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Defining some new n-tuple sequence spaces related to l_p space with the help of Orlicz function

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Abstract. In this paper, we introduce and study the n-sequence space $l^n_\infty(M, q)$ and $m^n(M, \phi, q)$ by using the Orlicz function M. We show that the spaces are seminormed and $m^n(M, \phi, q)$ is complete. The inclusion relations involving the spaces have also been obtained. Further, we relate the space $m^n(M, \phi, q)$ to p-summable spaces.

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1. Introduction

The Banach space gave birth to many useful concept in mathematics, Orlicz space is no different. After the development of Lebesgue theory of integration, Z. W Birnbaum and W. Orlicz introduces Orlicz space as the generalization of $L^p,\ 1 < p < \infty$ [2]. In the definition of L^p, they replaced x^p by a more general convex function ϕ. Later Orlicz used this idea to construct the space L^M.

The space $m(\phi)$ (along with its dual space $n(\phi)$) was introduced by Sargent [11] and several interesting properties and results were discussed. This space $m(\phi)$ is very interesting and important space as it has all $l_p, (1 \leq p \leq \infty)$ spaces as special cases depending upon the choice of the sequence ϕ. Further these two spaces $m(\phi)$ and $n(\phi)$ were studied by several authors in [1, 3, 8, 14]. Malkowsky and Mursaleen [5, 6] gave the matrix transformation between these spaces. Mursaleen [7] also studied the geometrical properties related to l^p space.

Let w be the set of all complex sequences and $\phi = \{\phi \in w : 0 < \phi_1 \leq \phi_n \leq \phi_{n+1}$ and $(n+1)\phi_n \geq n\phi_n\}$. Further let P_s denotes the class of all subsets of $\mathbb{N}$ which do not contain more than s elements. For each $\phi \in \phi$, Sargent [14] defined the sequence space

$$m(\phi) = \left\{(x_k) \in w : \sup_{s \geq 1, \sigma \in P_s} \tfrac{1}{\phi_s} \sum_{k \in \sigma} |x_k| < \infty\right\}.$$

A comprehensive study of Orlicz space was done by Lindenstrauss and Tzafriri [4] as they construct the sequence space l^M,

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$$l^M = \left\{(x_k) \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right), \text{ for some } \rho > 0\right\},$$

and prove that it contains a subspace isomorphic to l_p $(1 \le p < \infty)$. Many others like Prashar and Chaudhry [10], Mursaleen et al. [9] have introduced different classes of sequence spaces defined by Orlicz function.

In 2016, Savas [12] introduced the double sequence space $m''(M, \phi, q)$. Tripathy et al. [13] found some interesting results related to the n-sequence space.

In this paper, we took the idea of $m(\phi)$ and generalize the concept to the n-sequence space and obtain some inclusion relation involving $m^n(M, \phi, q)$. Savas [12] proved that the result holds for the space of double sequences, here we show that it is, in fact, true for all $n \in \mathbb{N}$.

2. Definition and preliminaries

An Orlicz function is a function $M : [0, \infty) \longrightarrow [0, \infty)$ which is continuous, non-decreasing and convex with $M(0) = 0,\ M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$.

If convexity of M is replaced by $M(x + y) \le M(x) + M(y)$, then it is called a modulus function. An Orlicz function M can always be represented in the integral form $M(x) = \int_0^x \eta(t)dt$, where η is known as the kernel of M, is right differentiable for $t \ge 0,\ \eta(t) > 0,\ \eta$ is non-decreasing and $\eta(t) \longrightarrow \infty$ as $t \longrightarrow \infty$.

An Orlicz function M is said to satisfy Δ_2-condition for all values of x, if there exists a constant $K > 0$, such that $M(2x) \le KM(x)$ for all $x \ge 0$.

Remark 2.1. An Orlicz function M satisfies the inequality $M(\lambda x) \le \lambda M(x)$ for all λ with $0 < \lambda < 1$.

Throughout the article the set of all n-sequences will be denoted by w^n. Also whenever we say limit of n-sequence, we mean limit in Pringsheim's sense.

Definition 2.2. An n-sequence $x = (x_{i_1,i_2,\ldots,i_n})$ *such that* $i_1, i_2, ..., i_n \in \mathbb{N}$ is said to be bounded if $\sup\limits_{i_1,i_2,\ldots,i_n} |x_{i_1,i_2,\ldots,i_n}| < \infty$. The space of all bounded n-sequences is denoted by l_∞^n.

Definition 2.3. Consider an n-sequence $x = (x_{i_1,i_2,\ldots,i_n})$ *such that* $i_1, i_2, ..., i_n \in \mathbb{N}$. If for a given $\epsilon > 0,\ \exists\ n_0 = n_0(\epsilon) \in \mathbb{N}$ such that

$$|x_{i_1,i_2,\ldots,i_n} - L| < \epsilon, \quad \forall\ i_1, i_2, ..., i_n > n_0,$$

then L is called the limit of $(x_{i_1,i_2,\ldots,i_n})$ in Pringsheim's sense and we say that n-sequence x is convergent in Pringshiem's sense to the limit L and we write $P - \lim\limits_{i_1,i_2,\ldots,i_n} x = L$.

Definition 2.4. An n-sequence $x = (x_{i_1,i_2,\ldots,i_n,})$ is said to be a Cauchy sequence if for a given $\epsilon > 0$ there exists $n_0(\epsilon) \in \mathbb{N}$ such that

$$|x_{m_1,m_2,\ldots,m_n} - x_{i_1,i_2,\ldots,i_n}| < \epsilon, \quad m_j \ge i_j \ge n_0\ (1 \le j \le n).$$

3. Main Result

In this section, we introduce the sequence space $l^n_\infty(M,q)$ and $m^n(M,\phi,q)$ and prove some results about them.

The space of all convergent n-sequences in Pringsheim sense is denoted by c^n. Let $P_{r_1,r_2,...,r_n}$ denote the class of all subsets of $\mathbb{N}^n$ that do not contain more than $r_1 \cdot r_2 \cdot ... \cdot r_n$ elements. We take $\{\phi_{m_1,m_2,...,m_n}\}$ as a non-decreasing n-sequence of positive real numbers such that

$$(m_1,m_2,...,m_n)\phi_{m_1+1,m_2+1,...,m_n+1} \leq (m_1+1,m_2+1,...,m_n+1)\phi_{m_1,m_2,...,m_n},$$

for all $(m_1,m_2,...,m_n) \in \mathbb{N}^n$.

$w^n(X)$ and $l^n_\infty(X)$ denote the space of all n-sequences and bounded n-sequences, respectively, with elements in X, where (X,q) is a seminormed space. The zero sequence is denoted by $\bar{\theta} = (\theta,\theta,\theta,...)$, where θ is the zero element of X.

We first define the following spaces:

$$l^n_\infty(M,q) = \Bigg\{(x_{i_1,i_2,...,i_n}) \in w^n(X) : \sup_{i_1,i_2,...,i_n \geq 1} M\left(q\left(\frac{x_{i_1,i_2,...,i_n}}{\rho}\right)\right) < \infty, \text{ for some } \rho > 0\Bigg\},$$

$$m^n(M,\phi,q) = \Bigg\{(x_{i_1,i_2,...,i_n}) \in w^n(X) : \sup_{\substack{r_1,r_2,...,r_n \geq 1 \\ \sigma \in P_{r_1,r_2,...,r_n}}} \frac{1}{\phi_{r_1,r_2,...,r_n}} \sup_{i_1,i_2,...,i_n \geq 1} M\left(q\left(\frac{x_{i_1,i_2,...,i_n}}{\rho}\right)\right) < \infty, \text{ for some } \rho > 0\Bigg\}.$$

Theorem 3.1. *$m^n(M,\phi,q)$ and $l^n_\infty(M,q)$ are linear spaces.*

Proof. Let $(x_{i_1,i_2,...,i_n}),(y_{i_1,i_2,...,i_n}) \in m^n(M,\phi,q)$ and $\alpha,\beta \in \mathbb{C}$. Then there exist positive numbers ρ_1 and ρ_2 such that

$$\sup_{\substack{r_1,r_2,...,r_n \geq 1 \\ \sigma \in P_{r_1,r_2,...,r_n}}} \frac{1}{\phi_{r_1,r_2,...,r_n}} \sum_{i_1,i_2,...,i_n \in \sigma} M\left(q\left(\frac{x_{i_1,i_2,...,i_n}}{\rho_1}\right)\right) < \infty$$

and

$$\sup_{\substack{r_1,r_2,...,r_n \geq 1 \\ \sigma \in P_{r_1,r_2,...,r_n}}} \frac{1}{\phi_{r_1,r_2,...,r_n}} \sum_{i_1,i_2,...,i_n \in \sigma} M\left(q\left(\frac{y_{i_1,i_2,...,i_n}}{\rho_2}\right)\right) < \infty.$$

Let $\rho_3 = max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since q is a semi-norm and M is a non-decreasing convex function, we have

$$\sum_{i_1,i_2,...,i_n\in\sigma} M\Bigg(q\Bigg(\frac{\alpha x_{i_1,i_2,...,i_n}+\beta y_{i_1,i_2,...,i_n}}{\rho_3}\Bigg)\Bigg)$$
$$\leq \sum_{i_1,i_2,...,i_n\in\sigma} M\Bigg(q\Bigg(\frac{\alpha x_{i_1,i_2,...,i_n}}{\rho_3}\Bigg)+q\Bigg(\frac{\beta y_{i_1,i_2,...,i_n}}{\rho_3}\Bigg)\Bigg)$$
$$\leq \sum_{i_1,i_2,...,i_n\in\sigma} M\Bigg(q\Bigg(\frac{x_{i_1,i_2,...,i_n}}{\rho_1}\Bigg)\Bigg)+\sum_{i_1,i_2,...,i_n\in\sigma} M\Bigg(q\Bigg(\frac{y_{i_1,i_2,...,i_n}}{\rho_2}\Bigg)\Bigg).$$

Hence,

$$\sup_{\substack{r_1,r_2,...,r_n\geq 1\\ \sigma\in P_{r_1,r_2,...,r_n}}} \frac{1}{\phi_{r_1,r_2,...,r_n}}\sum_{i_1,i_2,...,i_n\in\sigma} M\Bigg(q\Bigg(\frac{\alpha x_{i_1,i_2,...,i_n}+\beta y_{i_1,i_2,...,i_n}}{\rho_3}\Bigg)\Bigg)$$
$$\leq \sup_{\substack{r_1,r_2,...,r_n\geq 1\\ \sigma\in P_{r_1,r_2,...,r_n}}} \frac{1}{\phi_{r_1,r_2,...,r_n}}\sum_{i_1,i_2,...,i_n\in\sigma} M\Bigg(q\Bigg(\frac{x_{i_1,i_2,...,i_n}}{\rho_1}\Bigg)\Bigg)$$
$$+ \sup_{\substack{r_1,r_2,...,r_n\geq 1\\ \sigma\in P_{r_1,r_2,...,r_n}}} \frac{1}{\phi_{r_1,r_2,...,r_n}}\sum_{i_1,i_2,...,i_n\in\sigma} M\Bigg(q\Bigg(\frac{y_{i_1,i_2,...,i_n}}{\rho_2}\Bigg)\Bigg)$$
$$< \infty.$$

Hence, $m^n(M,\phi,q)$ is a linear space. The proof of $l^n_\infty(M,q)$ can be done in a similar way. □

Theorem 3.2. *The space $m^n(M,\phi,q)$ is a seminormed space, seminormed by*

$$f(x_{i_1,i_2,...,i_n}) = \inf\Bigg\{\rho > o : \sup_{\substack{r_1,r_2,...,r_n\geq 1\\ \sigma\in P_{r_1,r_2,...,r_n}}} \frac{1}{\phi_{r_1,r_2,...,r_n}}\sum_{i_1,i_2,...,i_n\in\sigma} M\Bigg(q\Bigg(\frac{x_{i_1,i_2,...,i_n}}{\rho}\Bigg)\Bigg)\leq 1\Bigg\}.$$

Proof. Let $(x_{i_1,i_2,...,i_n})$ and $(y_{i_1,i_2,...,i_n})\in m^n(M,\phi,q)$.
Obviously, $f(x_{i_1,i_2,...,i_n})\geq 0$, for all $x_{i_1,i_2,...,i_n}\in m^n(M,\phi,q)$ and $f(\bar{\theta})=0$.
Let $\rho_1>0$ *and* $\rho_2>0$ be such that

$$\sup_{\substack{r_1,r_2,...,r_n\geq 1\\ \sigma\in P_{r_1,r_2,...,r_n}}} \frac{1}{\phi_{r_1,r_2,...,r_n}}\sum_{i_1,i_2,...,i_n\in\sigma} M\Bigg(q\Bigg(\frac{x_{i_1,i_2,...,i_n}}{\rho_1}\Bigg)\Bigg)\leq 1$$

and

$$\sup_{\substack{r_1,r_2,...,r_n\geq 1\\ \sigma\in P_{r_1,r_2,...,r_n}}} \frac{1}{\phi_{r_1,r_2,...,r_n}}\sum_{i_1,i_2,...,i_n\in\sigma} M\Bigg(q\Bigg(\frac{y_{i_1,i_2,...,i_n}}{\rho_2}\Bigg)\Bigg)\leq 1.$$

Let $\rho = \rho_1 + \rho_2$. Then we have

$$\begin{aligned}
&\sup_{\substack{r_1, r_2, ..., r_n \geq 1 \\ \sigma \in P_{r_1,r_2,\ldots,r_n}}} \frac{1}{\phi_{r_1,r_2,\ldots,r_n}} \sum_{i_1,i_2,\ldots,i_n \in \sigma} M\left(q\left(\frac{x_{i_1,i_2,\ldots,i_n} + y_{i_1,i_2,\ldots,i_n}}{\rho}\right)\right) \\
&= \sup_{\substack{r_1, r_2, ..., r_n \geq 1 \\ \sigma \in P_{r_1,r_2,\ldots,r_n}}} \frac{1}{\phi_{r_1,r_2,\ldots,r_n}} \sum_{i_1,i_2,\ldots,i_n \in \sigma} M\left(q\left(\frac{x_{i_1,i_2,\ldots,i_n} + y_{i_1,i_2,\ldots,i_n}}{\rho_1 + \rho_2}\right)\right) \\
&\leq \sup_{\substack{r_1, r_2, ..., r_n \geq 1 \\ \sigma \in P_{r_1,r_2,\ldots,r_n}}} \frac{1}{\phi_{r_1,r_2,\ldots,r_n}} \sum_{i_1,i_2,\ldots,i_n \in \sigma} \left\{\frac{\rho_1}{\rho_1 + \rho_2} M\left(q\left(\frac{x_{i_1,i_2,\ldots,i_n}}{\rho_1}\right)\right)\right\} \\
&+ \left\{\frac{\rho_2}{\rho_1 + \rho_2} M\left(q\left(\frac{y_{i_1,i_2,\ldots,i_n}}{\rho_2}\right)\right)\right\}. \\
&\leq \frac{\rho_1}{\rho_1 + \rho_2} \sup_{\substack{r_1, r_2, ..., r_n \geq 1 \\ \sigma \in P_{r_1,r_2,\ldots,r_n}}} \frac{1}{\phi_{r_1,r_2,\ldots,r_n}} \sum_{i_1,i_2,\ldots,i_n \in \sigma} M\left(q\left(\frac{x_{i_1,i_2,\ldots,i_n}}{\rho_1}\right)\right) \\
&+ \frac{\rho_2}{\rho_1 + \rho_2} \sup_{\substack{r_1, r_2, ..., r_n \geq 1 \\ \sigma \in P_{r_1,r_2,\ldots,r_n}}} \frac{1}{\phi_{r_1,r_2,\ldots,r_n}} \sum_{i_1,i_2,\ldots,i_n \in \sigma} M\left(q\left(\frac{y_{i_1,i_2,\ldots,i_n}}{\rho_2}\right)\right) \\
&\leq 1.
\end{aligned}$$

Since, the ρ's are non-negative, so we have

$$\begin{aligned}
f(x_{i_1,i_2,\ldots,i_n} + y_{i_1,i_2,\ldots,i_n}) &= \inf\Bigg\{\rho > o : \sup_{\substack{r_1, r_2, ..., r_n \geq 1 \\ \sigma \in P_{r_1,r_2,\ldots,r_n}}} \\
&\quad \frac{1}{\phi_{r_1,r_2,\ldots,r_n}} \sum_{i_1,i_2,\ldots,i_n \in \sigma} M\left(q\left(\frac{x_{i_1,i_2,\ldots,i_n} + y_{i_1,i_2,\ldots,i_n}}{\rho}\right)\right) \leq 1\Bigg\} \\
&\leq \inf\left\{\rho_1 > o : \sup_{\substack{r_1, r_2, ..., r_n \geq 1 \\ \sigma \in P_{r_1,r_2,\ldots,r_n}}} \frac{1}{\phi_{r_1,r_2,\ldots,r_n}} \sum_{i_1,i_2,\ldots,i_n \in \sigma} M\left(q\left(\frac{x_{i_1,i_2,\ldots,i_n}}{\rho_1}\right)\right) \leq 1\right\} \\
&+ \inf\left\{\rho_2 > o : \sup_{\substack{r_1, r_2, ..., r_n \geq 1 \\ \sigma \in P_{r_1,r_2,\ldots,r_n}}} \frac{1}{\phi_{r_1,r_2,\ldots,r_n}} \sum_{i_1,i_2,\ldots,i_n \in \sigma} M\left(q\left(\frac{y_{i_1,i_2,\ldots,i_n}}{\rho_2}\right)\right) \leq 1\right\} \\
&= f(x_{i_1,i_2,\ldots,i_n}) + f(y_{i_1,i_2,\ldots,i_n}).
\end{aligned}$$

Now for $\lambda \in \mathbb{C}$, without loss of generality, let $\lambda \neq 0$, then

$$f(\lambda(x_{i_1,i_2,...,i_n}))$$

$$= \inf\left\{\rho > o: \sup_{\substack{r_1,r_2,...,r_n \geq 1 \\ \sigma \in P_{r_1,r_2,...,r_n}}} \frac{1}{\phi_{r_1,r_2,...,r_n}} \sum_{i_1,i_2,...,i_n \in \sigma} M\left(q\left(\frac{\lambda x_{i_1,i_2,...,i_n}}{\rho}\right)\right) \leq 1\right\}$$

$$= \inf\left\{|\lambda|\ r > o: \sup_{\substack{r_1,r_2,...,r_n \geq 1 \\ \sigma \in P_{r_1,r_2,...,r_n}}} \frac{1}{\phi_{r_1,r_2,...,r_n}} \sum_{i_1,i_2,...,i_n \in \sigma} M\left(q\left(\frac{\lambda x_{i_1,i_2,...,i_n}}{r}\right)\right)\right.$$

$$\left. \leq 1, where\ \left\{r = \frac{\rho}{|\lambda|}\right\}\right.$$

$$= |\lambda| \inf\left\{r > o: \sup_{\substack{r_1,r_2,...,r_n \geq 1 \\ \sigma \in P_{r_1,r_2,...,r_n}}} \frac{1}{\phi_{r_1,r_2,...,r_n}} \sum_{i_1,i_2,...,i_n \in \sigma} M\left(q\left(\frac{\lambda x_{i_1,i_2,...,i_n}}{r}\right)\right) \leq 1\right\}$$

$$= |\lambda| f(x_{i_1,i_2,...,i_n}).$$

This shows that $m^n(M, \phi, q)$ is a seminormed space. □

Proposition 3.3. *The space $l^n_\infty(M, q)$ is a seminormed space, seminormed by*

$$g((x_{i_1,i_2,...,i_n})) = \inf\left\{\rho > o: \sup_{i_1,i_2,...,i_n \geq 1} M\left(q\left(\frac{x_{i_1,i_2,...,i_n}}{\rho}\right)\right) \leq 1\right\}.$$

Proof. The proof is similar to Theorem 3.2, hence skipped. □

Theorem 3.4. *$m^n(M, \phi, q) \subseteq m^n(M, \psi, q)$ if and only if $\sup_{r_1,r_2,...,r_n \geq 1} \frac{\phi_{r_1,r_2,...,r_n}}{\psi_{r_1,r_2,...,r_n}} < \infty$.*

Proof. Let $\sup_{r_1,r_2,...,r_n \geq 1} \frac{\phi_{r_1,r_2,...,r_n}}{\psi_{r_1,r_2,...,r_n}} < \infty$ and $(x_{i_1,i_2,...,i_n}) \in m^n(M, \phi, q)$. Then $\sup_{\substack{r_1,r_2,...,r_n \geq 1 \\ \sigma \in P_{r_1,r_2,...,r_n}}} \frac{1}{\phi_{r_1,r_2,...,r_n}} \sum_{i_1,i_2,...,i_n \in \sigma} M\left(q\left(\frac{x_{i_1,i_2,...,i_n}}{\rho}\right)\right) < \infty$, for some $\rho > 0$.
So we have

$$\sup_{\substack{r_1,r_2,...,r_n \geq 1 \\ \sigma \in P_{r_1,r_2,...,r_n}}} \frac{1}{\psi_{r_1,r_2,...,r_n}} \sum_{i_1,i_2,...,i_n \in \sigma} M\left(q\left(\frac{x_{i_1,i_2,...,i_n}}{\rho}\right)\right)$$

$$\leq \left\{\sup_{r_1,r_2,...,r_n \geq 1} \frac{\phi_{r_1,r_2,...,r_n}}{\psi_{r_1,r_2,...,r_n}}\right\}\left\{\sup_{\substack{r_1,r_2,...,r_n \geq 1 \\ \sigma \in P_{r_1,r_2,...,r_n}}} \frac{1}{\phi_{r_1,r_2,...,r_n}}\right.$$

$$\left.\left\{\sum_{i_1,i_2,...,i_n \in \sigma} M\left(q\left(\frac{x_{i_1,i_2,...,i_n}}{\rho}\right)\right)\right\}\right\}$$

$$\leq \infty.$$

Thus, $(x_{i_1,i_2,\ldots,i_n}) \in m^n(M,\psi,q)$ and therefore $m^n(M,\phi,q) \subseteq m^n(M,\psi,q)$.
Conversely, let $m^n(M,\phi,q) \subseteq m^n(M,\psi,q)$. Suppose that $\sup_{r_1,r_2,\ldots,r_n\geq 1} \frac{\phi_{r_1,r_2,\ldots,r_n}}{\psi_{r_1,r_2,\ldots,r_n}} = \infty$, then there exists a sequence of natural numbers $\{r_{k1}, r_{k2}\ldots, r_{kn}\}, k \in \mathbb{N}$ such that $\lim_{k\to\infty} \frac{\phi_{r_{k1},r_{k2},\ldots,r_{kn}}}{\psi_{r_{k1},r_{k2},\ldots,r_{kn}}} = \infty$.
Let $(x_{i_1,i_2,\ldots,i_n}) \in m^n(M,\phi,q)$. Then there exists $\rho > 0$ such that

$$\sup_{\substack{r_1,r_2,\ldots,r_n\geq 1\\ \sigma\in P_{r_1,r_2,\ldots,r_n}}} \frac{1}{\phi_{r_1,r_2,\ldots,r_n}} \sum_{i_1,i_2,\ldots,i_n\in\sigma} M\left(q\left(\frac{x_{i_1,i_2,\ldots,i_n}}{\rho}\right)\right) < \infty.$$

Now, we have

$$\begin{aligned}
&\sup_{\substack{r_1,r_2,\ldots,r_n\geq 1\\ \sigma\in P_{r_1,r_2,\ldots,r_n}}} \frac{1}{\psi_{r_1,r_2,\ldots,r_n}} \sum_{i_1,i_2,\ldots,i_n\in\sigma} M\left(q\left(\frac{x_{i_1,i_2,\ldots,i_n}}{\rho}\right)\right)\\
&\quad\geq \left\{\sup_{k\geq 1} \frac{\phi_{r_{k1},r_{k2},\ldots,r_{kn}}}{\psi_{r_{k1},r_{k2},\ldots,r_{kn}}}\right\}\\
&\quad\left\{\sup_{\substack{r_1,r_2,\ldots,r_n\geq 1\\ \sigma\in P_{r_1,r_2,\ldots,r_n}}} \frac{1}{\phi_{r_{k1},r_{k2},\ldots,r_{kn}}} \sum_{i_1,i_2,\ldots,i_n\in\sigma} M\left(q\left(\frac{x_{i_1,i_2,\ldots,i_n}}{\rho}\right)\right)\right\}\\
&\quad= \infty,
\end{aligned}$$

which is a contradiction.
Hence,

$$\sup_{r_1,r_2,\ldots,r_n\geq 1} \frac{\phi_{r_1,r_2,\ldots,r_n}}{\psi_{r_1,r_2,\ldots,r_n}} < \infty. \qquad \square$$

Corollary 3.5. *Let M be an Orlicz function. Then $m^n(M,\phi,q) = m^n(M,\psi,q)$ if and only if* $\sup_{r_1,r_2,\ldots,r_n\geq 1} \frac{\phi_{r_1,r_2,\ldots,r_n}}{\psi_{r_1,r_2,\ldots,r_n}} < \infty$ *and* $\sup_{r_1,r_2,\ldots,r_n\geq 1} \frac{\psi_{r_1,r_2,\ldots,r_n}}{\phi_{r_1,r_2,\ldots,r_n}} < \infty$.

Theorem 3.6. *Let M, M_1, M_2 be Orlicz functions satisfying Δ_2-condition. Then*

(i) $m^n(M_1,\phi,q) \subseteq m^n(M\ o\ M_1,\phi,q)$,

(ii) $m^n(M_1,\phi,q) \cap m^n(M_2,\phi,q) \subseteq m^n(M_1+M_2,\phi,q)$.

Proof. (i) Let $(x_{i_1,i_2,\ldots,i_n}) \in m^n(M_1,\phi,q)$. Then there exists $\rho > 0$ such that

$$\sup_{\substack{r_1,r_2,\ldots,r_n\geq 1\\ \sigma\in P_{r_1,r_2,\ldots,r_n}}} \frac{1}{\phi_{r_1,r_2,\ldots,r_n}} \sum_{i_1,i_2,\ldots,i_n\in\sigma} M_1\left(q\left(\frac{x_{i_1,i_2,\ldots,i_n}}{\rho}\right)\right) < \infty.$$

Let $0 < \epsilon < 1$ and $0 < \delta < 1$ such that $M(t) < \epsilon$, for all $0 \le t < \delta$.
Suppose $y_{i_1,i_2,...,i_n} = M_1\bigg(q\bigg(\frac{x_{i_1,i_2,...,i_n}}{\rho}\bigg)\bigg)$ and for any $\sigma \in P_{r_1,r_2,...,r_n}$, let

$$\sum_{i_1,i_2,...,i_n \in \sigma} M(y_{i_1,i_2,...,i_n}) = \sum_{y_{i_1,i_2,...,i_n} \le \delta} M(y_{i_1,i_2,...,i_n}) + \sum_{y_{i_1,i_2,...,i_n} > \delta} M(y_{i_1,i_2,...,i_n}).$$

By Remark 2.1, we have

(3.1)
$$\sum_{y_{i_1,i_2,...,i_n} \le \delta} M(y_{i_1,i_2,...,i_n}) \le M(1) \sum_{y_{i_1,i_2,...,i_n} \le \delta} (y_{i_1,i_2,...,i_n}) + M(2) \sum_{y_{i_1,i_2,...,i_n} > \delta} (y_{i_1,i_2,...,i_n}).$$

For $y_{i_1,i_2,...,i_n} > \delta$,
$$y_{i_1,i_2,...,i_n} < \frac{y_{i_1,i_2,...,i_n}}{\delta} \le 1 + \frac{y_{i_1,i_2,...,i_n}}{\delta}.$$
Since M is a non-decreasing and convex, so
$$M(y_{i_1,i_2,...,i_n}) < M\bigg(1 + \frac{y_{i_1,i_2,...,i_n}}{\delta}\bigg) < \frac{1}{2}M(2) + \frac{1}{2}M\bigg(\frac{2y_{i_1,i_2,...,i_n}}{\delta}\bigg).$$
Since M satisfies Δ_2-condition, so

$$\begin{aligned} M(y_{i_!,i_2,...,i_n}) &< \frac{1}{2}K\frac{y_{i_1,i_2,...,i_n}}{\delta}M(2) + \frac{1}{2}K\frac{y_{i_1,i_2,...,i_n}}{\delta}M(2) \\ &= K\frac{y_{i_1,i_2,...,i_n}}{\delta}M(2). \end{aligned}$$

Therefore,
(3.2)
$$\sum_{y_{i_1,i_2,...,i_n} > \delta} M(y_{i_1,i_2,...,i_n}) \le \max\big(1, K\delta^{-1}M(2)\big) \sum_{y_{i_1,i_2,...,i_n} > \delta} (y_{i_1,i_2,...,i_n}).$$

Now, from (3.1) and (3.2) one can say that $(x_{i_1,i_2,...,i_n}) \in m^n(M\ o\ M_1, \phi, q)$ and hence
$m^n(M_1, \phi, q) \subseteq m^n(M\ o\ M_1, \phi, q)$.
(ii) Let $(x_{i_1,i_2,...,i_n}) \in m^n(M_1, \phi, q) \cap m^n(M_2, \phi, q)$, then there exists $\rho_1, \rho_2 > 0$ such that

$$\sup_{\substack{r_1, r_2, ..., r_n \ge 1 \\ \sigma \in P_{r_1,r_2,...,r_n}}} \frac{1}{\phi_{r_1,r_2,...,r_n}} \sum_{i_1,i_2,...,i_n \in \sigma} M_1\bigg(q\bigg(\frac{x_{i_1,i_2,...,i_n}}{\rho_1}\bigg)\bigg) < \infty$$

and

$$\sup_{\substack{r_1, r_2, ..., r_n \ge 1 \\ \sigma \in P_{r_1,r_2,...,r_n}}} \frac{1}{\phi_{r_1,r_2,...,r_n}} \sum_{i_1,i_2,...,i_n \in \sigma} M_2\bigg(q\bigg(\frac{y_{i_1,i_2,...,i_n}}{\rho_2}\bigg)\bigg) < \infty.$$

Let $\rho = \max\{\rho_1, \rho_2\}$. Then

$$\sup_{\substack{r_1, r_2, ..., r_n \geq 1 \\ \sigma \in P_{r_1,r_2,\dots,r_n}}} \sum_{i_1,i_2,\dots,i_n \in \sigma} (M_1 + M_2)\bigg(q\bigg(\frac{x_{i_1,i_2,\dots,i_n}}{\rho}\bigg)\bigg)$$

$$\leq \sup_{\substack{r_1, r_2, ..., r_n \geq 1 \\ \sigma \in P_{r_1,r_2,\dots,r_n}}} \sum_{i_1,i_2,\dots,i_n \in \sigma} M_1\bigg(q\bigg(\frac{x_{i_1,i_2,\dots,i_n}}{\rho_1}\bigg)\bigg)$$

$$+ \sup_{\substack{r_1, r_2, ..., r_n \geq 1 \\ \sigma \in P_{r_1,r_2,\dots,r_n}}} \sum_{i_1,i_2,\dots,i_n \in \sigma} M_2\bigg(q\bigg(\frac{x_{i_1,i_2,\dots,i_n}}{\rho_2}\bigg)\bigg).$$

Hence the theorem is proved. □

Corollary 3.7. *Let M be an Orlicz function satisfying Δ_2-condition. Then $m^n(\phi, q) \subseteq m^n(M, \phi, q)$*

Proof. The result follows from Theorem 3.6-(i) by taking $M_1(x) = x$ in it. □

Corollary 3.8. *Let M be an Orlicz function satisfying the Δ_2-condition. Then $m^n(\phi, q) \subseteq m^n(M, \phi, q)$ if and only if* $\sup_{r_1,r_2,\dots,r_n \geq 1} \dfrac{\phi_{r_1,r_2,\dots,r_n}}{\psi_{r_1,r_2,\dots,r_n}} < \infty.$

Theorem 3.9. $l_1^n(M, q) \subseteq m^n(M, \phi, q) \subseteq l_\infty^n(M, q)$, *where*

$$l_1^n(M,q) = \bigg\{(x_{i_1,i_2,\dots,i_n}) \in w^n(X) : \sum_{i_1,i_2,\dots,i_n=1,1,\dots,1}^{\infty,\infty,\dots,\infty} M\bigg(q\bigg(\frac{x_{i_1,i_2,\dots,i_n}}{\rho}\bigg)\bigg) < \infty, \text{ for some } \rho > o\bigg\}.$$

Proof. Let $(x_{i_1,i_2,\dots,i_n}) \in l_1^n(M, q)$. Then we have

$$\sum_{i_1,i_2,\dots,i_n=1,1,\dots,1}^{\infty,\infty,\dots,\infty} M\bigg(q\bigg(\frac{x_{i_1,i_2,\dots,i_n}}{\rho}\bigg)\bigg) < \infty, \text{ for some } \rho > o. \tag{3.3}$$

Since, $(\phi_{m_1,m_2,\dots,m_n})$ is monotonic increasing sequence, so we have

$$\frac{1}{\phi_{r_1,r_2,\dots,r_n}} \sum_{i_1,i_2,\dots,i_n \in \sigma} M\bigg(q\bigg(\frac{x_{i_1,i_2,\dots,i_n}}{\rho}\bigg)\bigg)$$

$$\leq \frac{1}{\phi_{1,1,\dots,1}} \sum_{i_1,i_2,\dots,i_n \in \sigma} M\bigg(q\bigg(\frac{x_{i_1,i_2,\dots,i_n}}{\rho}\bigg)\bigg)$$

$$\leq \frac{1}{\phi_{1,1,\dots,1}} \sum_{i_1,i_2,\dots,i_n=1,1,\dots,1}^{\infty,\infty,\dots,\infty} M\bigg(q\bigg(\frac{x_{i_1,i_2,\dots,i_n}}{\rho}\bigg)\bigg)$$

$$< \infty.$$

Thus,

$$\sup_{\substack{r_1, r_2, ..., r_n \geq 1 \\ \sigma \in P_{r_1,r_2,\ldots,r_n}}} \frac{1}{\phi_{r_1,r_2,\ldots,r_n}} \sum_{i_1,i_2,\ldots,i_n \in \sigma} M\left(q\left(\frac{x_{i_1,i_2,\ldots,i_n}}{\rho}\right)\right) < \infty.$$

So,

$$(x_{i_1,i_2,\ldots,i_n}) \in m^n(M,\phi,q).$$

Hence,

$$l_1^n(M,q) \subseteq m^n(M,\phi,q).$$

Now, let $(x_{i_1,i_2,\ldots,i_n}) \in m^n(M,\phi,q)$. Then we have

$\sup_{\substack{r_1, r_2, ..., r_n \geq 1 \\ \sigma \in P_{r_1,r_2,\ldots,r_n}}} \frac{1}{\phi_{r_1,r_2,\ldots,r_n}} \sum_{i_1,i_2,\ldots,i_n \in \sigma} M\left(q\left(\frac{x_{i_1,i_2,\ldots,i_n}}{\rho}\right)\right) < \infty$, for some $\rho > 0$.

Take cardinality of σ as 1, then

$\sup_{i_1,i_2,\ldots,i_n \in \mathbb{N}^n} \frac{1}{\phi_{1,1,\ldots,1}} M\left(q\left(\frac{x_{i_1,i_2,\ldots,i_n}}{\rho}\right)\right) < \infty$, for some $\rho > 0$,

$\Rightarrow x_{i_1,i_2,\ldots,i_n} \in l_\infty^n(M,q)$.

Therefore,

$$m^n(M,\phi,q) \subseteq l_\infty^n(M,q). \qquad \square$$

Theorem 3.10. *Let (X,q) be complete. Then $m^n(M,\phi,q)$ is also complete.*

Proof. If we consider a normed linear space $(X, \|.\|)$ instead of a seminormed space (X,q) in Theorem 3.2, then we will get $m^n(M,\phi,q)$ as a normed space normed by

$$\|(x_{i_1,i_2,\ldots,i_n})\| = \inf\Bigg\{\rho > 0 : \sup_{\substack{r_1, r_2, ..., r_n \geq 1 \\ \sigma \in P_{r_1,r_2,\ldots,r_n}}} \frac{1}{\phi_{r_1,r_2,\ldots,r_n}} \sum_{i_1,i_2,\ldots,i_n \in \sigma} M\left(\frac{\|(x_{i_1,i_2,\ldots,i_n})\|}{\rho}\right) \leq 1\Bigg\}.$$

The space $m^n(M,\phi,\|.\|)$ will be a Banach space, if X is a Banach space. $\square$

4. l_p space: A special case of the space $m^n(M,\phi,q)$

In this section, we show how l_p space is related to our main space $m^n(M,\phi,q)$. We know that l_p spaces are a class of p-summable sequences spaces, so for n-sequences we write

$$l_p = \{x_{i_1,i_2,\ldots,i_n} \in w_n : \sum_{i_1,i_2,\ldots,i_n} |x_{i_1,i_2,\ldots,i_n}|^p < \infty\}.$$

(4.1)

$$m^n(M,\phi,q)=\Big\{(x_{i_1,i_2,\dots,i_n})\in w^n(X):\sup_{\substack{r_1,r_2,\dots,r_n\geq 1\\ \sigma\in P_{r_1,r_2,\dots,r_n}}}\frac{1}{\phi_{r_1,r_2,\dots,r_n}}\Big\{\sum_{i_1,i_2,\dots,i_n\in\sigma}M\Big(q\Big(\frac{x_{i_1,i_2,\dots,i_n}}{\rho}\Big)\Big)<\infty,\ \text{for some } \rho>0\Big\}\Big\}.$$

The notations used here are same as in the third section.
For $j=1$ to n, take $r_j=1$. Then for the seminorm $q(x)=x$ and Orlicz function $M(x)=x^p$, the space $m^n(M,\phi,q)$ will be an l_p space. To show this, first consider the set $P_{r_1,r_2,\dots,r_n}$. From the definition of $P_{r_1,r_2,\dots,r_n}$ in the third section,

$$P_{r_1,r_2,\dots,r_n}=\cup\{A\subset\mathbb{N}^n:|A|\leq r_1\cdot r_2\cdot\dots\cdot r_n\}.$$

Since we are taking r_j's as 1, we get

$$P_{r_1,r_2,\dots,r_n}=\cup\{A\subset\mathbb{N}^n:|A|\leq 1\}=\mathbb{N}^n.$$

Also,

$$\phi_{r_1,r_2,\dots,r_n}=\phi_{1,1,\dots,1},$$

which is a constant and hence will not affect the space. Substituting all the values in the definition of $m^n(M,\phi,q)$ (4.1), we get an l_p space.

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