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On split equality monotone Yosida variational inclusion and fixed point problems for countable infinite families of certain nonlinear mappings in Hilbert spaces

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Abstract. In this article, we introduce a split equality monotone Yosida variational inclusion problem which is more general than the split equality monotone variational inclusion problem, split equality variational inclusion problem and Yosida inclusion problem. We develop an iterative algorithm for approximating a common solution of split equality monotone Yosida variational inclusion problem and split equality fixed point problems for infinite family of generalized k-strictly pseudocontractive multivalued mappings and infinite family of L-Lipschitzian and quasi-pseudocontractive mappings in the settings of infinite-dimensional Hilbert spaces. Using our iterative algorithm, we state and prove a strong convergence theorem for approximating an element in the intersection of the solution set of the aforementioned problems. Our iterative algorithm is designed in such a way that it does not require prior knowledge of the operator norm. We apply our result to solve a variational inequality problem. Our result extends and complements some related results in the literature.

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1. Introduction

Let C be a nonempty, closed and convex subset of a real Hilbert space H endowed with inner product $\langle ., . \rangle$ and induced norm $||.||$, with $\mathbb{R}$ the set of real numbers and $Dom(T)$ the domain of T.

A point $x \in H$ is called a fixed point of a mapping $T : H \to H$, if $Tx = x$. However, if $T : H \to 2^H$ is a multi-valued mapping, then a point $x \in H$ is called a fixed point of T if $x \in Tx$. We denote by $Fix(T)$, the collection of all fixed points of T.

Definition 1.1. A mapping $T : H \to H$ is called
(i) nonexpansive, if

$$||Tx - Ty|| \leq ||x - y||, \ \forall \ x, y \in H;$$

(ii) strongly nonexpansive, if T is nonexpansive and

$$\lim_{n\to\infty} ||(x_n - y_n) - (Tx_n - Ty_n)|| = 0;$$

whenever $\{x_n\}$ and $\{y_n\}$ are bounded sequences in H and

$$\lim_{n\to\infty} (||x_n - y_n|| - ||(Tx_n - Ty_n)||) = 0;$$

(iii) averaged nonexpansive if it can be written as $T = (1 - \alpha)I + \alpha S$, where $\alpha \in (0, 1)$, I is the identity mapping on H, and $S : H \to H$ is a nonexpansive mapping;
(iv) k-strictly pseudo-contractive, if there exists a constant $k \in (0, 1)$ such that

$$||Tx - Ty||^2 \leq ||x - y||^2 + k||(I - T)x - (I - T)y||^2, \ \forall \ x, y \in H;$$

(v) pseudo-contractive, if $k = 1$ in (iv);
(vi) quasi-pseudo-contractive [11], if $Fix(T) \neq \emptyset$ and

$$||Tx - x^*||^2 \leq ||x - x^*||^2 + ||Tx - x||^2, \ \forall \ x \in H \ \text{ and } \ x^* \in F(T).$$

Given a real Hilbert space H, we denote by $CB(H)$ the family of nonempty, closed and bounded subsets of H. It is well known that the Hausdorff distance defined by

$$D(A, B) := \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\right\},$$

is a metric on this family $CB(H)$, where $d(a, B) = \inf_{b\in B} d(a, b)$.

The pioneer work on fixed points of multi-valued mappings using the application of Hausdorff metric was done by Markin [30] in 1973, where he studied the fixed point of a nonexpansive multi-valued mapping. Since then, there have been many results in literature which have found applications in the field of pure and applied sciences. Using the concept of Hausdorff metric, Chidume et. al. [13] introduced a new class of mapping called k-strictly pseudocontractive mapping, which is defined as follows:

Definition 1.2. Let H be a real Hilbert space and C be a nonempty, open and convex subset of H. Let $T : \bar{C} \to CB(\bar{C})$ be a mapping. Then, T is called a multi-valued k-strictly pseudocontractive mapping if there exists $k \in (0,1)$ such that for all $x, y \in C(T)$, we have

$$D^2(Tx, Ty) \leq ||x-y||^2 + k||(x-u)-(y-v)||^2,$$

for all $u \in Tx, \quad v \in Ty$.

Recently, Chidume and Okpala [14] introduced a different class of multi-valued strictly pseudocontractive mappings which is a superset of the class introduced in [13], as follows:

Definition 1.3. Let H be a real Hilbert space and C be a nonempty subset of H. Let $T : C \to CB(C)$ be a multi-valued mapping. Then T is called a generalized k-strictly pseudocontractive multi-valued mapping if there exists $k \in (0,1)$ such that for all $x, y \in C(T)$, the following inequality holds:

$$D^2(Tx, Ty) \leq ||x-y||^2 + kD^2(Ax, Ay),$$

where A $:= I - T$ and I is the identity operator on C.

They proved the following theorem using this class of mappings:

Theorem 1.4. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $T : C \to CB(C)$ be a generalized k-strictly pseudocontractive multi-valued mapping such that $Fix(T) \neq \emptyset$. Assume $Tp = \{p\}$, $\forall\, p \in Fix(T)$. Define a sequence $\{x_n\}$ by $x_0 \in C$,*

$$x_{n+1} = (1-\lambda)x_n + \lambda y_n, \tag{1.1}$$

for $y_n \in U^n$ and $\lambda \in (0, 1-k)$. Then $d(x_n, Tx_n) \to 0$ as $n \to \infty$, where

$$U^n := \big\{y_n \in Tx_n : D^2(x_n, Tx_n) \leq ||x_n - y_n||^2 + \frac{1}{n^2}\big\}.$$

Definition 1.5. [16] Let H be a real Hilbert space and T be a multi-valued mapping. T is said to be strongly demiclosed at 0, if for any sequence $\{x_n\} \subset Dom(T)$ such that $x_n \to p$ and $\{d(x_n, Tx_n)\}$ converges strongly to 0, then $d(p, Tp) = 0$. If T is a single valued mapping, then we have conclude that $||p - Tp|| = 0$.

Let $A : H \to 2^H$ be a multi-valued mapping, then the Variational Inclusion Problem (VIP) is to find $x \in H$ such that $0 \in Ax$. The study of this problem has been given reasonable attention by many researchers due to its wide applications. For instance, many problems in physics, economics, management sciences and operation research can be formulated as VIP. It also covers other optimization problems such as equilibrium problem, variational inequalities, minimization or maximization problems to mention a few (see [2, 3, 1, 6, 7, 19, 20, 21, 23, 24, 25, 26, 27, 28, 31, 36, 40, 42, 46] and the references contained in them).

Let $A : H \to 2^H$ be a multi-valued mapping with graph $G(A) := \{(x, y) : y \in A(x)\}$. Then, A is called monotone if for all (x, u) and $(y, v) \in G(A)$, the following inequality holds:

$$\langle x - y, u - v\rangle \geq 0.$$

The monotone mapping A is said to be maximal if its graph $G(A)$ is not properly contained in the graph of any other monotone mapping. A single-valued mapping $A : H \to H$ is called α-inverse strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha||Tx - Ty||^2, \forall\, x, y \in H.$$

It is well known that monotone operators on real Hilbert spaces can be regularized into single-valued Lipschitzian monotone operators through a process known as the Yosida approximation (see [5]).

This class of monotone operators was introduced by Zarantonello [45] and Minty [32] and since then many authors have shown significant interest in it due to its firm relation with the following evolution equation:

$$\begin{cases} \frac{dx}{dt} + A(x) = 0; \\ x(0) = x_0; \end{cases} \tag{1.2}$$

which is a model for many physical problems of practical applications. If the function A in (1.2) is not continuous, then it will be very difficult to solve these types of models. To solve this problem, Yosida introduced a natural step, which is to find a sequence of Lipschitz functions that approximate A in some sense. It is well known that two quite useful single-valued Lipschitz continuous operators can be associated with a monotone operator, namely its resolvent operator and its Yosida approximation operator. The Yosida approximation operators are useful to approximate solutions of VIP using resolvent operators. Recently, many authors engaged the Yosida approximation operators to study some VIP using different techniques, (see [4, 9, 18]).

Very recently, Ahmad et. al. [5] introduced the following Yosida approximation inclusion problem which is to find $x \in X$ such that

$$0 \in J^{\mathcal{H}\ (.,.)}_{M,\lambda}(x) + M(x), \lambda > 0, \tag{1.3}$$

where X is a smooth Banach space, M is an $\mathcal{H}(.,.)$- accretive operator with respect to A and B, $A, B : X \to X$ are single-valued mappings, $\mathcal{H}(A, B)$ is α-strongly accretive with respect to A, β-relaxed accretive with respect to B, with $\alpha > \beta$ and $J^{\mathcal{H}\ (.,.)}_{M,\lambda}(x)$ is the generalized Yosida approximation operator defined by

$$J^{\mathcal{H}\ (.,.)}_{M,\lambda}(u) = \frac{1}{\lambda}\big[I - R^{\mathcal{H}\ (.,.)}_{M,\lambda}\big](u),\ \forall\, u \in X, \tag{1.4}$$

where I is the identity mapping on X and $R^{\mathcal{H}\ (.,.)}_{M,\lambda}$ is the resolvent operator associated with the mappings $\mathcal{H}(.,.)$ and M. It was shown in [5] that the resolvent operator

$$R^{\mathcal{H}\ (.,.)}_{M,\lambda}(u) = [\mathcal{H}(A,B) + \lambda M]^{-1}(u),\ \forall\ u \in X,\ \lambda > 0$$

and the generalized Yosida approximation operator in (1.4) are connected by the following relation

$$\lambda J^{\mathcal{H}\ (.,.)}_{M,\lambda}(x) \in [\lambda M + \mathcal{H}(A,B) - I](R^{\mathcal{H}\ (.,.)}_{M,\lambda}(x)).$$

In order to study the strong convergence characteristics of the solutions of Yosida inclusion (1.3), Ahmad et. al. [5] proposed the following iterative algorithm: For $x_0 \in X$, define the sequence $\{x_n\} \subset X$ by the following scheme:

$$x_{n+1} = R^{\mathcal{H}\ (.,.)}_{M_n,\lambda}\big[\mathcal{H}(A,B)x_n - \lambda J^{\mathcal{H}\ (.,.)}_{M_n,\lambda}(x_n)\big].$$

In 2014, Moudafi [33] introduced the following Split Monotone Variational Inclusion Problem (SMVIP) which is to find

$$x^* \in H_1 \text{ such that } f(x^*) + B_1(x^*) \ni 0, \tag{1.5}$$

such that

$$y^* = Ax^* \in H_2 \text{ solves } g(y^*) + B_2(y^*) \ni 0, \tag{1.6}$$

where $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ are two multi-valued monotone mappings on real Hilbert spaces H_1 and H_2, respectively, $A : H_1 \to H_2$ is a bounded linear operator, $f : H_1 \to H_1$ and $g : H_2 \to H_2$ are two single-valued mappings.

Based on the work of Moudafi [33], Rahaman et. al. [38] introduced the Split Monotone Yosida Variational Inclusion Problem (SMYVIP) which is to find a point $x^* \in H_1$ such that

$$0 \in f_1(x^*) + B_1(x^*) - J^{B_1}_{\lambda_1}(x^*), \tag{1.7}$$

and

$$y^* = Ax^* \in H_2 \text{ solves } 0 \in f_2(y^*) + B_2(y^*) - J^{B_2}_{\lambda_2}(y^*), \tag{1.8}$$

where $B_i : H_i \to 2^{H_i},\ i = 1,2$ are multivalued maximal monotone mappings, $f_i : H_i \to H_i$ are single-valued mappings, $J^{B_i}_{\lambda_i} = \frac{1}{\lambda_i}(I_i - R^{B_i}_{\lambda_i})$ are the Yosida approximation operators of the mappings B_i, $R^{B_i}_{\lambda_i} = (I_i + \lambda_i B_i)^{-1}$ is the resolvent of the multivalued maximal monotone mapping B_i for $\lambda_i > 0$ and I_i are the identity mappings on Hilbert spaces H_i.

Rahaman et. al. [38] presented the following Yosida approximation technique to approximate the solution of SMYVIP (1.7)-(1.8).

$$\begin{cases} x_0 \in H_1; \\ u_n = T[x_n + \gamma A^*(S - I)Ax_n]; \\ v_n = \delta_n u_n + \tau g_n(u_n); \\ x_{n+1} = (1 - \alpha_n E)v_n + \alpha_n \beta f(v_n); \end{cases}$$

where H_1 and H_2 are real Hilbert spaces, $A : H_1 \to H_2$ is a bounded linear operator with adjoint A^*, E is a strongly positive bounded linear operator on H_1 with coefficient $\bar{r} > 0$ and $\beta \in (0, \frac{\bar{r}}{k})$, $\{g_n\}$ is a family of k-demicontractive mappings and uniformly convergent for any $x \in K$, where K is any bounded subset of H_1, $f : H_1 \to H_1$ a ξ-contraction mapping, $\tau > 0$, $\{\alpha_n\}$ and $\{\delta_n\}$ are sequences in $[0, 1)$. Furthermore, they proved that the sequence $\{x_n\}$ converges strongly to a solution of SMYVIP (1.7)-(1.8).

Let H_1, H_2 and H_3 be real Hilbert spaces, $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be two bounded linear operators. Let C and Q be nonempty, closed and convex subsets of H_1 and H_2, respectively. The Split Equality Problem SEP which was introduced by Moudafi [34] is to find

$$x \in C, \ y \in Q \text{ such that } Ax = By. \tag{1.9}$$

The SEP allows asymmetric and partial relations between the variable x and y and it also covers many situations such as decomposition methods for PDEs, applications in game theory and intensity-modulated radiation therapy, (see [8, 7]). Since the inception of SEP, many other related optimization problems such as split equality minimization problem, split equality equilibrium problem, split equality fixed point problem, Split Equality Variational Inclusion Problem (SEVIP), and Split Equality Monotone Yosida Variational Inclusion Problem (SEMVIP) have been introduced by authors working in this direction, (see [3, 22, 29, 35, 41] and the references contained in them).

In 2015, Guo et. al. [17] proposed two different iterative algorithms and proved that they converge strongly to a common solution of SEVIP and fixed point problem for a family of nonexpansive mappings, which is a unique solution of a variational inequality problem as an optimality condition for a minimization problem.

Very recently, Eslamian and Fakhri [15] proved the following strong convergence theorem for finding an element in the zero point set of the sum of two monotone operators and in common fixed point set of a finite family of quasi-nonexpansive multi-valued mappings.

Theorem 1.6. *Let H_1, H_2 and H_3 be real Hilbert spaces, $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be bounded linear operators with adjoints A^* and B^*. Let $f : H_1 \to H_1$ and $g : H_2 \to H_2$ be, respectively, α and β-inverse strongly monotone operators and F, G be two maximal monotone operators on H_1, H_2. For $i \in \{1, 2, ..m\}$, $T_i : H_1 \to CB(H_1)$ and $S_i : H_2 \to CB(H_2)$ be two finite families of quasi-nonexpansive multi-valued mappings such that $S_i - I$ and $T_i - I$ are demiclosed at 0, where S_i and T_i satisfies the common end point condition. Suppose $\Omega := \{(x, y) : x \in \cap_{i=1}^m Fix(T_i) \cap (f + F)^{-1}(0), y \in Fix(S_i) \cap (g + G)^{-1}(0), Ax = By\} \neq \emptyset$. Let $\{(x_n, y_n)\}$ be sequences generated for $x_0, \theta \in H_1$,* *and* $y_0, \eta \in H_2$ *by*

$$\begin{cases} z_n = x_n - \gamma_n A^*(Ax_n - By_n); \\ u_n = J^F_{\lambda_n}(I - \lambda_n f)z_n; \\ x_{n+1} = \alpha_n\theta_n + \beta_n u_n + \sum_{i=1}^m \delta_{n,i}v_{n,i}; \\ w_n = y_n + \gamma_n B^*(Ax_n - By_n); \\ t_n = J^G_{\mu_n}(I - \mu_n g)w_n; \\ y_{n+1} = \alpha_n\eta + \beta_n t_n + \sum_{i=1}^m \delta_{n,i}s_{n,i} \ \forall \ n > 0, \end{cases}$$

where $v_{n,i} \in T_i u_n, s_{n,i} \in S_i t_n$ *and the step-size* γ_n *is chosen in such a way that*

$$\gamma_n \in \left(\epsilon, \frac{2||Ax_n - By_n||^2}{||B^*(Ax_n - By_n)||^2 + ||A^*(Ax_n - By_n)||^2} - \epsilon\right), n \in \pi, \tag{1.10}$$

otherwise $\gamma_n = \gamma$ *(*γ *being any nonnegative value), where the index set* $\pi = \{n : Ax_n - By_n \neq 0\}$. *Let the sequences* $\{\alpha_n\}, \{\beta_n\}, \{\delta_{n,i}\}$, $\{\lambda_n\}$ *and* $\{\mu_n\}$ *satisfy the following conditions:*
(i) $\alpha_n + \beta_n + \sum_{i=1}^m \delta_{n,i} = 1$ *and* $\liminf_n \beta_n\delta_{n,i} > 0$ *for each* $i \in \{1, 2, \ldots, m\}$*;*
(ii) $\{\lambda_n\} \subset [a, b] \subset (0, 2\alpha)$ *and* $\{\mu_n\} \subset [c, d] \subset (0, 2\beta)$*;*
(iii) $\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=0}^\infty \alpha_n = \infty$.
Then, the sequence $\{(x_n, y_n)\}$ *converges strongly to* $(x^*, y^*) \in \Omega$.

Motivated by the works of Rahaman et. al. [38], Guo et. al. [17], Eslamian and Fakhri [15] and other related works in this direction, we introduce the Split Equality Monotone Yosida Variational Inclusion Problem (SEMYVIP), which is defined as follows:

Let H_1, H_2 and H_3 be real Hilbert spaces, $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be bounded linear operators. Let $F : H_1 \to 2^{H_1}$ and $G : H_2 \to 2^{H_2}$ be multi-valued maximal monotone mappings with nonempty values and let $f : H_1 \to H_1$ and $g : H_2 \to H_2$ be nonlinear mappings. The SEMYVIP is to find $x \in H_1$ such that

$$x \in (f + F - J^F_\lambda)^{-1}(0); \tag{1.11}$$

and $y \in H_2$, solves

$$y \in (g + G - J^G_\mu)^{-1}(0) \text{ with } Ax = By; \tag{1.12}$$

where $J^F_\lambda = \frac{1}{\lambda}(I - R^F_\lambda)$ and $J^G_\mu = \frac{1}{\mu}(I - R^G_\mu)$ are the Yosida approximation operators of the mappings F and G, $R^F_\lambda = (I + \lambda F)^{-1}$ and $R^G_\lambda = (I + \mu G)^{-1}$ are the resolvent operators of the mappings F and G for $\lambda, \mu > 0$, and I is the identity mapping. Furthermore, we introduce an iterative algorithm to approximate a common solution of problem (1.11)-(1.12) which is also a common fixed point of a countable infinite families of quasi-pseudo-contractive mappings and generalized strictly pseudocontractive mappings in real Hilbert spaces. Using our iterative scheme, we prove a strong convergence theorem for approximating a common solution of the aforementioned problem. We apply our result to solve a variational inequality problem. Our result extends and complements the results of [15], [17] and other related results in the literature.

2. Preliminaries

We state some known and useful results which will be needed in the proof of our main results.

Lemma 2.1. *[12] Let H be a real Hilbert space. Then the following results hold for all $x, y \in H$ and $\lambda \in [0,1]$*

(i) $||\lambda x + (1-\lambda)y||^2 = \lambda||x||^2 + (1-\lambda)||y||^2 - \lambda(1-\lambda)||x-y||^2.$

(ii) $2\langle x, y\rangle = ||x||^2 + ||y||^2 - ||x-y||^2 = ||x+y||^2 - ||x||^2 - ||y||^2.$

Lemma 2.2. *[10] Let E be a uniformly convex real Banach space. For arbitrary $r > 0$, let $B_r(0) := \{x \in E : ||x|| \leq r\}$. Then, for any given sequence $\{x_i\}_{i=1}^{\infty} \subset B_r(0)$ and for any given sequence $\{\lambda_i\}_{i=1}^{\infty}$ in $(0,1)$ with $\sum_{i=1}^{\infty} \lambda_i = 1$, there exists a continuously strictly increasing convex function*

$$g : [0, 2r] \to \mathbb{R} \text{ with } g(0) = 0,$$

such that for any positive integers i, j with $i < j$, the following inequality holds

$$||\sum_{i=1}^{\infty} \lambda_i x_i||^2 = \sum_{i=1}^{\infty} \lambda_i ||x||^2 - \lambda_i \lambda_j g(||x_i - x_j||).$$

Lemma 2.3. *[14] Let H be a real Hilbert space, $T : H \to H$ be a L-Lipschitzian mapping with $L \geq 1$. Denote $K := (1-\theta)I + \theta T((1-\eta)I + \eta T)$.*

If $0 < \theta < \eta < \dfrac{1}{1+\sqrt{1+L^2}}$, then the following conclusion holds:

(i) $Fix(T) = Fix((1-\theta)I + \theta T((1-\eta)I + \eta I)) = Fix(K).$

(ii) If T is demiclosed at 0, then K is demiclosed at 0.

(iii) In addition, if $T : H \to H$ is quasi-pseudocontractive, then the mapping K is quasi-nonexpansive, that is

$$||Kx - x^*|| \leq ||x - x^*||$$

for all $x \in H$ and $x^* \in Fix(T) = Fix(K)$.

Lemma 2.4. *[14] Let H be a real Hilbert space and $\{x_i\}_{i\in\mathbb{N}}$ be a bounded sequence in H. For $\delta_i \in (0,1)$ such that $\sum_{i=1}^{\infty} \delta_1 = 1$, the following identity holds:*

$$||\sum_{i=1}^{\infty} \delta_i x_i||^2 = \sum_{i=1}^{\infty} \delta_i ||x_i||^2 - \sum_{1\leq i<j<\infty} \delta_i \delta_j ||x_i - x_j||^2.$$

Lemma 2.5. *[14] Let E be a normed linear space, $B_1, B_2 \in CB(E)$ and $x, y \in E$ arbitrary. Then, the following hold:*

(a) $D(B_1, B_2) = D(x + B_1, x + B_2).$

(b) $D(B_1, B_2) = D(-B_1, -B_2)$.

(c) $D(x + B_1, y + B_2) \leq ||x - y|| + D(B_1, B_2)$.

(d) $D(\{x\}, B_1) = \sup_{b_1 \in B_1} ||x - b_1||$.

(e) $D(\{x\}, B_1) = D(0, x - B_1)$.

Lemma 2.6. *[14] Let C be a nonempty and close subset of a real Hilbert space H and let $T : C \to CB(C)$ be a generalized k-strictly pseudocontractive multivalued mapping. Then, $(I - T)$ is strongly demiclosed at zero.*

Lemma 2.7. *(Demiclosedness principle) [37] Let C be a nonempty, closed and convex subset of a real Hilbert space H and $T : C \to C$ be a nonexpansive operator with $Fix(T) \neq \emptyset$. If the sequence $\{x_n\} \subseteq C$ converges weakly to x and the sequence $\{(I - T)x_n\}$ converges strongly to y, then $(I - T)x = y$. In particular, if $y = 0$ then $x \in Fix(T)$.*

Lemma 2.8. *[44] Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n \delta_n, \quad \forall\ n \geq 0,$$

where $\{\alpha_n\}$ is a sequence in (0,1) and $\{\delta_n\}$ is a sequence such that

$$(i) \sum_{n=1}^{\infty} \alpha_n = \infty, \qquad (ii) \limsup_{n \to \infty} \delta_n \leq 0 \quad or \quad \sum_{n=1}^{\infty} |\alpha_n \delta_n| < \infty.$$

Then, $\lim_{n \to \infty} s_n = 0$.

Given a countably infinite family $\{T_i\}_{i \geq 1}$ of generalized k_i-strictly pseudocontractive multivalued mappings and an arbitrary sequence $\{x_n\} \subset C$, we denote by U_n^i the set of inexact distal points of x_n with respect to the set $T_i x_n$, that is

$$U_n^i := \{u_n^i \in T_i x_n : D^2(\{x_n\}, T_i x_n) \leq ||x_n - u_n^i||^2 + \frac{1}{n^2}\}, \quad \text{(see [14])}. \tag{2.1}$$

3. Main Results

In this section, we state and prove our main results.

Lemma 3.1. *Let H_1, H_2 and H_3 be real Hilbert spaces, $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be two bounded linear operators with adjoints A^* and B^*, respectively. For $i, j = 1, 2, \cdots$, let $T_i : H_1 \to H_1$ be a countable infinite family of L-Lipschitizian and quasi-pseudocontractive mappings with $L \geq 1$ and let $S_j : H_2 \to CB(H_2)$ be a countable infinite family of generalized k_j-strictly pseudocontractive multi-valued mappings such that for some $k \in (0,1)$, $k_j \in (0, k]$. Let $F : H_1 \to 2^{H_1}$ and $G : H_2 \to 2^{H_2}$ be two multivalued maximal monotone mappings with nonempty values, $f : H_1 \to H_1$ and $g : H_2 \to H_2$ be two inverse strongly monotone mappings. Assume that $\Gamma := \{(p, q) : p \in \cap_{i=1}^{\infty} Fix(T_i) \cap$*

$(f+F-J_\lambda^F)^{-1}(0),\ q\in\cap_{j=1}^{\infty}Fix(S_j)\cap(g+G-J_\mu^G)^{-1}(0), Ap=Bq\}\neq\emptyset$. Let $\{(x_n,y_n)\}$ be the sequence generated for $x_0,u\in H_1$ and $y_0,v\in H_2$ by

$$(3.1)\qquad \begin{cases} u_n = x_n-\gamma_n A^*(Ax_n-By_n);\\ z_n = R_\lambda^F[I+\lambda(J_\lambda^F-f)]u_n;\\ x_{n+1} = \alpha_n u+\beta_n x_n+\delta_n(\sigma_{n,0}z_n\\ \qquad +(\sum\limits_{i=1}^{\infty}\sigma_{n,i}(1-\theta)I+\theta T_i((1-\eta)I+\eta T_i))z_n);\\ v_n = y_n+\gamma_n B^*(Ax_n-By_n);\\ w_n = R_\mu^G[I+\mu(J_\mu^G-g)]v_n;\\ y_{n+1} = \alpha_n v+\beta_n y_n+\delta_n(t_{n,0}w_n+(\sum\limits_{j=1}^{\infty}t_{n,j})g_n^j);\quad g_n^j\in S_jw_n, \end{cases}$$

where the step-size γ_n is chosen in such a way that

$$(3.2)\qquad \gamma_n\in\left(\epsilon,\frac{2||Ax_n-By_n||^2}{||B^*(Ax_n-By_n)||^2+||A^*(Ax_n-By_n)||^2}-\epsilon\right), n\in\pi,$$

otherwise $\gamma_n=\gamma$ (γ being any nonnegative value), where the index set $\pi=\{n: Ax_n-By_n\neq 0\}$. Let λ,μ be positive parameters and let $\{\alpha_n\},\{\beta_n\},\{\delta_n\},\{\sigma_{n,i}\}$ and $\{t_{n,j}\}$ be sequences in $(0,1)$ satisfying

- *(i)* $\alpha_n+\beta_n+\delta_n=1$;
- *(ii)* $\sum\limits_{i=0}^{\infty}\sigma_{n,i}=1=\sum\limits_{j=0}^{\infty}t_{n,j}$, *with* $t_{n,0}\in(k_j,1)$;
- *(iii)* $0<\theta<\eta<\frac{1}{1+\sqrt{1+L^2}}$.

Then the sequence $\{(x_n,y_n)\}$ is bounded.

Proof. It is well known that R_λ^F is firmly nonexpansive and hence averaged. Since the composition of averaged mappings is average, therefore $R_\lambda^F(I+\lambda(J_\lambda^F-f))$ is averaged and hence nonexpansive. It follows that $R_\lambda^F(I+\lambda(J_\lambda^F-f))$ is strongly nonexpansive. Similarly, $R_\mu^G(I+\mu(J_\mu^G-g))$ is also strongly nonexpansive. Take $(p,q)\in\Gamma$ and define $a_n:=\sigma_{n,0}z_n+(\sum\limits_{i=1}^{\infty}K_i)z_n)$, $K_i=(1-\theta)I+\theta T_i((1-\eta)I+\eta T_i))$ and $b_n:=t_{n,0}w_n+(\sum\limits_{j=1}^{\infty}t_{n,j})g_n^j$. It is obvious that $R_\lambda^F(I+\lambda(J_\lambda^F-f))p=p$, then we have from (3.1) that

$$\begin{aligned} ||a_n-p||^2 &= ||\sigma_{n,0}z_n+\sum_{i=1}^{\infty}\sigma_{n,i}K_iz_n-p||^2\\ &\leq \sigma_{n,0}||z_n-p||^2+\sum_{i=1}^{\infty}\sigma_{n,i}||K_iz_n-p||^2\\ &- \sigma_{n,0}\sigma_{n,i}g(||z_n-K_iz_n||) \end{aligned}$$

$$\begin{aligned} &= ||z_n - p||^2 - \sigma_{n,0}\sigma_{n,i} g(||z_n - K_i z_n||) \\ &\leq ||z_n - p||^2. \end{aligned} \tag{3.3}$$

Also by using Lemma 2.4, 2.5(e) and equation (2.1), we obtain

$$\begin{aligned} ||b_n - q||^2 =& ||t_{n,0}(w_n - q) + \sum_{j=1}^{\infty} t_{n,j}(g_n^j - q)||^2 \\ =& t_{n,0}||w_n - q||^2 + \sum_{j=1}^{\infty} t_{n,j}||g_n^j - q|| \\ & - \sum_{i=1}^{\infty} t_{n,0}t_{n,j}||w_n - g_n^j||^2 \\ & - \sum_{1\leq j<m<n} t_{n,j}t_{n,m}||g_n^j - g_n^m||^2 \\ \leq& t_{n,0}||w_n - q||^2 + \sum_{j=1}^{\infty} t_{n,j} D^2(S_j w_n, S_j q) \\ & - \sum_{j=1}^{\infty} t_{n,0}t_{n,j}||w_n - g_n^j||^2 \\ \leq& t_{n,0}||w_n - q||^2 \\ & + \sum_{j=1}^{\infty} t_{n,j}(||w_n - q||^2 + k_j D^2(\{0\}, w_n - S_j w_n)) \\ & - \sum_{j=1}^{\infty} t_{n,0}t_{n,j}||w_n - g_n^j||^2 \\ \leq& \sum_{j=0}^{\infty} t_{n,j}||w_n - q||^2 \\ & + \sum_{j=1}^{\infty} t_{n,j}k_j D^2(\{w_n\}, S_j w_n) \\ & - \sum_{j=1}^{\infty} t_{n,0}t_{n,j}||w_n - g_n^j||^2 \\ \leq& \sum_{j=0}^{\infty} t_{n,j}||w_n - q||^2 \\ & + \sum_{j=1}^{\infty} t_{n,j}k_j(||w_n - g_n^j|| + \frac{1}{n^2}) \\ & - \sum_{j=1}^{\infty} t_{n,0}t_{n,j}||w_n - g_n^j||^2 \end{aligned}$$

$$\leq ||w_n - q||^2 + \frac{k}{n^2} - \sum_{j=1}^{\infty} t_{n,j}(t_{n,0} - k)||w_n - g_n^j||^2$$

$$\leq ||w_n - q||^2 + \frac{k}{n^2}. \tag{3.4}$$

Adding (3.3) and (3.4), we have

$$||a_n - p||^2 + ||b_n - q||^2 \leq ||z_n - p||^2 + ||w_n - q|| + \frac{k}{n^2}. \tag{3.5}$$

Now,

$$\begin{aligned}
||z_n - p||^2 &= ||R_\lambda^F(I + \lambda(J_\lambda^F - f))u_n - R_\lambda^F(I + \lambda(J_\lambda^F - f))p||^2 \\
&\leq ||u_n - p||^2 \\
&= ||x_n - p - \gamma_n A^*(Ax_n - By_n)||^2 \\
&= ||x_n - p||^2 - 2\gamma_n \langle x_n - p, A^*(Ax_n - By_n)\rangle \\
&+ \gamma_n^2 ||A^*(Ax_n - By_n)||^2 \\
&= ||x_n - p||^2 + \gamma_n^2 ||A^*(Ax_n - By_n)||^2 \\
&- \gamma_n ||Ax_n - Ap||^2 - \gamma_n ||Ax_n - By_n||^2 + \gamma_n ||By_n - Ap||^2.
\end{aligned} \tag{3.6}$$

Following a similar approach to the proof of (3.6), we have

$$\begin{aligned}
||w_n - q||^2 &= ||y_n - q||^2 + \gamma_n^2 ||B^*(Ax_n - By_n)||^2 - \gamma_n ||By_n - Bq||^2 \\
&\quad -\gamma_n ||Ax_n - By_n||^2 + \gamma_n ||By_n - Ap||^2.
\end{aligned} \tag{3.7}$$

Adding (3.6) and (3.7) and using (3.2) with $Ap = Bq$, we obtain

$$\begin{aligned}
||z_n - p||^2 + ||w_n - q||^2 &= ||x_n - p||^2 + ||y_n - q||^2 \\
&- \gamma_n(2||Ax_n - By_n||^2 \\
&- \gamma_n(||A^*(Ax_n - By_n)||^2 \\
&+ ||B^*(Ax_n - By_n)||^2)|| \\
&\leq ||x_n - p||^2 + ||y_n - q||^2.
\end{aligned} \tag{3.8}$$

Substituting (3.8) into (3.5), we get

$$||a_n - p||^2 + ||b_n - q||^2 \leq ||x_n - p||^2 + ||y_n - q||^2 + \frac{k}{n^2}. \tag{3.9}$$

Observe from (3.1) and Lemma 2.4, that

$$\begin{aligned}
||x_{n+1} - p||^2 &= ||\alpha_n u + \beta_n x_n + \delta_n a_n - p||^2 \\
&\leq \alpha_n ||u - p||^2 + \beta_n ||x_n - p||^2 + \delta_n ||a_n - p||^2.
\end{aligned} \tag{3.10}$$

Similarly,

$$||y_{n+1} - q||^2 \leq \alpha_n ||v - q||^2 + \beta_n ||y_n - q||^2 + \delta_n ||b_n - q||^2. \tag{3.11}$$

Adding (3.10) and (3.11) and using (3.9), we obtain

$$\begin{aligned}
&||x_{n+1} - x_n||^2 + ||y_{n+1} - y_n||^2 \\
\leq\ & \alpha_n[||u-p||^2 + ||v-q||^2] \\
+\ & \beta_n[||x_n - p||^2 + ||y_n - q||^2] \\
& +\delta_n[||a_n - p||^2 + ||b_n - q||^2] \\
\leq\ & \alpha_n[||u-p||^2 + ||v-q||^2] \\
+\ & \beta_n[||x_n - p||^2 + ||y_n - q||^2] \\
& +\delta_n[||x_n - p||^2 + ||y_n - q||^2 + \frac{k}{n^2}] \\
=\ & \alpha_n[||u-p||^2 + ||v-q||^2] \\
+\ & (1-\alpha_n)[||x_n - p||^2 + ||y_n - q||^2] + \frac{\delta_n k}{n^2} \\
\leq\ & \max\{[||u-p||^2 + ||v-q||^2], \\
& [||x_n - p||^2 + ||y_n - q||^2]\} + \frac{\delta_n k}{n^2} \\
\vdots\ & \\
\leq\ & \max\{[||u-p||^2 + ||v-q||^2], \\
& [||x_0 - p||^2 + ||y_0 - q||^2]\} + \frac{\delta_n k}{n^2}, \quad n > 0.
\end{aligned}$$

Therefore, $\{||x_n - p||^2 + ||y_n - q||^2\}$ is bounded. Thus, the sequences $\{x_n\}$ and $\{y_n\}$ are bounded. Consequently, the sequences $\{u_n\}, \{v_n\}, \{w_n\}$ and $\{z_n\}$ are all bounded.

Theorem 3.2. *Let H_1, H_2 and H_3 be real Hilbert spaces, $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be two bounded linear operators with adjoints A^* and B^*, respectively. For $i, j = 1, 2, \cdots$, let $T_i : H_1 \to H_1$ be a countable infinite family of L-Lipschitizian and quasi-pseudocontractive mappings with $L \geq 1$ such that T_i is demiclosed at 0, and let $S_j : H_2 \to CB(H_2)$ be a countable infinite family of generalized k_j-strictly pseudocontractive multi-valued mappings such that for some $k \in (0,1)$, $k_j \in (0, k]$. Let $F : H_1 \to 2^{H_1}$ and $G : H_2 \to 2^{H_2}$ be two multi-valued maximal monotone mappings with nonempty values, $f : H_1 \to H_1$ and $g : H_2 \to H_2$ be two inverse strongly monotone mappings. Assume that $\Gamma := \{(p,q) : p \in \cap_{i=1}^{\infty} Fix(T_i) \cap (f + F - J_\lambda^F)^{-1}(0),\ q \in \cap_{j=1}^{\infty} Fix(S_j) \cap (g + G - J_\mu^G)^{-1}(0), Ap = Bq\} \neq \emptyset$. Let $\{(x_n, y_n)\}$ be the sequence generated by (3.1), where the step-size γ_n is chosen in such a way that*

$$\gamma_n \in \left(\epsilon, \frac{2||Ax_n - By_n||^2}{||B^*(Ax_n - By_n)||^2 + ||A^*(Ax_n - By_n)||^2} - \epsilon\right), n \in \pi, \tag{3.12}$$

otherwise $\gamma_n = \gamma$ (γ being any nonnegative value), where the index set $\pi = \{n : Ax_n - By_n \neq 0\}$. Let λ, μ be positive parameters, $\{\alpha_n\}, \{\beta_n\}, \{\delta_n\}$ be sequences in $(0,1)$ such that $\alpha_n + \beta_n + \delta_n = 1$, $\{\sigma_{n,i}\}$ and $\{t_{n,j}\}$ be sequences in $(0,1)$, with the following conditions satisfied:

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\sum_{i=0}^{\infty} \sigma_{n,i} = 1 = \sum_{j=0}^{\infty} t_{n,j}$, *with* $t_{n,0} \in (k_j, 1)$;

(iii) $0 < \theta < \eta < \frac{1}{1+\sqrt{1+L^2}}$;

(iv) $0 < a \leq \beta_n\delta_n \leq b < 1$;

(v) $\lim_{n\to\infty} \frac{1}{n^2\alpha_n} = 0$.

Then the sequence $\{(x_n, y_n)\}$ *converges strongly to* $(\bar{x}, \bar{y}) \in \Gamma$.

Proof. Observe from (3.1), (3.6) and Lemma 2.4, that

$$\begin{aligned} ||x_{n+1} - p||^2 &= ||\alpha_n u + \beta_n x_n + \delta_n a_n - p||^2 \\ &\leq \alpha_n||u-p||^2 + \beta_n||x_n - p||^2 \\ &+ \delta_n||a_n - p||^2 - \beta_n\delta_n||x_n - a_n||^2 \\ &\leq \alpha_n||u-p||^2 + \beta_n||x_n - p||^2 \\ &+ \delta_n||z_n - p||^2 - \beta_n\delta_n||x_n - a_n||^2 \\ &\leq \alpha_n||u-p||^2 + \beta_n||x_n - p||^2 \\ &+ \delta_n[||x_n - p||^2 + \gamma_n^2||A^*(Ax_n - By_n)|| \\ &- \gamma_n||Ax_n - Ap||^2 \\ &\quad -\gamma_n||Ax_n - By_n||^2 + \gamma_n||By_n - Ap||^2] \\ &- \beta_n\delta_n||x_n - a_n||^2 \\ &= \alpha_n||u-p||^2 + (1-\alpha_n)||x_n - p||^2 \\ &+ \delta_n[\gamma_n^2||A^*(Ax_n - By_n)|| - \gamma_n||Ax_n - Ap||^2 \\ &\quad -\gamma_n||Ax_n - By_n||^2 + \gamma_n||By_n - Ap||^2] \\ &- \beta_n\delta_n||x_n - a_n||^2. \end{aligned} \tag{3.13}$$

Similarly, we obtain by using (3.1), (3.7) and Lemma 2.4, that

$$\begin{aligned} ||y_{n+1} - q||^2 &= \alpha_n||v-q||^2 + (1-\alpha_n)||y_n - q||^2 \\ &+ \delta_n[\gamma_n^2||B^*(Ax_n - By_n)|| - \gamma_n||By_n - Bq||^2 \\ &\quad -\gamma_n||Ax_n - By_n||^2 + \gamma_n||Ax_n - Bq||^2] \\ &- \beta_n\delta_n||y_n - b_n||^2 + \frac{\delta_n k}{n^2}. \end{aligned} \tag{3.14}$$

Adding, (3.13) and (3.14), we obtain

$$\begin{aligned} &||x_{n+1} - p||^2 + ||y_{n+1} - y_n||^2 \\ &\leq \alpha_n[||u-p||^2 + ||v-q||^2] \\ &+ (1-\alpha_n)[||x_n - p||^2 + ||y_n - q||^2]) - \delta_n\gamma_n[2||Ax_n - By_n||^2 \end{aligned}$$

$$
\begin{aligned}
&- \quad \gamma_n||A^*(Ax_n - By_n)|| - \gamma_n||B^*(Ax_n - By_n)||^2] \\
(3.15) \qquad &- \quad \beta_n\delta_n[||x_n - a_n||^2 + ||y_n - b_n||^2] + \frac{\delta_n k}{n^2}.
\end{aligned}
$$

We now divide the rest of the proof into two cases.
Case 1: Assume that $\{||x_n - p||^2 + ||y_n - q||^2\}$ is monotonically nonincreasing, we have that $\{||x_n - p||^2 + ||y_n - q||^2\}$ is convergent. Hence,

$$(||x_n - p||^2 + ||y_n - q||^2) - (||x_{n+1} - p||^2 + ||y_{n+1} - q||^2) \to 0 \quad \text{as} \quad n \to \infty.$$

□

From (3.15), we have

$$
\begin{aligned}
& \beta_n\delta_n[||x_n - a_n||^2 + ||y_n - b_n||^2] + \delta_n\gamma_n[2||Ax_n - By_n||^2 \\
& - \gamma_n||A^*(Ax_n - By_n)||^2 - \gamma_n||B^*(Ax_n - By_n)||] \\
\leq \ & \alpha_n[||u - p||^2 + ||v - q||^2] + (1 - \alpha_n)[||x_n - p||^2 + ||y_n - q||^2] \\
(3.16) \qquad & - [||x_{n+1} - p||^2 + ||y_{n+1} - q||^2] + \frac{k}{n^2}.
\end{aligned}
$$

By letting $n \to \infty$ in (3.16), we obtain

$$\lim_{n\to\infty}(||x_n - a_n||^2 + ||y_n - b_n||^2) = 0.$$

That is

$$(3.17) \qquad \lim_{n\to\infty} ||x_n - a_n|| = 0.$$

and

$$(3.18) \qquad \lim_{n\to\infty} ||y_n - b_n|| = 0.$$

Also,

$$(3.19) \qquad \lim_{n\to\infty}(||A^*(Ax_n - By_n)||^2 + ||B^*(Ax_n - By_n)||^2) = 0.$$

Note that $Ax_n = By_n$ if $n \notin \Omega$. Thus,

$$(3.20) \qquad \lim_{n\to\infty} ||A^*(Ax_n - By_n)|| = \lim_{n\to\infty} ||B^*(Ax_n - By_n)|| = 0.$$

By using (3.1) and the firmly nonexpansive property of $R^F_\lambda\left[I + \lambda\left(J^F_\lambda - f\right)\right]$, we get

$$(3.21) \qquad ||z_n - p||^2 \leq ||u_n - p||^2 - ||z_n - u_n||^2.$$

Similarly, we get

$$(3.22) \qquad ||w_n - p||^2 \leq ||v_n - p||^2 - ||w_n - v_n||^2.$$

Adding inequalities (3.21) and (3.22), we have

$$
\begin{aligned}
||z_n - p||^2 + ||w_n - q||^2 &\leq ||u_n - p||^2 + ||v_n - q||^2 \\
&- ||z_n - u_n||^2 - ||w_n - v_n||^2. \qquad (3.23)
\end{aligned}
$$

Now from (3.1), we have

$$
\begin{aligned}
||x_{n+1} - p||^2 &= ||\alpha_n u + \beta_n x_n + \delta_n a_n - p||^2 \\
&\leq \alpha_n ||u - p||^2 + \beta_n ||x_n - p||^2 \\
&+ \delta_n ||a_n - p||^2 - \beta_n \delta_n ||x_n - a_n||^2 \\
&\leq \alpha_n ||u - p||^2 + \beta_n ||x_n - p||^2 \\
&+ \delta_n ||z_n - p||^2 - \beta_n \delta_n ||x_n - a_n||^2. \qquad (3.24)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
||y_{n+1} - q||^2 &\leq \alpha_n ||v - q||^2 + \beta_n ||y_n - q||^2 + \delta_n ||w_n - q||^2 \\
&- \beta_n \delta_n ||y_n - b_n||^2 + \frac{\delta_n k}{n^2}. \qquad (3.25)
\end{aligned}
$$

Adding (3.24) and (3.25), and using (3.23), we obtain

$$
\begin{aligned}
&||x_{n+1} - p||^2 + ||y_{n+1} - q||^2 \\
&\leq \alpha_n [||u - p||^2 + ||v - p||^2] \\
&+ \beta_n [||x_n - p||^2 + ||y_n - q||^2] \\
&+ \delta_n [||z_n - p||^2 + ||w_n - q||^2] \\
&- \beta_n \delta_n [||x_n - a_n||^2 + ||y_n - b_n||^2] + \frac{\delta_n k}{n^2} \\
&\leq \alpha_n [||u - p||^2 + ||v - q||^2] \\
&+ \beta_n [||x_n - p||^2 + ||y_n - q||^2] + \delta_n [||u_n - p||^2 \\
&+ ||v_n - q|| - ||z_n - u_n||^2 - ||w_n - v_n||^2] \\
&- \beta_n \delta_n [||x_n - a_n||^2 + ||y_n - b_n||^2] + \frac{\delta_n k}{n^2} \\
&\leq \alpha_n [||u - p||^2 + ||v - q||^2] \\
&+ \beta_n [||x_n - p||^2 + ||y_n - q||^2] + \delta_n [||x_n - p||^2 \\
&+ ||y_n - q||] - \delta_n [||z_n - u_n||^2 - ||w_n - v_n||^2] \\
&- \beta_n \delta_n [||x_n - a_n||^2 + ||y_n - b_n||^2] + \frac{\delta_n k}{n^2} \\
&= \alpha_n [||u - p||^2 + ||v - q||^2] \\
&+ (1 - \alpha_n) [||x_n - p||^2 + ||y_n - q||^2] \\
&- \delta_n [||z_n - u_n||^2 + ||w_n - v_n||^2] \\
&- \beta_n \delta_n [||x_n - a_n||^2 + ||y_n - b_n||^2] + \frac{\delta_n k}{n^2}, \qquad (3.26)
\end{aligned}
$$

which implies that

$$
\delta_n [||z_n - u_n||^2 + ||w_n - v_n||^2]
$$

$$\leq \alpha_n[||u-p||^2 + ||v-q||^2] + (1-\alpha_n)[||x_n-p||^2 + ||y_n-q||^2]$$
$$-[||x_{n+1}-x_n||^2 + ||y_{n+1-y_n}||^2]$$
$$-\beta_n\delta_n[||x_n-a_n||^2 + ||y_n-b_n||^2] + \frac{k}{n^2}. \tag{3.27}$$

Taking limit of (3.27) as $n\to\infty$, we obtain

$$\lim_{n\to\infty}(||z_n-u_n||^2 + ||w_n-v_n||^2) = 0. \tag{3.28}$$

Hence,

$$\lim_{n\to\infty}||z_n-u_n|| = ||R^F_\lambda(I+\lambda(J^F_\lambda - f))u_n - u_n|| = 0 \tag{3.29}$$

and

$$\lim_{n\to\infty}||w_n-v_n|| = ||R^F_\mu(I+\mu(J^F_\mu - g))v_n - v_n|| = 0. \tag{3.30}$$

From (3.1) and (3.20), we obtain

$$||u_n-x_n|| = \gamma_n||A^*(Ax_n-By_n)|| \to 0 \text{ as } n\to\infty. \tag{3.31}$$

Similarly, we obtain

$$||v_n-y_n|| = \gamma_n||B^*(Ax_n-By_n)|| \to 0 \text{ as } n\to\infty. \tag{3.32}$$

Using (3.29) and (3.31), we have

$$||z_n-x_n|| \leq ||z_n-u_n|| + ||u_n-x_n|| \to 0 \text{ as } n\to\infty. \tag{3.33}$$

Also, by using (3.30) and (3.32), we obtain

$$||w_n-y_n|| \leq ||w_n-v_n|| + ||v_n-y_n|| \to 0 \text{ as } n\to\infty. \tag{3.34}$$

We also see that

$$\begin{cases} ||a_n-z_n|| \leq ||a_n-x_n|| + ||x_n-z_n|| \to 0 \text{ as } n\to\infty, \\ ||b_n-w_n|| \leq ||b_n-y_n|| + ||y_n-w_n|| \to 0 \text{ as } n\to\infty. \end{cases} \tag{3.35}$$

Next, we show that $||K_iz_n - z_n|| \to 0$ as $n\to\infty$. Indeed, from (3.3) and (3.35), we have

$$\begin{aligned} &\sigma_{n,0}\sigma_{n,i}g(||z_n-K_iz_n||) \\ &= ||z_n-p||^2 - ||a_n-p||^2 \\ &\leq ||z_n-a_n||(||z_n-p|| + ||a_n-p||) \to 0 \text{ as } n\to\infty. \end{aligned}$$

Since $\sigma_{n,0}\sigma_{n,i}\neq 0$, we obtain from the property of g that

$$\lim_{n\to\infty}||K_iz_n - z_n|| = 0. \tag{3.36}$$

Similarly, from (3.4) and (3.35), we have

$$\begin{aligned}\sum_{j=1}^{\infty} t_{n,j}(t_{n,0}-k)||w_n - g_n^j|| &\leq ||w_n - q||^2 - ||b_n - q||^2 + \frac{k}{n^2} \\ &\leq ||w_n - b_n||(||w_n - q|| + ||b_n - q||) \\ &+ \frac{k}{n^2} \to 0 \text{ as } n \to \infty. \end{aligned} \tag{3.37}$$

Using the fact that, $\sum\limits_{j=1}^{\infty} t_{n,j} \neq 0$ and $d(w_n, S_j w_n) \leq ||w_n - g_n^j||$, we get that

$$\lim_{n\to\infty} d(w_n, S_j w_n) = 0. \tag{3.38}$$

Furthermore, we obtain

$$\begin{aligned}||x_{n+1} - x_n|| &= ||\alpha_n u + \beta_n x_n + \delta_n a_n - x_n|| \\ &\leq \alpha_n ||u - x_n|| + \delta_n ||a_n - x_n||,\end{aligned}$$

which, by condition (i) and (3.17), implies that

$$\lim_{n\to\infty} = ||x_{n+1} - x_n|| = 0. \tag{3.39}$$

Also, by using condition (i) and (3.18), we obtain

$$\lim_{n\to\infty} ||y_{n+1} - y_n|| = 0. \tag{3.40}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \bar{x}$. By (3.31) and (3.32), we have that $u_{n_k} \rightharpoonup \bar{x}$ and $z_{n_k} \rightharpoonup \bar{x}$. Using (3.36) and the demiclosed property of K_i and Lemma 2.3, we have $\bar{x} \in \cap_{i=1}^{\infty} Fix(K_i) = \cap_{i=1}^{\infty} Fix(T_i)$. Also, since $\{y_n\}$ is bounded, there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} \rightharpoonup \bar{y}$. Using (3.38) and the fact that S_j is demiclosed at 0 for each j, we obtain $\bar{y} \in \cap_{j=1}^{\infty} F(S_j)$. We have from (3.29) and (3.30), that $\bar{x} \in (f + F - J_\lambda^F)^{-1}(0)$ and $\bar{y} \in (f + G - J_\mu^G)^{-1}(0)$, respectively. Hence, we have that $(\bar{x}, \bar{y}) \in \Gamma$.

Now, since A and B are bounded linear operators, we have that $\{Ax_n\}$ converges weakly to $A\bar{x}$ and $\{By_n\}$ converges weakly to $B\bar{y}$.

Next, we show that $A\bar{x} = B\bar{y}$.

$$\begin{aligned}&||A\bar{x} - B\bar{y}||^2 \\ &= \langle A\bar{x} - B\bar{y}, A\bar{x} - B\bar{y}\rangle \\ &= \langle A\bar{x} - B\bar{y}, A\bar{x} - B\bar{y} + Ax_n - Ax_n + By_n - By_n\rangle \\ &= \langle A\bar{x} - B\bar{y}, A\bar{x} - Ax_n\rangle + \langle A\bar{x} - B\bar{y}, Ax_n - By_n\rangle \\ &+ \langle A\bar{x} - B\bar{y}, By_n - B\bar{y}\rangle \\ &= \langle A\bar{x} - B\bar{y}, A\bar{x} - Ax_n\rangle + \langle A\bar{x}, Ax_n - By_n\rangle - \langle B\bar{y}, Ax_n - By_n\rangle \\ &+ \langle A\bar{x} - B\bar{y}, By_n - B\bar{y}\rangle\end{aligned}$$

$$
\begin{aligned}
&= \langle A\bar{x} - B\bar{y}, A\bar{x} - Ax_n\rangle + \langle \bar{x}, A^*(Ax_n - By_n)\rangle \\
&- \langle \bar{y}, B^*(Ax_n - By_n)\rangle + \langle A\bar{x} - B\bar{y}, By_n - B\bar{y}\rangle \\
&\le \langle A\bar{x} - B\bar{y}, A\bar{x} - Ax_n\rangle + ||\bar{x}|| ||A^*(Ax_n - By_n)|| \\
&+ ||\bar{y}|| \, ||B^*(Ax_n - By_n)|| \\
&+ \langle A\bar{x} - B\bar{y}, By_n - B\bar{y}\rangle \to 0, \; n \to \infty.
\end{aligned}
$$

This implies that $||A\bar{x} - B\bar{y}|| = 0$. Hence, $A\bar{x} = B\bar{y}$.

Next, we show that $\{(x_n, y_n)\}$ converges strongly to $(\bar{x}, \bar{y})$.

From (3.1), we have that

$$
\begin{aligned}
&||x_{n+1} - \bar{x}||^2 \\
&= ||\alpha_n u + \beta_n x_n + \delta_n a_n - \bar{x}||^2 \\
&= ||\alpha_n(u - \bar{x}) + \beta_n(x_n - \bar{x}) + \delta_n(a_n - \bar{x})||^2 \\
&\le ||\beta_n(x_n - \bar{x}) + \delta_n(a_n - \bar{x})||^2 + 2\alpha_n\langle x_{n+1} - \bar{x}, u - \bar{x}\rangle \\
&\le \beta_n||x_n - \bar{x}||^2 + \delta_n||z_n - \bar{x}||^2 + 2\alpha_n\langle x_{n+1} - x_n, u - \bar{x}\rangle. \qquad (3.41)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&||y_{n+1} - \bar{y}||^2 \\
&= ||\alpha_n v + \beta_n y_n + \delta_n b_n - \bar{y}||^2 \\
&= ||\alpha_n(v - \bar{y}) + \beta_n(y_n - \bar{y}) + \delta_n(b_n - \bar{y})||^2 \\
&\le ||\beta_n(y_n - \bar{y}) + \delta_n(b_n - \bar{y})||^2 + 2\alpha_n\langle y_{n+1} - \bar{y}, v - \bar{y}\rangle \\
&\le \beta_n||y_n - \bar{y}||^2 + \delta_n||w_n - \bar{y}||^2 + \frac{\delta_n k}{n^2} + 2\alpha_n\langle y_{n+1} - y_n, v - \bar{y}\rangle. \qquad (3.42)
\end{aligned}
$$

By adding, (3.41) and (3.42), we obtain

$$
\begin{aligned}
&||x_{n+1} - \bar{x}||^2 + ||y_{n+1} - \bar{y}||^2 \\
&= \beta_n[||x_n - \bar{x}||^2 + ||y_n - \bar{y}||^2] \\
&+ \delta_n[||z_n - \bar{x}||^2 + ||w_n - \bar{y}||^2] + \frac{\delta_n k}{n^2} \\
&+ 2\alpha_n(\langle x_{n+1} - \bar{x}, u - \bar{x}\rangle + \langle y_{n+1} - \bar{y}, v - \bar{y}\rangle) \\
&\le \beta_n[||x_n - \bar{x}||^2 + ||y_n - \bar{y}||^2] \\
&+ \delta_n[||x_n - \bar{x}||^2 + ||y_n - \bar{y}||^2] + \frac{\delta_n k}{n^2} \\
&+ 2\alpha_n(\langle x_{n+1} - \bar{x}, u - \bar{x}\rangle + \langle y_{n+1} - \bar{y}, v - \bar{y}\rangle) \\
&\le (1 - \alpha_n)[||x_n - \bar{x}||^2 + ||y_n - \bar{y}||^2] \\
&+ 2\alpha_n\left(\langle x_{n+1} - \bar{x}, u - \bar{x}\rangle + \langle y_{n+1} - \bar{y}, v - \bar{y}\rangle + \frac{k}{\alpha_n n^2}\right). \qquad (3.43)
\end{aligned}
$$

Since $x_n \rightharpoonup \bar{x}$ and $y_n \rightharpoonup \bar{y}$, then $x_{n+1} \rightharpoonup \bar{x}$ and $y_{n+1} \rightharpoonup \bar{y}$. Thus, using Lemma 2.8, condition (v) and (3.43), we obtain $||x_n - \bar{x}||^2 + ||y_n - \bar{y}||^2 \to 0$ as $n \to \infty$.

Hence,

$$\lim_{n\to\infty} ||x_n - \bar{x}|| = 0 = \lim_{n\to\infty} ||y_n - \bar{y}||. \tag{3.44}$$

Therefore, $(x_n, y_n) \to (\bar{x}, \bar{y}) \in \Gamma$.

Case 2: Assume that $\{||x_n - p||^2 + ||y_n - q||^2\}$ is not monotone decreasing. Set $\Upsilon_n := ||x_n - p||^2 + ||y_n - q||^2$ and let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping defined for $n \geq n_0$ (for some large n_0) by

$$\tau(n) := \max\{k \in \mathbb{N} : k \leq n, \Upsilon_k \leq \Upsilon_{k+1}\}.$$

Clearly, τ is a non-decreasing sequence such that $\tau(n) \to \infty$, as $n \to \infty$ and

$$\Upsilon_{\tau(n)} \leq \Upsilon_{\tau(n)+1},\ n \geq n_0.$$

From (3.16), we have that

$$\begin{aligned} &\epsilon^2\big(||A^*(Ax_{\tau(n)} - By_{\tau n})||^2 + ||B^*(Ax_{\tau n} - By_{\tau n}||^2\big) \\ \leq\ &\alpha_{\tau(n)}\big[||u-p||^2 + ||v-q||^2\big] + (1-\alpha_{\tau(n)})[||x_{\tau(n)} - p||^2 \\ &+ ||y_{\tau(n)} - q||^2] - [||x_{\tau(n)+1} - p||^2 + ||y_{\tau(n)} - q||^2]. \end{aligned}$$

Using condition (i) of (3.1), we have that

$$(||A^*(Ax_{\tau(n)} - By_{\tau(n)})||^2 + ||B^*(Ax_{\tau(n)} - By_{\tau(n)})||^2) \to 0, \quad \text{as} \quad n \to \infty.$$

Note $Ax_{\tau(n)} - By_{\tau n} = 0$, if $\tau(n) \notin \pi$. Hence

$$\lim_{\tau(n)\to\infty} ||A^*(Ax_{\tau(n)} - By_{\tau n})|| = 0, \tag{3.45}$$

and

$$\lim_{\tau(n)\to\infty} ||B^*(Ax_{\tau(n)} - By_{\tau(n)})|| = 0. \tag{3.46}$$

Following the same argument as in Case 1, we have that $(\{x_{\tau(n)}, \{y_{\tau(n)}\}\})$ converges weakly to $(\bar{x}, \bar{y}) \in \Gamma$.

Now, for all $n \geq n_0$,

$$\begin{aligned} 0 &\leq [||x_{\tau(n)+1} - \bar{x}||^2 + ||y_{\tau_{(n)}+1} - \bar{y}||^2] - [||x_{\tau(n)} - \bar{x}||^2 + ||y_{\tau(n)} - \bar{x}||^2] \\ &\leq (1-\alpha_{\tau(n)})[||x_{\tau(n)} - \bar{x}||^2 + ||y_{\tau(n)} - \bar{y}||^2] \\ &- 2\alpha_{\tau(n)}\left(\langle x_{\tau_{(n)}+1} - \bar{x}, u - \bar{x}\rangle + \langle y_{\tau_{(n)}+1} - \bar{y}, v - \bar{y}\rangle + \frac{k}{\alpha_{\tau(n)}n^2}\right) \\ &- [||x_{\tau_{(n)}} - \bar{x}||^2 + ||y_{\tau_{(n)}} - \bar{y}||^2], \end{aligned}$$

which implies

$$\alpha_{\tau(n)}[||x_{\tau_{(n)}-\bar{x}}|| + ||y_{\tau_{(n)}} - \bar{y}||^2]$$

$$\leq 2\alpha_{\tau(n)}\left(\langle x_{\tau_{(n)}}-\bar{x},u-\bar{x}\rangle+\langle y_{\tau_{(n)}}-\bar{y},v-\bar{y}\rangle+\frac{k}{\alpha_{\tau(n)}n^2}\right). \tag{3.47}$$

Now, we note that $\alpha_{\tau(n)}>0$, and hence we get

$$\lim_{n\to\infty}\left(||x_{\tau(n)}-\bar{x}||^2+||y_{\tau(n)}-\bar{y}||^2\right)=0.$$

Therefore,

$$\lim_{n\to\infty}\Upsilon_{\tau(n)}=\lim_{n\to\infty}\Upsilon_{\tau(n)+1}=0.$$

For $n\geq n_0$, it is clear that $\Upsilon_{\tau(n)}\leq\Upsilon_{\tau(n)+1}$ if $n\neq\tau(n)$ (that is $\tau(n)<n$) because $\Upsilon_j>\Upsilon_{j+1}$ for $\tau(n)+1\leq j\leq n$.
Consequently, for all $n\geq n_0$

$$0<\Upsilon_n\leq\max\{\Upsilon_{\tau(n)},\Upsilon_{\tau_n+1}\}=\Upsilon_{\tau_n+1}.$$

Hence, $\lim_{n\to\infty}\Upsilon_n=0$. Therefore, we conclude that $\{(x_n,y_n)\}$ converges strongly to $(\bar{x},\bar{y})$. □

Corollary 3.3. *Let H_1, H_2 and H_3 be real Hilbert spaces, $A:H_1\to H_3$ and $B:H_2\to H_3$ be two bounded linear operators with adjoints A^* and B^*, respectively. Let $F:H_1\to 2^{H_1}$ and $G:H_2\to 2^{H_2}$ be two multi-valued maximal monotone mappings with nonempty values, $f:H_1\to H_1$ and $g:H_2\to H_2$ be two inverse strongly monotone mappings. Assume that $\Gamma:=\{(p,q):p\in(f+F-J_\lambda^F)^{-1}(0),\ q\in(g+G-J_\mu^G)^{-1}(0),Ap=Bq\}\neq\emptyset$. Let $\{(x_n,y_n)\}$ be the sequence generated for $x_0,u\in H_1$ and $y_0,v\in H_2$, by*

$$\begin{cases}u_n=x_n-\gamma_nA^*(Ax_n-By_n);\\ z_n=R_\lambda^F[I+\lambda(J_\lambda^F-f)]u_n;\\ x_{n+1}=\alpha_nu+\beta_nx_n+\delta_nz_n;\\ v_n=y_n+\gamma_nB^*(Ax_n-By_n);\\ w_n=R_\mu^G[I+\mu(J_\mu^G-g)]v_n;\\ y_{n+1}=\alpha_nv+\beta_ny_n+\delta_nw_n,\end{cases} \tag{3.48}$$

where the step-size γ_n is chosen in such a way that

$$\gamma_n\in\left(\epsilon,\frac{2||Ax_n-By_n||^2}{||B^*(Ax_n-By_n)||^2+||A^*(Ax_n-By_n)||^2}-\epsilon\right),n\in\pi, \tag{3.49}$$

otherwise $\gamma_n=\gamma$ (γ being any nonnegative value), where the index set $\pi=\{n:Ax_n-By_n\neq0\}$. Let λ,μ be positive parameters, $\{\alpha_n\},\{\beta_n\},\{\delta_n\}$ be sequences in $(0,1)$ such that $\alpha_n+\beta_n+\delta_n=1$ satisfying the following conditions:

(i) $\lim_{n\to\infty}\alpha_n=0,\ \sum_{n=1}^{\infty}\alpha_n=\infty;$

(ii) $0<a\leq\beta_n\delta_n\leq b<1;$

(iii) $\lim_{n\to\infty} \frac{1}{n^2\alpha_n} = 0.$

Then the sequence $\{(x_n, y_n)\}$ *converges strongly to* $\{(\bar{x}, \bar{y})\} \in \Gamma$.

Corollary 3.4. *Let* H_1, H_2 *and* H_3 *be real Hilbert spaces,* $A : H_1 \to H_3$ *and* $B : H_2 \to H_3$ *be two bounded linear operators with adjoints* A^* *and* B^*, *respectively. For* $i, j = 1, 2, \cdots$, *let* $T_i : H_1 \to H_1$ *be a countable infinite family of L-Lipschitizian and quasi-pseudocontractive mappings with* $L \geq 1$ *such that* T_i *is demiclosed at* 0 *and let* $S_j : H_2 \to CB(H_2)$ *be a countable infinite family of generalized* k_j*-strictly pseudocontractive multi-valued mappings such that for some* $k \in (0,1)$, $k_j \in (0, k]$. *Let* $F : H_1 \to 2^{H_1}$ *and* $G : H_2 \to 2^{H_2}$ *be two multi-valued maximal monotone mappings with nonempty values. Assume that* $\Gamma := \{(p.q) : p \in \cap_{i=1}^{\infty} Fix(T_i) \cap F^{-1}(0),\ q \in \cap_{j=1}^{\infty} F(S_j) \cap G^{-1}(0) : Ap = Bq\} \neq \emptyset$. *Let* $\{(x_n, y_n)\}$ *be the sequences generated by* $x_0, u \in H_1$ *and* $y_0, v \in H_2$ *defined by*

$$\begin{cases} u_n = x_n - \gamma_n A^*(Ax_n - By_n); \\ z_n = (I + \lambda F)^{-1} u_n; \\ x_{n+1} = \alpha_n u + \beta_n x_n + \delta_n(\sigma_{n,0} z_n \\ \qquad + (\sum_{i=1}^{\infty}(1-\theta)I + \theta T_i((1-\eta)I + \eta T_i))z_n); \\ v_n = y_n + \gamma_n B^*(Ax_n - By_n); \\ w_n = (I + \mu G)^{-1} v_n; \\ y_{n+1} = \alpha_n v + \beta_n y_n + \delta_n(t_{n,0} w_n + (\sum_{j=1}^{\infty} t_{n,j}) g_n^j); \quad g_n^j \in S_j w_n, \end{cases} \tag{3.50}$$

where the step-size γ_n *is chosen in such a way that*

$$\gamma_n \in \left(\epsilon, \frac{2||Ax_n - By_n||^2}{||B^*(Ax_n - By_n)||^2 + ||A^*(Ax_n - By_n)||^2} - \epsilon\right), n \in \pi, \tag{3.51}$$

otherwise $\gamma_n = \gamma$ *(* γ *being any nonnegative value), where the index set* $\pi = \{n : Ax_n - By_n \neq 0\}$. *Let* λ, μ *be positive parameters,* $\{\alpha_n\}, \{\beta_n\}, \{\delta_n\}$ *be sequences in* $(0,1)$ *such that* $\alpha_n + \beta_n + \delta_n = 1$, $\{\sigma_{n,i}\}$ *and* $\{t_{n,j}\}$ *be sequences in* $(0,1)$ *with the following conditions satisfied*

(i) $\lim_{n\to\infty} \alpha_n = 0,\ \sum_{n=1}^{\infty} \alpha_n = \infty;$

(ii) $\sum_{i=0}^{\infty} \sigma_{n,i} = 1 = \sum_{j=0}^{\infty} t_{n,j}, \quad$ *with* $\ t_{n,0} \in (k_j, 1);$

(iii) $0 < \theta < \eta < \frac{1}{1+\sqrt{1+L^2}};$

(iv) $0 < a \leq \beta_n \delta_n \leq b < 1;$

(v) $\lim_{n\to\infty} \frac{1}{n^2\alpha_n} = 0.$

Then the sequence $\{(x_n, y_n)\}$ *converges strongly to* $\{(\bar{x}, \bar{y})\} \in \Gamma$.

4. Application and Numerical Example

4.1. Variational Inequality Problem

The variational inequality problem is defined as:

$$\text{(4.1)} \qquad \text{Find } \ x \in C \quad \text{such that} \quad \langle Ax, y - x\rangle \geq 0, \quad \forall\, y \in C,$$

where $A : C \to H$ is a nonlinear operator- We denote the set of solutions of VIP (4.1) by $VI(C, A)$.

Let H be a real Hilbert space and h be a proper, convex and lower semi-continuous function of H into $\mathbb{R}$. Then the subdifferential δh of h is defined as follows:

$$\text{(4.2)} \qquad \delta h(x) = \{z \in H : h(x) + \langle z, u - x\rangle \leq h(u), \forall\, u \in H\},$$

for all $x \in H$. We know that δh is a maximal monotone operator (see [39]). Let C be a nonempty, closed and convex subset of H and ι_C be the indicator function of C which is defined by

$$\iota_C = \begin{cases} 0, & if\ x \in C, \\ +\infty, & if\ x \notin C. \end{cases}$$

Then, ι_C is a proper, convex and lower semicontinuous function on H. So, we can define the resolvent operator as $R_\lambda^{\delta_{\iota_C}}$ of δ_{ι_C} for $\lambda > 0$, i.e.

$$R_\lambda^{\delta_{\iota_C}}(x) = (I + \lambda\delta_{\iota_C})^{-1}(x), \ x \in H.$$

We know that $R_\lambda^{\delta_{\iota_C}}(x) = P_C(x)$ for all $x \in H$ and $\lambda > 0$ (see [43]). Moreover, for a single-valued operator $f : H \to H$, we have that

$$x \in (f + \delta_{\iota_C})^{-1}(0) \Longleftrightarrow x \in VI(C, f).$$

Let C be a nonempty, closed and convex subset of a real Hilbert space H. For each $x \in H$, it is known that there exists a unique element $P_C x$ of C such that

$$||x - P_C x|| = \inf\{||x - y|| : y \in C\},$$

where P_C is referred to as the nearest point mapping or metric projection from H onto C.

Now, we present an application of our main theorem.

Theorem 4.1. *Let H_1, H_2 and H_3 be real Hilbert spaces, $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be two bounded linear operators with adjoints A^* and B^*, respectively. For $i, j = 1, 2, \cdots$, let $T_i : H_1 \to H_1$ be a countable infinite family of L-Lipschitizian and quasi-pseudocontractive mappings with $L \geq 1$ such that T_i is demiclosed at 0 and let $S_j : H_2 \to CB(H_2)$ be a countable infinite family of generalized k_j-strictly pseudocontractive multi-valued mappings such* *that for some* $k \in (0,1)$, $k_j \in (0,k]$. *Let* $f : H_1 \to H_1$ *and* $g : H_2 \to H_2$ *be* ϕ, φ- *inverse strongly monotone mappings. Assume that* $\Gamma := \{(p,q) : p \in \cap_{i=1}^{\infty} Fix(T_i) \cap VI(C,f),\ q \in \cap_{j=1}^{\infty} Fix(S_j) \cap VI(Q,g), Ap = Bq\} \neq \emptyset$. *Let* $\{(x_n, y_n)\}$ *be the sequences generated by* $x_0, u \in H_1$ *and* $y_0, v \in H_2$ *defined by*

$$\tag{4.3} \begin{cases} u_n = x_n - \gamma_n A^*(Ax_n - By_n); \\ z_n = P_C(I - \lambda_n f)u_n; \\ x_{n+1} = \alpha_n u + \beta_n x_n + \delta_n(\sigma_{n,0} z_n \\ \qquad + (\sum\limits_{i=1}^{\infty}(1-\theta)I + \theta T_i((1-\eta)I + \eta T_i))z_n); \\ v_n = y_n + \gamma_n B^*(Ax_n - By_n); \\ w_n = P_Q(I - \mu_n g)u_n;; \\ y_{n+1} = \alpha_n v + \beta_n y_n + \delta_n(t_{n,0} w_n + (\sum\limits_{j=1}^{\infty} t_{n,j})g_n^j); \quad g_n^j \in S_j w_n, \end{cases}$$

where the step-size γ_n *is chosen in such a way that*

$$\tag{4.4} \gamma_n \in \left(\epsilon, \frac{2||Ax_n - By_n||^2}{||B^*(Ax_n - By_n)||^2 + ||A^*(Ax_n - By_n)||^2} - \epsilon\right), n \in \pi,$$

otherwise $\gamma_n = \gamma$ *(* γ *being any nonnegative value), where the index set* $\pi = \{n : Ax_n - By_n \neq 0\}$. *Let* λ, μ *be positive parameters,* $\{\alpha_n\}, \{\beta_n\}, \{\delta_n\}$ *be sequences in* $(0,1)$ *such that* $\alpha_n + \beta_n + \delta_n = 1$, $\{\sigma_{n,i}\}$ *and* $\{t_{n,j}\}$ *be sequences in* $(0,1)$ *with the following conditions satisfied:*

(i) $\lim\limits_{n\to\infty} \alpha_n = 0,\ \sum\limits_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\sum\limits_{i=0}^{\infty} \sigma_{n,i} = 1 = \sum\limits_{j=0}^{\infty} t_{n,j}$, *with* $t_{n,0} \in (k_j, 1)$;

(iii) $0 < \theta < \eta < \frac{1}{1+\sqrt{1+L^2}}$;

(iv) $0 < a \leq \beta_n \delta_n \leq b < 1$;

(v) $\lim\limits_{n\to\infty} \frac{1}{n^2 \alpha_n} = 0$.

Then the sequence $\{(x_n, y_n)\}$ *converges strongly to* $\{(\bar{x}, \bar{y})\} \in \Gamma$.

The following is a consequence of Theorem 4.1.

Corollary 4.2. *Let* H_1, H_2 *and* H_3 *be real Hilbert spaces,* $A : H_1 \to H_3$ *and* $B : H_2 \to H_3$ *be two bounded linear operators with adjoints* A^* *and* B^**, respectively. For* $i, j = 1, 2, \cdots$, *let* $T : H_1 \to H_1$ *and* $S : H_2 \to H_2$ *be nonexpansive mappings. Let* $f : H_1 \to H_1$ *and* $g : H_2 \to H_2$ *be* ϕ, φ- *inverse strongly monotone mappings. Assume that* $\Gamma := \{(p,q) : p \in Fix(T) \cap VI(C,f),\ q \in$

$Fix(S) \cap VI(Q,g), Ap = Bq\} \neq \emptyset$. Let $\{(x_n, y_n)\}$ be the sequences generated by $x_0, u \in H_1$ and $y_0, v \in H_2$ and

$$(4.5)\qquad \begin{cases} u_n = x_n - \gamma_n A^*(Ax_n - By_n); \\ z_n = P_C(I - \lambda_n f)u_n; \\ x_{n+1} = \alpha_n u + \beta_n x_n + \delta_n T z_n; \\ v_n = y_n + \gamma_n B^*(Ax_n - By_n); \\ w_n = P_Q(I - \mu_n g)u_n; \\ y_{n+1} = \alpha_n v + \beta_n y_n + \delta_n S w_n \end{cases}$$

where the step-size γ_n is chosen in such a way that

$$(4.6)\qquad \gamma_n \in \left(\epsilon, \frac{2||Ax_n - By_n||^2}{||B^*(Ax_n - By_n)||^2 + ||A^*(Ax_n - By_n)||^2} - \epsilon\right), n \in \pi,$$

otherwise $\gamma_n = \gamma$ (γ being any nonnegative value), where the index set $\pi = \{n : Ax_n - By_n \neq 0\}$. Let λ, μ be positive parameters, $\{\alpha_n\}, \{\beta_n\}$ and $\{\delta_n\}$ be sequences in $(0,1)$ such that $\alpha_n + \beta_n + \delta_n = 1$, satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < a \leq \beta_n \delta_n \leq b < 1$;

(iii) $\lim_{n\to\infty} \frac{1}{n^2 \alpha_n} = 0$.

Then the sequence $\{(x_n, y_n)\}$ converges strongly to $\{(\bar{x}, \bar{y})\} \in \Gamma$.

4.2. Numerical Example

We consider a numerical example in $(\mathbb{R}^2, ||.||_2)$ (where $\mathbb{R}^2$ is the Euclidean plane). Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ and $G : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $F(x_1, x_2) = (2x_1 - x_2,\ x_1 + 2x_2)$ and $G(x_1, x_2) = (x_1 + x_2,\ 2x_1 - x_2)$, respectively. Clearly, F and G are maximal monotone mappings. Also, let $f : \mathbb{R}^2 \to \mathbb{R}^2$ and $g : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x_1, x_2) = (\frac{x_1}{34}, \frac{x_2}{17})$ and $g(x_1, x_2) = (\frac{x_1}{2}, \frac{x_2}{3})$, respectively, then f and g are inverse strongly monotone mappings. Take $\lambda = \mu = \frac{1}{2}$, then we compute the resolvent operators and the Yosida approximation operator as follows:

$$R^F_\lambda = (I + \lambda F)^{-1} = \left(\frac{8x_1}{17} + \frac{2x_2}{17},\ \frac{8x_2}{17} - \frac{2x_1}{17}\right).$$

$$R^G_\mu = (I + \mu G)^{-1} = (2x_1 - 2x_2,\ 6x_2 - 4x_1).$$

$$J^F_\lambda = \frac{1}{\lambda}(I - R^F_\lambda) = \left(\frac{9x_1}{34} - \frac{x_2}{17},\ \frac{x_1}{17} + \frac{9x_2}{34}\right).$$

$$J^G_\mu = \frac{1}{\mu}(I - R^G_\mu) = \left(x_2 - \frac{x_1}{2},\ 2x_1 - \frac{5x_2}{2}\right).$$

Therefore, we have that

$$R_\lambda^F[I+\lambda(J_\lambda^F-f)] = \left(\frac{9x_1}{17}+\frac{61x_2}{578},\ \frac{156x_2}{289}-\frac{2x_1}{17}\right)$$

and

$$R_\mu^G[I+\mu(J_\mu^G-g)] = \left(\frac{x_1}{2}+\frac{x_2}{3},\ x_1-\frac{3x_2}{2}\right).$$

Let the sequences $\alpha_n = \frac{1}{3(n+3)}$, $\beta_n = \frac{n+1}{3(n+3)}$, $\delta_n = \frac{2n+7}{3(n+3)}$ and the step size γ_n be chosen in such a way that for some $\epsilon > 0$,

$$\gamma_n \in \left(\epsilon, \frac{2||Ax_n - By_n||^2}{||B^*(Ax_n-By_n)||^2 + ||A^*(Ax_n-By_n)||^2} - \epsilon\right), n \in \pi,$$

otherwise $\gamma_n = \gamma$ (γ being any nonnegative value), where the index set $\pi = \{n : Ax_n - By_n \neq 0\}$.

Let $u, x_0 \in \mathbb{R}^2$ and $v, y_0 \in \mathbb{R}^2$ be arbitrary. Then, our Algorithm (3.48) becomes

$$\begin{cases} u_n = x_n - \gamma_n A^*(Ax_n - By_n); \\ z_n = R_\lambda^F[I+\lambda(J_\lambda^F - f)]u_n; \\ x_{n+1} = \frac{1}{3(n+3)}u + \frac{n+1}{3(n+3)}x_n + \frac{2n+7}{3(n+3)}z_n; \\ v_n = y_n + \gamma_n B^*(Ax_n - By_n); \\ w_n = R_\mu^G[I+\mu(J_\mu^G - g)]v_n; \\ y_{n+1} = \frac{1}{3(n+3)}v + \frac{n+1}{3(n+3)}y_n + \frac{2n+7}{3(n+3)}w_n; \end{cases}$$

Furthermore, let $A : \mathbb{R}^2 \to \mathbb{R}^2$ and $B : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$A(x) = \begin{pmatrix} 4 & -2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{and} \quad B(\bar{y}) = \begin{pmatrix} 5 & 8 \\ 7 & 4 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

We now consider the following cases for our numerical experiments.

Case 1
Take $x_0 = (-1, -0.5)^T$, $y_0 = (-1, -0.5)^T$, $u = (-1, 0)^T$ and $v = (-1, 0)^T$.

Case 2
Take $x_0 = (-1, -0.5)^T$, $y_0 = (-1, -0.5)^T$, $u = (1, 2)^T$ and $v = (1, 2)^T$.

Case 3
Take $x_0 = (1, 0.5)^T$, $y_0 = (-1, -0.5)^T$, $u = (-1, 0)^T$ and $v = (-1, 0)^T$.

Case 4
Take $x_0 = (-1, -0.5)^T$, $y_0 = (1, 0.5)^T$, $u = (2, -3)^T$ and $v = (-2, 3)^T$.

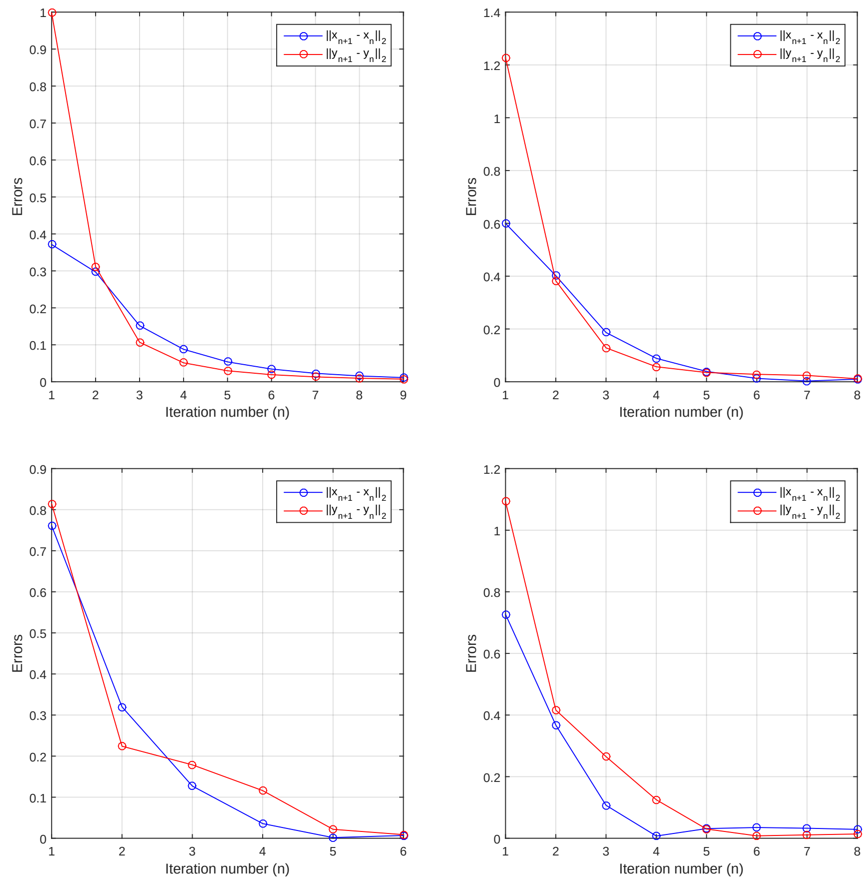

Figure 1: Errors vs Iteration numbers(n): **Case 1** (top left); **Case 2** (top right); **Case 3** (bottom left); **Case 4** (bottom right).

5. Conclusion

In this paper, we introduce the SEMYVIP (1.11)-(1.12) and present an iterative algorithm to approximate a common solution of SEMYVIP (1.11)-(1.12) and infinite families of quasi-pseudo-conctractive and multi-valued generalized strictly pseudocontrative mapping in real Hilbert spaces. The SEMYVIP (1.11)-(1.12) and the map considered in this article generalizes the ones considered in [15], [17] and [33]. Our results do not require any compactness condition nor any prior knowledge of operator norm.

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