \in B$ and $z \in A, w \in B$. Then the above metric is clearly a rectangular *b*-metric with $s = \frac{4}{3}$. Now, define a mapping $S : M \to M$ by:

$$Sz = \begin{cases} \frac{1}{5}, & \text{if } z \in [3,4] \cup \{\frac{1}{3}\};\\ 0, & \text{if } z \in \{0,\frac{1}{7}\};\\ \frac{1}{7}, & \text{if } z = \frac{1}{5} \end{cases}$$

and a binary relation \mathcal{R} by: $\mathcal{R} = \{(0,0), (\frac{1}{5}, \frac{1}{5}), (\frac{1}{7}, \frac{1}{7}), (0, \frac{1}{5}), (0, \frac{1}{7}), (\frac{1}{5}, \frac{1}{7})\} \cup \{(z,w): z, w \in B\}$. Then we have $\rho_b(Sz, Sw) \leq \lambda \rho_b(Sz, Sw)$, for all $\lambda \in [\frac{1}{3}, 1)$ and $z, w \in M$ with $(z,w) \in \mathcal{R}^{\neq}$. Hence, all the requirements of Corollary 4.2 are fulfilled. Thus, S has a unique fixed point. But for $z = \frac{1}{3}$ and $w = \frac{1}{5}$, we obtain

$$3 = \rho_b\left(\frac{1}{5}, \frac{1}{7}\right) = \rho_b\left(S\left(\frac{1}{3}\right), S\left(\frac{1}{5}\right)\right) \le \lambda\rho_b\left(\frac{1}{3}, \frac{1}{5}\right) = \lambda.1 = \lambda,$$

where $\lambda \in [0, 1)$, which is a contradiction. Hence, the result (viz. Theorem 2.1) of [20] can not be applied.

5. Results on rectangular *b*-metric spaces endowed with graph

Let (M, ρ_b) be a rectangular *b*-metric space and \mathcal{D} the diagonal of the Cartesian product $M \times M$. Consider a directed graph \mathcal{G} such that the set $V(\mathcal{G})$ of all vertices of \mathcal{G} coincides with M and the set $E(\mathcal{G})$ of all edges contains all the loops, i.e., $\mathcal{D} \subset E(\mathcal{G})$. We assume that \mathcal{G} has no parallel edges and identify \mathcal{G} by the pair $(V(\mathcal{G}), E(\mathcal{G}))$. For details, we refer the reader to [13, 1, 7, 17] and reference therein.

Remark 5.1. We can say that $E(\mathcal{G}) \subseteq M \times M$ is a binary relation which is reflexive.

In the sequel, we denote a rectangular *b*-metric space (M, ρ_b) endowed with a graph \mathcal{G} by (M, ρ_b, \mathcal{G}) . Also, we denote \mathcal{G}^{\neq} by $(V(\mathcal{G}), E^{\neq}(\mathcal{G}))$, where $E^{\neq}(\mathcal{G}) = \{(z, w) \in E(\mathcal{G}) : z \neq w\}$.

We define the following notions in rectangular *b*-metric spaces.

Definition 5.1. Let M be a non-empty set and \mathcal{G} a graph on M. A sequence $\{z_n\} \subseteq M$ is said to be \mathcal{G} -preserving if $(z_n, z_{n+1}) \in E(\mathcal{G})$ or $(z_{n+1}, z_n) \in E(\mathcal{G})$, for all $n \in \mathbb{N}$.

Definition 5.2. Let (M, ρ_b, \mathcal{G}) be a rectangular *b*-metric space endowed with graph \mathcal{G} and S a self-mapping on M. Then

- (a) (M, ρ_b, \mathcal{G}) is said to be \mathcal{G} -complete if every \mathcal{G} -preserving Cauchy sequence $\{z_n\}$ in M converges in M.
- (b) (M, ρ_b, \mathcal{G}) is called \mathcal{G} -regular if for each \mathcal{G} -preserving sequence $\{z_n\} \subseteq M$ such that $\{z_n\} \to z$, for some $z \in M$, we have $(z_n, z) \in E(\mathcal{G})$ or $(z, z_n) \in E(\mathcal{G})$, for all $n \in \mathbb{N}$.

Definition 5.3. Let (M, ρ_b, \mathcal{G}) be rectangular *b*-metric space with graph \mathcal{G} . Then

(a) a multi-valued mapping $S: M \to 2^M$ is \mathcal{G} -continuous iff for each $z \in M$ and \mathcal{R} -preserving sequence $\{z_n\}$ such that $\{z_n\} \to z$, we have

$$\lim_{n \to \infty} H_{\rho_b}(Sz_n, Sz) = 0.$$

(b) a multi-valued mapping $S: M \to 2^M$ is said to be \mathcal{G} -h-upper semicontinuous iff for each $z \in M$ and \mathcal{G} -preserving sequence $\{z_n\} \subseteq M$ such that $\{z_n\} \to z$, we have

$$\lim_{n \to \infty} \delta_{\rho_b}(Sz_n, Sz) = 0.$$

Tiammee and Suantai [28] introduced the idea of graph preserving mappings as follows:

Definition 5.4. [28] Let M be a non-empty set endowed with graph \mathcal{G} . Then a mapping $S: M \to 2^M$ is called graph preserving if

 $(z, w) \in E(\mathcal{G})$ implies $(u, v) \in E(\mathcal{G})$, for all $u \in Sz$ and $v \in Sw$.

We define a slightly more generalized and weaker version of the above notion, namely weak \mathcal{G}^{\neq} -graph preserving.

Definition 5.5. Let M be non-empty set endowed with graph \mathcal{G} and $S: M \to 2^M$. Then we say that the mapping S is weak \mathcal{G}^{\neq} -graph preserving if for each $x \in M$ and $y \in Sx$ with $(x, y) \in E^{\neq}(\mathcal{G})$, we have $(y, z) \in E^{\neq}(\mathcal{G})$, for all $z \neq y \in Sy$.

Remark 5.2. Every graph preserving mapping is weak \mathcal{G}^{\neq} -graph preserving.

The below example shows that our Definition is weaker than Definition 5.4.

Example 5.1. Let $\mathcal{G} = (\mathbb{N}, E(\mathcal{G}))$, where $E(\mathcal{G}) = \{(2n-1, 2n) : n \in \mathbb{N}\} \cup \{(2n, 1) : n \in \mathbb{N}\} \cup (1, 3)$. Define $S : \mathbb{N} \to 2^{\mathbb{N}}$ by:

$$Sn = \begin{cases} \{2k, 2k+1\}, & \text{if } n = 2k-1, \ k \in \mathbb{N}, \\ \{1\}, & \text{if } n = 2k, \ k \in \mathbb{N}. \end{cases}$$

S is weak \mathcal{G}^{\neq} -graph preserving.

Indeed, for n = 2k-1, there exists $2k \in S(2k-1)$ such that $(2k-1, 2k) \in E^{\neq}(\mathcal{G})$ and we have $(2k, 1) \in E^{\neq}(\mathcal{G})$.

For n = 2k, there exists $1 \in S(2k)$ such that $(2k, 1) \in E^{\neq}(\mathcal{G})$. We have $S1 = \{2, 3\}$ and $(1, 2), (1, 3) \in E^{\neq}(\mathcal{G})$.

But S is not graph preserving as $(1,3) \in E(\mathcal{G})$, $S1 = \{2,3\}$ and $S3 = \{4,5\}$ but $(2,4), (2,5), (3,5) \notin E(\mathcal{G})$.

Now, in view of Remark 5.1, we present the main result of this section.

Theorem 5.1. Let (M, ρ_b, \mathcal{G}) be a rectangular b-metric space with coefficient $b \geq 1$ endowed with graph \mathcal{G} such that ρ_b is continuous on M and $S : M \to CL^{\rho_b}(M)$ a multi-valued mapping satisfying the following assertions:

- (i) there exists $z_0 \in M$ and $z_1 \in Sz_0$ such that $(z_0, z_1) \in E^{\neq}(\mathcal{G})$;
- (ii) S is weak \mathcal{G}^{\neq} -graph preserving;
- (iii) (M, ρ_b, \mathcal{G}) is \mathcal{G}^{\neq} -complete;
- (iv) S satisfies

 $\phi(H_{\rho_b}(Sz, Sw), \rho_b(z, w), \rho_b(z, Sz), \rho_b(w, Sw)) \le 0,$

for all $z, w \in M$ such that $(z, w) \in E^{\neq}(\mathcal{G})$;

(v) (M, ρ_b, \mathcal{G}) is \mathcal{G}^{\neq} -regular.

Then S has a fixed point.

Proof. Let us define a binary relation \mathcal{R} on M in the following way:

 $(z, w) \in \mathcal{R}$ if and only if $(z, w) \in E(\mathcal{G})$,

for all $z, w \in M$. Then clearly all the conditions of Theorem 4.1 are satisfied. Thus, the existence of a fixed point of S is followed.

Similarly, we give an analogous result of Theorem 4.2 in the following way:

Theorem 5.2. Let (M, ρ_b, \mathcal{G}) be a rectangular b-metric space with coefficient $b \geq 1$ endowed with graph \mathcal{G} and $S : M \to CL^{\rho_b}(M)$ a multi-valued mapping. If we replace condition (v) of Theorem 5.1 ((with all others remaining the same) by the following:

 $(v)^*$ either S is \mathcal{G}^{\neq} -h-upper semicontinuous or ρ_b^S is lower semicontinuous. Then S has a fixed point.

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