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Green's function and an inequality of Lyapunov-type for conformable boundary value problem

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Abstract. In this article, we consider a conformable boundary value problem associated with Robin type boundary conditions and present a Lyapunov-type inequality for the same. Further, we attain a lower bound on the smallest eigenvalue for the corresponding conformable eigenvalue problem using the established result, semi maximum norm and Cauchy–Schwartz inequality.

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1. Introduction

The theory of arbitrary order integrals and derivatives is one of the oldest branches of mathematics that dates back to the works of Euler. Many scientists gave valuable contributions to its development [15, 19]. In this process, they proposed several types of arbitrary order derivatives including the definition proposed in [13], which is the focus of this article.

Interestingly, each definition of arbitrary order derivative captures only a few of the properties of the classical derivative. Recently, in [20], the author proposed the principle of nonlocality in the context of a number of familiar fractional derivatives and proved that the derivative presented in [13] cannot be considered as a fractional derivative. At the same time, several authors have argued that that there is a significant value in exploring conformable derivatives.

The study of conformable derivatives was initiated in [13]. Following this work, the basic concepts of conformable calculus were developed in [1]. Subsequently, the definition of conformable derivative was generalized in many ways [9, 14, 24]. Several authors have explored properties and physical applications of this derivative [3, 4, 6]. Recently, the authors of [5] showed that the definitions of conformable derivatives in [13] and [12] are equivalent to the simple

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change of variables when applied to differentiable functions and discussed the significance of these derivatives in exploring the physical applications.

On the other hand, Theorem 1.1 was established in [16] for the following boundary-value problem (BVP):

$$\begin{cases} y''(t) + r(t)y(t) = 0, & a < t < b, \\ y(a) = y(b) = 0. \end{cases} \tag{1.1}$$

Theorem 1.1. *[16] Assume $r \in C\big([a,b],\mathbb{R}\big)$ and (1.1) possesses a nontrivial solution. We have*

$$\int_a^b |r(s)|ds > \frac{4}{(b-a)}. \tag{1.2}$$

This inequality (1.2) is called the Lyapunov inequality. Due to its applicability in the field of differential equations, several mathematicians have generalized the Lyapunov inequality in many forms. We refer [7, 17, 18, 21, 23, 22] and the references therein for a detailed discussion on this topic.

In particular, the authors of [2] and [8] independently generalized Theorem 1.1 for conformable derivatives as follows:

Assume $r \in C\big([a,b],\mathbb{R}\big)$ and consider the BVP

$$\begin{cases} \big(T^{\alpha}_{a+}y\big)(t) + r(t)y(t) = 0, & a < t < b, \\ y(a) = y(b) = 0. \end{cases} \tag{1.3}$$

Here $1 < \alpha \leq 2$ and T^{α}_{a+} denotes the α^{th}-order conformable differential operator.

Theorem 1.2. *[2] Assume (1.3) possesses a nontrivial solution. We have*

$$\int_a^b |r(s)|ds > \frac{\alpha^{\alpha}}{(\alpha-1)^{\alpha-1}(b-a)^{\alpha-1}}. \tag{1.4}$$

Theorem 1.3. *[8] If (1.3) possesses a nontrivial solution, then*

$$\int_a^b |r(s)|(s-a)^{\alpha-2}ds > \frac{4}{b-a}. \tag{1.5}$$

Recently, the authors of [10, 11] derived a few inequalities of the Lyapunov type for conformable BVPs associated with left focal, right focal and anti-periodic boundary conditions. In line with this approach, here we obtain a Lyapunov-type inequality for the following two-point conformable BVP:

$$\begin{cases} \big(T^{\alpha}_{0+}y\big)(t) + r(t)y(t) = 0, & 0 < t < T, \\ y(0) - y'(0) = y(T) + y'(T) = 0. \end{cases} \tag{1.6}$$

Here $1 < \alpha \leq 2$, $r \in C\big([0,T],\mathbb{R}\big)$ and T^{α}_{0+} denotes the α^{th}-order conformable differential operator.

2. Preliminaries

This section contains preliminaries on conformable calculus.

Definition 2.1. [1] Let $y : [a,\infty) \to \mathbb{R}$ and $0 < \alpha \leq 1$. The α^{th}-order conformable derivative of y starting from a is defined by

$$\big(T^{\alpha}_{a+}y\big)(t) = \lim_{\varepsilon\to 0}\Big[\frac{y\big(t+\varepsilon(t-a)^{1-\alpha}\big)-y(t)}{\varepsilon}\Big], \quad t \in (a,\infty).$$

If $\big(T^{\alpha}_{a+}y\big)$ exists on (a,b) then, $\big(T^{\alpha}_{a+}y\big)(t) \to \big(T^{\alpha}_{a+}y\big)(a)$ as $t \to a^+$.

Definition 2.2. [1] Let $y : [a,\infty) \to \mathbb{R}$, $\alpha > 0$ and take $n \in \mathbb{N}_1$ with $n-1 < \alpha \leq n$. Assume $y^{(n-1)}$ exists on (a,∞). The α^{th}-order conformable derivative of y starting from a is defined by

$$\begin{aligned}\big(T^{\alpha}_{a+}y\big)(t) &= \big(T^{\alpha-n+1}_{a+}y^{(n-1)}\big)(t)\\ &= \lim_{\varepsilon\to 0}\Big[\frac{y^{(n-1)}\big(t+\varepsilon(t-a)^{n-\alpha}\big)-y^{(n-1)}(t)}{\varepsilon}\Big], \quad t \in (a,\infty).\end{aligned}$$

If $y^{(n)}$ exists on (a,∞), then

$$\big(T^{\alpha}_{a+}y\big)(t) = (t-a)^{n-\alpha}y^{(n)}(t), \quad t \in (a,\infty).$$

Definition 2.3. [1] Let y be a real valued function defined on $[a,b]$, $\alpha > 0$ and take $n \in \mathbb{N}_1$ with $n-1 < \alpha \leq n$. The α^{th}-order conformable integral of y starting from a is defined by

$$\big(I^{\alpha}_{a+}y\big)(t) = \frac{1}{(n-1)!}\int_a^t (t-s)^{n-1}(s-a)^{\alpha-n}y(s)ds, \quad t \in [a,b].$$

Theorem 2.4. *[1] Let $y : [a,b] \to \mathbb{R}$, $\alpha > 0$ and take $n \in \mathbb{N}_1$ with $n-1 < \alpha \leq n$. If $y^{(n-1)}$ exists on (a,b), then*

$$\big(I^{\alpha}_{a+}T^{\alpha}_{a+}y\big)(t) = y(t) - \sum_{k=0}^{n-1}\frac{y^{(k)}(a)(t-a)^k}{k!}, \quad t \in (a,b).$$

3. Main Results

Assume $1 < \alpha \leq 2$ and $h \in C\big([0,T],\mathbb{R}\big)$. Consider the conformal BVP

$$\begin{cases}\big(T^{\alpha}_{0+}y\big)(t) + h(t) = 0, \quad 0 < t < T,\\ y(0) - y'(0) = y(T) + y'(T) = 0.\end{cases} \tag{3.1}$$

First, we derive the Green's function for (3.1) and obtain a few of its properties.

Theorem 3.1. *The unique solution of the conformable BVP* (3.1) *is given by*

$$y(t) = \int_0^T G(t,s)h(s)ds. \tag{3.2}$$

Here

$$G(t,s) = \begin{cases} \frac{(T-s+1)(t+1)}{(T+2)} s^{\alpha-2}, & 0 < t \leq s \leq T, \\ \frac{(T-t+1)(s+1)}{(T+2)} s^{\alpha-2}, & 0 < s \leq t \leq T. \end{cases} \tag{3.3}$$

Proof. First, we apply the α^{th}-order conformable integral operator on (3.1). Then, from Theorem 2.4, we obtain

$$y(t) = C_1 + C_2 t - \int_0^t (t-s)s^{\alpha-2}h(s)ds. \tag{3.4}$$

Differentiating (3.4) w.r.t. t, we get

$$y'(t) = C_2 - \int_0^t s^{\alpha-2}h(s)ds. \tag{3.5}$$

Since $y(0) - y'(0) = 0$, we have

$$C_1 - C_2 = 0. \tag{3.6}$$

The second boundary condition and (3.6) yield

$$C_2 = \int_0^T \left(\frac{T-s+1}{T+2}\right) s^{\alpha-2}h(s)ds. \tag{3.7}$$

From (3.6) and (3.7), we have

$$C_1 = \int_0^T \left(\frac{T-s+1}{T+2}\right) s^{\alpha-2}h(s)ds. \tag{3.8}$$

Substituting the values of C_1 and C_2 from (3.7) and (3.8) in (3.4), we obtain

$$\begin{aligned} y(t) &= \int_0^T \left(\frac{T-s+1}{T+2}\right) s^{\alpha-2}h(s)ds + t\int_0^T \left(\frac{T-s+1}{T+2}\right) s^{\alpha-2}h(s)ds \\ &\quad - \int_0^t (t-s)s^{\alpha-2}h(s)ds \\ &= (t+1)\int_0^t \left(\frac{T-s+1}{T+2}\right) s^{\alpha-2}h(s)ds \\ &\quad + (t+1)\int_t^T \left(\frac{T-s+1}{T+2}\right) s^{\alpha-2}h(s)ds - \int_0^t (t-s)s^{\alpha-2}h(s)ds \\ &= \int_0^t \frac{(T-t+1)(s+1)}{(T+2)} s^{\alpha-2}h(s)ds \\ &\quad + \int_t^T \frac{(T-s+1)(t+1)}{(T+2)} s^{\alpha-2}h(s)ds = \int_0^T G(t,s)h(s)ds. \end{aligned}$$

The proof is complete. □

Lemma 3.2. *The Green's function $G(t,s)$ has the following properties:*

1. *$G(t,s) > 0$ for all $(t,s) \in (0,T] \times (0,T]$.*
2. *$G(t,s) \leq G(s,s)$ for all $(t,s) \in (0,T] \times (0,T]$.*
3. *$\max_{s\in[0,T]} s^{2-\alpha}G(s,s) = \frac{T+2}{4}$.*
4. *$\max_{t\in[0,T]} \int_0^T G(t,s)ds = \frac{T^\alpha}{\alpha}\left[\frac{(T+\alpha)}{\alpha(T+2)}\right]^{\frac{\alpha}{(\alpha-1)}} + \frac{T^{\alpha-1}(T+\alpha)}{\alpha(\alpha-1)(T+2)}$.*
5. *$\max_{t\in[0,T]} \int_0^T s^{2-\alpha}G(t,s)ds = \frac{T^2+4T}{8}$.*

Proof. Define the functions

$$G_1(t,s) = \frac{(T-s+1)(t+1)}{(T+2)}s^{\alpha-2} \text{ and } G_2(t,s) = \frac{(T-t+1)(s+1)}{(T+2)}s^{\alpha-2}.$$

The proof of (1) is trivial. We can easily check that $G_1(t,s)$ increases with respect to t. Differentiating $G_2(t,s)$ with respect to t for every fixed s, we observe that $G_2(t,s)$ is a decreasing function of t. Thus, we have (2). Clearly from (3.3), we have

$$s^{2-\alpha}G(s,s) = \frac{(T-s+1)(s+1)}{(T+2)}, \quad s \in [0,T]. \tag{3.9}$$

Define $H(s)$ as the right hand side of (3.9). Now, differentiating $H(s)$ with respect to s and equating it to 0, we obtain $s = \frac{T}{2}$. Again, differentiating $H'(s)$ with respect to s, we observe that $H''(s) < 0$ at $s = \frac{T}{2}$. So, $H(s)$ attains its maximum at $s = \frac{T}{2}$. Substituting $s = \frac{T}{2}$ in (3.9), we have (3). For (4), consider

$$\begin{aligned}\int_0^T G(t,s)ds &= \int_0^t \frac{(T-t+1)(s+1)}{(T+2)}s^{\alpha-2}ds + \int_t^T \frac{(T-s+1)(t+1)}{(T+2)}s^{\alpha-2}ds \\ &= \left(\frac{T-t+1}{T+2}\right)\left(\frac{t^{\alpha-1}}{\alpha-1} + \frac{t^\alpha}{\alpha}\right) + \frac{(T+1)(t+1)}{(T+2)} \\ &\quad \left[\frac{T^{\alpha-1}}{\alpha-1} - \frac{t^{\alpha-1}}{\alpha-1}\right] - \left(\frac{t+1}{T+2}\right)\left[\frac{T^\alpha}{\alpha} - \frac{t^\alpha}{\alpha}\right] \\ &= -\frac{t^\alpha}{\alpha(\alpha-1)} + \frac{(t+1)(T+\alpha)T^{\alpha-1}}{(T+2)\alpha(\alpha-1)}. \end{aligned} \tag{3.10}$$

Define $H_1(t)$ as the right hand side of (3.10). Now, differentiating $H_1(t)$ with respect to t and equating it to 0, we obtain $t = T\left[\frac{T+\alpha}{\alpha(T+2)}\right]^{\frac{1}{(\alpha-1)}}$. Again, differentiating ${H_1}'(t)$ with respect to t, we observe that ${H_1}''(t) \leq 0$ at $t = T\left[\frac{T+\alpha}{\alpha(T+2)}\right]^{\frac{1}{(\alpha-1)}}$. So, $H_1(t)$ attains its maximum at $t = T\left[\frac{T+\alpha}{\alpha(T+2)}\right]^{\frac{1}{(\alpha-1)}}$. Substituting $t = T\left[\frac{T+\alpha}{\alpha(T+2)}\right]^{\frac{1}{(\alpha-1)}}$ in (3.10), we have (4). Consider

$$\begin{aligned}\int_0^T s^{2-\alpha}G(t,s)ds &= \int_0^t \frac{(T-t+1)(s+1)}{T+2}ds + \int_t^T \frac{(T-s+1)(t+1)}{T+2}ds \\ &= \left(\frac{T-t+1}{T+2}\right)\left(t+\frac{t^2}{2}\right) + \frac{(T+1)(t+1)(T-t)}{(T+2)} \\ &\quad - \frac{(t+1)(T^2-t^2)}{2(T+2)} \\ &= -\frac{t^2}{2} + \frac{T(t+1)}{2}. \end{aligned} \tag{3.11}$$

Define $H_2(t)$ as the right hand side of (3.11). Now, differentiating $H_2(t)$ with respect to t and equating it to 0, we obtain $t = \frac{T}{2}$. Again, differentiating ${H_2}'(t)$ with respect to t, we observe that ${H_2}''(t) < 0$ at $t = \frac{T}{2}$. So, $H_2(t)$ attains its maximum at $t = \frac{T}{2}$. Substituting $t = \frac{T}{2}$ in (3.11), we have (6). The proof is complete. □

Theorem 3.3. *Suppose* (1.6) *possesses a nontrivial solution. We have*

$$\int_0^T s^{\alpha-2}|r(s)|ds > \frac{4}{T+2}. \tag{3.12}$$

Proof. Clearly, $C([0,T],\mathbb{R})$ is a Banach space with the norm

$$\|y\|_C = \max_{t\in[0,T]} |y(t)|.$$

Every solution of (1.6) satisfies the following equation:

$$y(t) = \int_0^T G(t,s)r(s)y(s)ds.$$

Consider

$$\begin{aligned}|y(t)| &= \left|\int_0^T G(t,s)r(s)y(s)ds\right| \\ &\le \int_0^T |G(t,s)||r(s)||y(s)|ds \\ &\le \|y\| \int_0^T |G(t,s)||r(s)|ds \\ &= \|y\| \int_0^T |s^{2-\alpha}G(t,s)||s^{\alpha-2}r(s)|ds,\end{aligned}$$

implying that

$$\|y\| \le \|y\| \max_{s\in[0,T]} \left[|s^{2-\alpha}G(t,s)|\right]\left[\int_0^T |s^{\alpha-2}r(s)|ds\right].$$

An application of Lemma 3.2 yields the result. The proof is complete. □

4. Application

We consider the eigenvalue problem corresponding to (1.6) and find an estimate on a lower bound for the smallest eigenvalue using three different methods.

Definition 4.1. A Lyapunov Inequality Lower Bound (LILB) is defined as a lower estimate for the smallest eigenvalue obtained from (3.12) by setting $r(s) = \lambda$, that is,

$$\lambda \geq \frac{1}{T\ G_{max}}. \tag{4.1}$$

Definition 4.2. A Cauchy-Schwartz Inequality Lower Bound (CSILB) is given by

$$\lambda \geq \left[\int_0^T \int_0^T G^2(t,s)dsdt\right]^{-\frac{1}{2}}. \tag{4.2}$$

Definition 4.3. A Semi Maximum Norm Lower Bound (SMNLB) is given by

$$\lambda \geq \frac{1}{\max_{0\leq t\leq T}\int_0^T |G(t,s)|ds}. \tag{4.3}$$

Theorem 4.4. *Assume y be a nontrivial solution of the conformable eigenvalue problem*

$$\begin{cases} \left(T_{0+}^{\alpha}y\right)(t) + \lambda y(t) = 0, \quad 0 < t < T, \\ y(0) - y'(0) = y(T) + y'(T) = 0, \end{cases}$$

where $y(t) \neq 0$ for each $t \in (0,T)$. Then,

$$|\lambda|_{(LILB)} > \frac{4(\alpha-1)}{T^{\alpha-1}(T+2)}, \quad 1 < \alpha \leq 2, \tag{4.4}$$

$$|\lambda|_{(CSILB)} \geq \frac{4\sqrt{(2\alpha-3)}}{(T+2)T^{\alpha-1}}, \quad \frac{3}{2} \leq \alpha \leq 2, \tag{4.5}$$

$$|\lambda|_{(SMNLB)} \geq \frac{1}{\frac{T^{\alpha}}{\alpha}\left[\frac{(T+\alpha)}{\alpha(T+2)}\right]^{\frac{\alpha}{\alpha-1}} + \frac{T^{\alpha-1}(T+\alpha)}{\alpha(\alpha-1)(T+2)}}, \quad 1 < \alpha \leq 2. \tag{4.6}$$

Proof. We take $r(s) = \lambda$ in (3.12). Then, we obtain

$$\int_0^T s^{\alpha-2}|\lambda|ds > \frac{4}{T+2},$$

implying that

$$|\lambda|\left(\frac{T^{\alpha-1}}{\alpha-1}\right) > \frac{4}{T+2}.$$

This proves the result (4.4). Consider,

$$\begin{aligned} \lambda &\geq \left(\int_0^T \int_0^T \left|G(t,s)\right|^2 dsdt\right)^{-\frac{1}{2}} \\ &= \left(\int_0^T \int_0^T \left|s^{2-\alpha} G(t,s) s^{\alpha-2}\right|^2 dsdt\right)^{-\frac{1}{2}} \\ &= \left(\int_0^T \int_0^T \left|s^{2-\alpha} G(t,s)\right|^2 s^{2\alpha-4} dsdt\right)^{-\frac{1}{2}} \\ &\geq \left(\int_0^T \int_0^T \left[\frac{T+2}{4}\right]^2 s^{2\alpha-4} dsdt\right)^{-\frac{1}{2}} \\ &= \frac{4}{T+2}\left(\int_0^T \frac{T^{2\alpha-3}}{2\alpha-3} dt\right)^{-\frac{1}{2}}, \end{aligned}$$

implying that

$$\lambda \geq \frac{4\sqrt{(2\alpha-3)}}{(T+2)T^{\alpha-1}}.$$

Thus, we have (4.5). The proof of (4.6) follows from (4.3) and Lemma 3.2. The proof is complete. □

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