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Existence and multiplicity of solutions for singular fractional elliptic system via the Nehari manifold approach

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Abstract. In this paper, we study the existence and multiplicity of positive nontrivial solutions of the following singular fractional elliptic system

$$\begin{cases} (-\Delta)_p^s u = a(x)u^{-\gamma} + \lambda f(x,u,v) \text{ in } \Omega, \\ (-\Delta)_p^s v = b(x)v^{-\gamma} + \lambda g(x,u,v) \text{ in } \Omega, \\ u = v = 0 \text{ on } \mathbb{R}^N \backslash \Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N > ps$ with $s \in (0,1)$, λ is a positive parameter and $0 < \gamma < 1 < p < r < p_s^* - 1, p_s^* = \frac{Np}{N-ps}$. $a, b \in L^\infty(\Omega, \mathbb{R}_*^+), f, g \in C(\Omega \times \mathbb{R} \times \mathbb{R}, \mathbb{R}^+)$ are positively homogeneous functions of degree $(r-1)$. The results are obtained by using the fibering method, Nehari Manifold technique and applying Ekeland's variational principle.

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1. Introduction

The paper focuses on the variational case to prove the existence and multiplicity of nontrivial positive solutions. We consider the following singular fractional elliptic system

$$(P_\lambda) \qquad \begin{cases} (-\Delta)_p^s u = a(x)u^{-\gamma} + \lambda f(x,u,v) \text{ in } \Omega, \\ (-\Delta)_p^s v = b(x)v^{-\gamma} + \lambda g(x,u,v) \text{ in } \Omega, \\ u = v = 0 \text{ on } \mathbb{R}^N \backslash \Omega, \end{cases}$$

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where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N > ps$ with $s \in (0,1)$, λ is a positive parameter, $0 < \gamma < 1 < p < r < p_s^* - 1$, and $p_s^* = \frac{Np}{N-ps}$ is the fractional critical Sobolev exponent. $a, b \in L^\infty(\Omega, \mathbb{R}_*^+)$, $f, g \in C(\Omega \times \mathbb{R} \times \mathbb{R}, \mathbb{R}^+)$ are positively homogeneous functions of degree $(r-1)$ such that

$$\left\{ \begin{array}{l} f(x,tu,tv) = t^{r-1} f(x,u,v), (x,u,v) \in \Omega \times \mathbb{R} \times \mathbb{R}, t > 0, \\ \\ g(x,tu,tv) = t^{r-1} g(x,u,v), (x,u,v) \in \Omega \times \mathbb{R} \times \mathbb{R}, t > 0. \end{array} \right. \tag{1.1}$$

Let $(-\Delta)_p^s$ be the fractional p-Laplacian operator defined on smooth functions by

$$(-\Delta)_p^s u = 2 \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y))}{|x-y|^{N+ps}} dy, x \in \mathbb{R}^N,$$

Our aim in the present work is to study the existence and multiplicity of nontrivial positive solutions by combining the variational method with Nehari Manifold, fibering maps and Ekeland's variational principle, see [2], [4], [3],[9], [10].

Precisely, we assume there exists a function $H : \bar{\Omega} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfying

$$H_u(u,v) = f(u,v) \text{ and } H_v(u,v) = g(u,v).$$

From (1.1), we can easily deduce that H is homogeneous of degree r which satisfies the following assumptions

$$\begin{aligned} H(x,tu,tv) &= t^r H(x,u,v), t > 0, \\ rH(x,u,v) &= uf(x,u,v) + vg(x,u,v) \text{ (Euler identity)}, \\ |H(x,u,v)| &\leq K(|u|^r + |v|^r), \text{ for some constant } K > 0. \end{aligned} \tag{1.2}$$

$$H^{\pm}(x,u,v) = \max(\pm H(x,u,v), 0) \neq 0 \text{ for all } (u,v) \neq (0,0). \tag{1.3}$$

Then, we have the following result of existence.

Theorem 1.1. *Suppose that* $0 < \gamma < 1 < p < r < p_s^* - 1$ *and the assumptions* (1.1)-(1.3) *hold. Then, there exists* $\Gamma > 0$ *such that, for all* $\lambda \in (0, \Gamma)$*, the system* (P_λ) *has at least two positive nontrivial solutions.*

There is a growing interest in the equations of this type, since they arise in many fields of sciences, physical phenomena, probability, stochastic calculus, and finance. For more details, one can see [5], [6], [14], [19].

This method has been applied to partial differential equations, nonlinear, semilinear and quasilinear systems, involving Laplacian and fractional p-Laplacian. Among that the many studies have been published, we refer the reader to see [8], [15],[17], [27], [18], [1]. At this point, we briefly recall literature concerning related singular problems and systems. In [20], by K. Saoudi & A. Ghanmi, they studied the following singular problem

$$\begin{cases} (-\Delta)_p^s u = \frac{a(x)}{u^\gamma} + \lambda f(x,u) \text{ in } \Omega, \\ \\ u = 0 \text{ on } \mathbb{R}^N \backslash \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N, N \geq 2$, is a bounded smooth domain, $a \in L^\infty(\Omega)$, λ is a positive parameter, $p \geq 2$ such that $N \geq ps$ and $0 < \gamma < 1 < p < r < p_s^*$ where $p_s^* = \frac{Np}{N-ps}$ and r is the homogeneity degree of the function f. Under appropriate assumptions on the function f, the authors have used the method of Nehari manifold combined with the fibering maps and they shown the existence of $T_{p,r,\gamma}$ such that for all $\lambda \in (0, T_{p,r,\gamma})$, the problem has at least two positive solutions.

Moreover in [2] Adimurthi and Giacomoni proved the multiplicity of positive solutions for a singular and critical elliptic problem in $\mathbb{R}^2$, and in [22], [13] Mohammed, Coclit and Palmieri established the existence of positive solutions of p-Laplace equation with singular nonlinearity. Then, in [11], Caffarelli and Silvestre studied the fractional Laplacian through extension theory. In [12] Chen, Hajaiej and Wang, studied the existence, non-existence and uniqueness of positive weak solutions of the semilinear fractional equation.

Some other results dealing with the existence of solutions concerning the singular problem have been achived by Ghanmi, Saoudi and Kratou (see [21], [23], [24]). They studied the existence and multiplicity of solutions of the semilinear singular elliptic equations involving the fractional Laplace and fractional p-Laplacian operator using Nehari manifold, the fibering method and applying Ekeland's variational principle with boundary conditions. See, for instance, [9], [2],[25], [26] and the references therein.

The outline of this paper is as follows. In the second section, we present some basic notations and preliminaries. In the third section, we apply the decomposition of Nehari Manifold, define energy functional associated with system (P_λ), prove that it is bounded below, coercive in $\mathcal{N}_\lambda$ and analyze the fibering method. In the last section, using Ekland's variational principle, we prove the existence and multiplicity of nontrivial positive solutions.

2. Notations and preliminaries

First, we introduce the following spaces. For $s \in (0,1)$, we define the fractional Sobolev space see [16].

$$W^{s,p}(\Omega) = \left\{ u \in L^p(\Omega) : \frac{u(x)-u(y)}{|x-y|^{\frac{N+ps}{p}}} \in L^p(\Omega) \right\},$$

with the norm

$$\|u\|_{W^{s,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \left(\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy \right)^{\frac{1}{p}}.$$

Consider the space

$$X = \left\{ u : \mathbb{R}^N \to \mathbb{R}, u \in L^p(\Omega) \text{ and } \frac{u(x) - u(y)}{|x - y|^{\frac{N+ps}{p}}} \in L^p(\Sigma) \right\},$$

with the norm

$$\|u\|_X = \|u\|_{L^p(\Omega)} + \left(\int_\Sigma \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} \right)^{\frac{1}{p}},$$

where $\Sigma = \mathbb{R}^{2N} \backslash ((\mathbb{R}^N \backslash \Omega) \times (\mathbb{R}^N \backslash \Omega))$.

Let X_0 denote the usual space defined as follow

$$X_0 = \left\{ u \in X : u = 0 \text{ a.e. in } \mathbb{R}^N \backslash \Omega \right\},$$

with the norm

$$\|u\|_{X_0} = (T(u,u))^{\frac{1}{p}} = \left(\int_\Sigma \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} dxdy \right)^{\frac{1}{p}},$$

where

$$T(u,z) = \int_\Sigma \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y))(z(x) - z(y))}{|x - y|^{N+ps}} dxdy.$$

Setting $E = X_0 \times X_0$ a reflexive Banach space, with the norm

$$\|(u,v)\|_E = \left(\|u\|_{X_0}^p + \|v\|_{X_0}^p \right)^{\frac{1}{p}} = (T(u,u) + T(v,v))^{\frac{1}{p}},$$

where $\mu_{s,p}$ is the best Sobolev constant of the embedding from E into $L^{p_s^*}(\Omega)$ given by

$$\mu_{s,p} = \inf_{u \in X_0} \left(\frac{\|u\|_{X_0}}{\|u\|_{L^{p_s^*}}} \right)^p. \tag{2.1}$$

Then, we give the definition of a weak solution of problem (P_λ).

Definition 2.1. We say that $(u,v) \in E$ is a weak solution of system (P_λ) if for every $(z,w) \in E$, we have

$$\begin{aligned} T(u,z) + T(v,w) \quad &= \int_\Omega \left(a(x) u^{-\gamma} z dx + b(x) v^{-\gamma} w \right) dx \\ &+ \lambda \int_\Omega (z f(x,u,v) + w g(x,u,v))\, dx. \end{aligned} \tag{2.2}$$

3. Nehari Manifold and Fibering maps

The energy functional $J_\lambda : E \to \mathbb{R}$ associated to the problem (P_λ) is given by

$$J_\lambda(u,v) = \frac{1}{p}\left\|(u,v)\right\|_E^p - \frac{1}{1-\gamma}\int_\Omega (a(x)u^{1-\gamma} + b(x)v^{1-\gamma})dx - \lambda\int_\Omega H(x,u,v)dx.$$

note that the critical points of J are weak solutions of (P_λ).

Let Γ be a constant defined by

(3.1)

$$\Gamma = \frac{|\Omega|^{\frac{(r+\gamma-1)(p-p_s^*)}{p_s^*(p+\gamma-1)}}}{rK}\left(\frac{p+\gamma-1}{r+\gamma-1}\right)\left[\frac{(r+\gamma-1)\max(\left\|a\right\|_\infty, \left\|b\right\|_\infty)}{(r-p)\,\mu_{s,p}^{\frac{1-\gamma}{p}}}\right]^{\frac{p-r}{p-1+\gamma}}.$$

For our convenience, we put

$$\begin{aligned} Q &= Q\,(u,v) = \left\|(u,v)\right\|_E^p = T(u,u) + T(v,v), \\ R &= R\,(u,v) = \int_\Omega (a(x)u^{1-\gamma} + b(x)v^{1-\gamma})dx, \\ S &= S\,(u,v) = \int_\Omega H(x,u,v)dx, \end{aligned}$$

$$Q_0 = Q\,(u_0,v_0)\,, R_0 = R\,(u_0,v_0)\,, S_0 = S\,(u_0,v_0)\,, (u_0,v_0) \in \mathcal{N}_\lambda$$

and for all sequence $\{(u_n,v_n)\} \in E$, we set

$$Q_n = Q\,(u_n,v_n)\,, R_n = R\,(u_n,v_n)\,, S_n = S\,(u_n,v_n)\,,$$

Therefore, we can write J_λ as follows

$$J_\lambda(u,v) = \frac{1}{p}Q - \frac{1}{1-\gamma}R - \lambda S.$$

Thus, $J_\lambda \in C^1(E,\mathbb{R}), J_\lambda^{'} : E \to E^{'}$ for every $(u,v) \in E$ and we have

$$\left\langle J_\lambda^{'}(u,v),(u,v)\right\rangle = Q - R - \lambda r S.$$

To find the critical points of J_λ , we will minimize the energy functional J_λ on the constraint of Nehari manifold

$$\mathcal{N}_\lambda = \left\{(u,v) \in E\backslash\left\{(0,0)\right\} : \left\langle J_\lambda^{'}(u,v),(u,v)\right\rangle = 0\right\}.$$

Then, $(u,v) \in \mathcal{N}_\lambda$ if and only if

$$Q - R - \lambda r S = 0 \text{ for all } (u,v) \in E \setminus (0,0). \tag{3.2}$$

We define $\phi_{u,v}(t) : \mathbb{R}^+ \to \mathbb{R}$ as follows

$$\phi_{u,v}(t) = J_\lambda(tu,tv) = \frac{1}{p}Qt^p - \frac{1}{1-\gamma}Rt^{1-\gamma} - \lambda S t^r.$$

Then, the first and the second derivative of the map $\phi_{u,v}(t)$ is given by

$$\phi'_{u,v}(t) = Qt^{p-1} - Rt^{-\gamma} - \lambda r S t^{r-1},$$

and

$$\phi''_{u,v}(t) = (p-1)Qt^{p-2} + \gamma R t^{-\gamma-1} - \lambda r(r-1)St^{r-2}.$$

So, it is easy to see that

$$(tu, tv) \in \mathcal{N}_\lambda \text{ if and only if } \phi'_{u,v}(t) = 0,$$

and

$$(u, v) \in \mathcal{N}_\lambda \text{ if and only if } \phi'_{u,v}(1) = 0.$$

Hence, for $(u, v) \in \mathcal{N}_\lambda$, we obtain

$$\begin{aligned} \phi''_{u,v}(1) \quad &= (p-1)Q + \gamma R - \lambda r(r-1)S \\ &= (r+\gamma-1)R - (r-p)Q \\ &= (p+\gamma-1)Q - \lambda r(r+\gamma-1)S \\ &= (p+\gamma-1)R - \lambda r(r-p)S. \end{aligned}$$

Now, we decompose $\mathcal{N}_\lambda$ into three parts $\mathcal{N}_\lambda^+, \mathcal{N}_\lambda^-, \mathcal{N}_\lambda^0$ corresponding, respectively, to local minima, local maxima and points of inflection of $\phi_{u,v}$ defined as follows

$$\begin{aligned} \mathcal{N}_\lambda^0 \quad &= \left\{(u, v) \in \mathcal{N}_\lambda, \phi''_{u,v}(1) = 0\right\}, \\ \mathcal{N}_\lambda^+ \quad &= \left\{(u, v) \in \mathcal{N}_\lambda, \phi''_{u,v}(1) > 0\right\}, \\ \mathcal{N}_\lambda^- \quad &= \left\{(u, v) \in \mathcal{N}_\lambda, \phi''_{u,v}(1) < 0\right\}. \end{aligned}$$

Furthermore, if $(u, v) \in \mathcal{N}_\lambda^+$, we have

$$R > \frac{r-p}{r+\gamma-1} Q > 0. \tag{3.3}$$

Now, we prove the following useful lemmas.

Lemma 3.1. *Suppose that (u_0, v_0) is a local minimizer of J_λ on $\mathcal{N}_\lambda$ and $(u_0, v_0) \notin \mathcal{N}_\lambda^0$. Then (u_0, v_0) is a critical point of J_λ.*

Proof. According to Theorem of Lagrange multiplier, there exist a real constant η such that

$$J'_\lambda(u_0, v_0) = \eta \xi'(u_0, v_0),$$

$$(u_0, v_0) \in \mathcal{N}_\lambda : \langle J'_\lambda(u_0, v_0), (u_0, v_0)\rangle = \eta \langle \xi'(u_0, v_0), (u_0, v_0)\rangle = 0,$$

where the constraint is

$$\xi(u, v) = Q - R - \lambda r S.$$

For all $(u,v) \in \mathcal{N}_\lambda$ and $(u_0,v_0) \notin \mathcal{N}_\lambda^0$, we have

$$\begin{aligned}\langle \xi'(u,v),(u,v)\rangle \qquad &= pQ-(1-\gamma)R-\lambda r^2 S\\ &= (p-1)Q+\gamma R-\lambda r(r-1)S\\ &= \phi''_{u,v}(1)\neq 0,\end{aligned}$$

which implies that $\eta = 0$, and $J'_\lambda(u_0,v_0) = 0$. $\square$

Lemma 3.2. *$J_\lambda$is coercive and bounded from below on $\mathcal{N}_\lambda$.*

Proof. Let $(u,v) \in \mathcal{N}_\lambda$. From (2.1) and by Hölder's inequality, we can write

$$\begin{aligned} R \quad &\le \quad \mu_{s,p}^{\frac{\gamma-1}{p}} |\Omega|^{\frac{p_s^*+\gamma-1}{p_s^*}} \max(\|a\|_\infty, \|b\|_\infty) \|(u,v)\|_E^{1-\gamma}\\ &= \quad \mu_{s,p}^{\frac{\gamma-1}{p}} |\Omega|^{\frac{p_s^*+\gamma-1}{p_s^*}} \max(\|a\|_\infty, \|b\|_\infty) Q^{\frac{1-\gamma}{p}}, \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} S \quad &\le \quad K\int_\Omega (|u|^r+|v|^r)\,dx \le K|\Omega|^{\frac{p_s^*-r}{p_s^*}}\left(\|u\|_{p_s^*}^r+\|v\|_{p_s^*}^r\right)\\ &\le \quad K|\Omega|^{\frac{p_s^*-r}{p_s^*}} \mu_{s,p}^{\frac{-r}{p}} \|(u,v)\|_E^r = K|\Omega|^{\frac{p_s^*-r}{p_s^*}} \mu_{s,p}^{\frac{-r}{p}} Q^{\frac{r}{p}}. \end{aligned} \tag{3.5}$$

Then, using the assumptions (3.2) and (3.4), we obtain

$$\begin{aligned} J_\lambda(u,v) \qquad &= \frac{1}{p}Q - \frac{1}{1-\gamma}R - \lambda S\\ &= \left(\frac{r-p}{rp}\right)Q - \left(\frac{\gamma+r-1}{r(1-\gamma)}\right)R\\ &\ge \left(\frac{r-p}{rp}\right)Q - \left(\frac{\gamma+r-1}{r(1-\gamma)}\right)\mu_{s,p}^{\frac{\gamma-1}{p}}(\|a\|_\infty+\|b\|_\infty)|\Omega|^{\frac{p_s^*+\gamma-1}{p_s^*}} Q^{\frac{1-\gamma}{p}}. \end{aligned}$$

Finaly, since $0<\gamma<1$ and $r>p>1>1-\gamma$, the functional J_λ is coercive and bounded from below on $\mathcal{N}_\lambda$. $\square$

Lemma 3.3. *Let $\lambda \in (0,\Gamma)$. Then, there exist two numbers denoted t_1 and t_2 such that*

$$\phi'_{u,v}(t_1) = \phi'_{u,v}(t_2) = 0,$$

and

$$\phi''_{u,v}(t_1) < 0 < \phi''_{u,v}(t_1),$$

that is $(t_1u,t_1v) \in \mathcal{N}_\lambda^+$ and $(t_2u,t_2v)\in\mathcal{N}_\lambda^-$.

Proof. We define $\psi_{u,v}(t):\mathbb{R}^+ \to \mathbb{R}$ by

$$\psi_{u,v}(t) = Qt^{p-r} - Rt^{1-\gamma-r} - \lambda r S.$$

Then, we have

$$\begin{aligned}\psi'_{u,v}(t) \qquad &= (r+\gamma-1)Rt^{-\gamma-r}-(r-p)Qt^{p-r-1}\\ &= (r+\gamma-1)Rt^{-\gamma-r}-(r-p)Qt^{p-r-1},\end{aligned}$$

then $\psi'_{u,v}(t)=0$ if and only if $\psi_{u,v}(t)$ has a unique critical point at

$$t_{\max}=\left(\frac{(r+\gamma-1)R}{(r-p)Q}\right)^{\frac{1}{p+\gamma-1}}. \tag{3.6}$$

Thus

$$\lim_{t\to+\infty}\psi_{u,v}(t)=-\lambda rS \text{ and } \lim_{t\to 0^+}\psi_{u,v}(t)=-\infty.$$

Moreover, $\psi'_{u,v}(t)>0$ for all $0<t<t_{\max}$, $\psi'_{u,v}(t)<0$ for all $t>t_{\max}$, and

$$\begin{aligned}\psi_{u,v}(t_{\max}) &= Qt_{\max}^{p-r}-Rt_{\max}^{-\gamma-r+1}-\lambda rS\\ &= \left(\frac{\gamma+r-1}{r-p}\right)^{\frac{p-r}{p-1+\gamma}}\left(\frac{p+\gamma-1}{\gamma+r-1}\right)Q^{\frac{\gamma+r-1}{p+\gamma-1}}R^{\frac{p-r}{p+\gamma-1}}-\lambda rS. \end{aligned}\tag{3.7}$$

From (3.1), (3.4) and (3.5) one sees that

$$\begin{aligned}&\psi(t_{\max})\\ \geq\ &\left(\frac{\gamma+r-1}{r-p}\right)^{\frac{p-r}{p+\gamma-1}}\left(\frac{p+\gamma-1}{r+\gamma-1}\right)\\ &\cdot\left(\frac{\mu_{s,p}^{-\frac{(\gamma-1)(r-p)}{p}}|\Omega|^{-\frac{(p_s^*+\gamma-1)(r-p)}{p_s^*}}Q^{r+\gamma-1}}{\left(\max(\|a\|_\infty,\|b\|_\infty)Q^{\frac{1-\gamma}{p}}\right)^{r-p}}\right)^{\frac{1}{p+\gamma-1}}\\ &-\lambda rK|\Omega|^{\frac{p_s^*-r}{p_s^*}}\mu_{s,p}^{\frac{-r}{p}}Q^{\frac{r}{p}}\\ \geq\ &(\Gamma-\lambda)\,rK\mu_{s,p}^{\frac{-r}{p}}|\Omega|^{\frac{p_s^*-r}{p_s^*}}\|(u,v)\|_E^r.\end{aligned}$$

Next, for all $\lambda\in(0,\Gamma)$, we obtain

$$\psi(t_{\max})\geq 0.$$

Consequently, there exist t_1 and t_2 such that $0<t_1<t_{\max}<t_2$ verify

$$\psi_{u,v}(t_1)=\psi_{u,v}(t_2)=0,$$

and

$$\psi'_{u,v}(t_1)<0<\psi'_{u,v}(t_2).$$

Finaly, we can conclude that $(t_1u,t_1v)\in\mathcal{N}_\lambda^+$ and $(t_2u,t_2v)\in\mathcal{N}_\lambda^-$. □

Corollary 3.4. *For all $\lambda\in(0,\Gamma)$, $\mathcal{N}_\lambda^\pm\neq\emptyset$ and $\mathcal{N}_\lambda^0=\{(0,0)\}$. Moreover $\mathcal{N}_\lambda^-$ is a closed set in $E-$topology.*

Proof. First, according to Lemma 3.3, $\mathcal{N}_\lambda^\pm$ are non-empty for $\lambda\in(0,\Gamma)$. Now, we proceed by contradiction to establish that $\mathcal{N}_\lambda^0=\{(0,0)\}$ for $\lambda\in(0,\Gamma)$. To do this, let us suppose that there exists $(0,0)\neq(u_0,v_0)\in\mathcal{N}_\lambda^0$, then we have

$$(p+\gamma-1)Q_0-\lambda r(r+\gamma-1)S_0=0, \tag{3.8}$$

and

$$0 = Q_0 - R_0 - \lambda r S_0 = \left(\frac{r-p}{r+\gamma-1}\right) Q_0 - R_0. \tag{3.9}$$

Hence, from (3.8) and (3.9), we obtain

$$\begin{aligned} 0 \quad & < \psi_{u_0,v_0}(t_{\max}) - \lambda r S_0 \\ & = \left(\frac{\gamma+r-1}{r-p}\right)^{\frac{p-r}{p-1+\gamma}} \left(\frac{p-1+\gamma}{\gamma+r-1}\right) \frac{(Q_0)^{\frac{\gamma+r-1}{p+\gamma-1}}}{(R_0)^{\frac{r-p}{p+\gamma-1}}} - \lambda r S_0 \\ \leq \quad & \left(\frac{r+\gamma-1}{r-p}\right)^{\frac{p-r}{p-1+\gamma}} \left(\frac{p+\gamma-1}{r+\gamma-1}\right) \left(\frac{r+\gamma-1}{r-p}\right)^{\frac{r-p}{p+\gamma-1}} Q_0 \\ & -(\frac{p+\gamma-1}{r+\gamma-1})Q_0 \\ & = 0, \end{aligned}$$

for all $\lambda \in (0, \Gamma)$, which is impossible. Thus $\mathcal{N}_\lambda^0 = \{(0,0)\}$.

Next, we prove that $\mathcal{N}_\lambda^-$ is closed for all $\lambda \in (0, \Gamma)$. So, we consider a sequence $\{(u_n, v_n)\} \subset \mathcal{N}_\lambda^-$ such that $(u_n, v_n) \to (u, v)$ in E as $n \to +\infty$, and prove that $(u, v) \subset \mathcal{N}_\lambda^-$. By using the definition of $\mathcal{N}_\lambda^-$, we have

$$Q_n - R_n - \lambda r S_n = 0,$$

and

$$(p-1)Q_n + \gamma R_n - \lambda r(r-1)S_n < 0.$$

Hence, we get that

$$Q - R - \lambda r S = 0,$$

and

$$(p-1)Q + \gamma R - \lambda r(r-1)S \leq 0.$$

Therefore, $(u, v) \in \mathcal{N}_\lambda^0 \cup \mathcal{N}_\lambda^- = \mathcal{N}_\lambda^-$. □

Lemma 3.5. *Let $(u, v) \in \mathcal{N}_\lambda^+$ (respectively $\mathcal{N}_\lambda^-$), with $u, v \geq 0$, then for any $(z, w) \in E$ with $z, w \geq 0$, there exists $\varepsilon > 0$ and a continuous function $h : B(0, \varepsilon) \to \mathbb{R}$ such that*

$$h(0) = 1 \text{ and } h(s)(u + sz, v + sw) \in \mathcal{N}_\lambda^+ \text{ (respectively } \mathcal{N}_\lambda^-).$$

Proof. We give the proof only for the case $(u, v) \in \mathcal{N}_\lambda^+$. We define $\Psi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ as follows

$$\begin{aligned} \Psi(t,s) \quad & = Q(u+sz, v+sw)\, t^{p+\gamma-1} - \lambda r S(u+sz, v+sw) t^{r+\gamma-1} \\ & -R(u+sz, v+sw). \end{aligned}$$

Then, we have

$$\begin{aligned} \Psi_t(t,s) \quad & = (p+\gamma-1)Q(u+sz, v+sw)\, t^{p+\gamma-2} \\ & -\lambda r(r+\gamma-1)S(u+sz, v+sw)t^{r+\gamma-2}. \end{aligned}$$

and the function $\Psi_t(t,s)$ is continuous in $\mathbb{R}^+ \times \mathbb{R}^+$. Moreover, since $(u,v) \in \mathcal{N}_\lambda^+ \subset \mathcal{N}_\lambda$, we obtain

$$\Psi(1,0) = Q - R - \lambda r S = 0,$$

and

$$\Psi_t(1,0) = (p+\gamma-1)Q - \lambda r(r-1)S > 0.$$

Thus, using the implicit function theorem at the point $(1,0)$, we obtain that there exist $\delta > 0$ and positive continuous function $h : B(0,\varepsilon) \to \mathbb{R}$ such that

$$h(0) = 1, h(s)(u+sz, v+sw) \in \mathcal{N}_\lambda, \text{ for all } s \text{ in } B(0,\delta).$$

Hence, taking $\varepsilon > 0$ smaller enough, we get

$$h(s)(u+sz, v+sw) \in \mathcal{N}_\lambda^+, \text{ for all } s \text{ in } B(0,\varepsilon).$$

The proof of Lemma 3.5 is completed. □

4. Existence and multiplicity results

In this section we prove the theorem of existence and multiplicity of positive nontrivial solutions to system (P_λ) for all $\lambda \in (0,\Gamma)$ in $\mathcal{N}_\lambda^\pm$.

4.1. Positive solutions in $\mathcal{N}_\lambda^+$

Since $J_\lambda(u,v) = J_\lambda(|u|, |v|)$ for all $\lambda \in (0,\Gamma)$, we can assume that all elements in $\mathcal{N}_\lambda$ are nonnegative. Then from Lemma 3.2 and Lemma 3.3, we can find α^+, α^- such that

$$\alpha^+ = \inf_{(u,v)\in\mathcal{N}_\lambda^+} J_\lambda(u,v) \text{ and } \alpha^- = \inf_{(u,v)\in\mathcal{N}_\lambda^-} J_\lambda(u,v),$$

for all $(u,v) \in \mathcal{N}_\lambda^+$. Consequently since $r > p > 1 > 1-\gamma$ and $0 < \gamma < 1$, we have

$$J_\lambda(u,v) = \frac{1}{p}Q - \frac{1}{1-\gamma}R - \lambda S = -\frac{(p+\gamma-1)(r-p)}{rp(1-\gamma)}Q < 0,$$

which means that

$$\alpha^+ = \inf_{(u,v)\in N_\lambda^+} J_\lambda(u,v) < 0 \text{ for all } \lambda \in (0,\Gamma). \tag{4.1}$$

Proof of Theorem 1.1. The proof is split into two steps.
Step 1: Let us consider the minimizing sequence $\{(u_n,v_n)\} \in \mathcal{N}_\lambda^+$ and applying Ekeland's variational principle (see [7]), we obtain

i) $J_\lambda(u_n,v_n) < \alpha^+ + \frac{1}{n}$,

ii) $J_\lambda(u,v) \geq J_\lambda(u_n,v_n) - \frac{1}{n}\|(u-u_n, v-v_n)\|_E$ for all $(u,v) \in \mathcal{N}_\lambda^+$.

As J_λ is coercive, then $\{(u_n, v_n)\}$ is bounded in E for $u_n, v_n \geq 0$. So there exists a subsequence denoted by $\{(u_n, v_n)\}$ and $u_0, v_0 \geq 0$ such that $(u_n, v_n) \rightharpoonup (u_0, v_0)$ weakly in E, strongly in $L^q(\Omega) \times L^q(\Omega), 1 < q < p_s^*$, and $u_n(x) \to u_0(x), v_n(x) \to v_0(x)$, a.e in Ω. Therefore, from (4.1) and by using the weak lower semi-continuity of the norm, we obtain

$$J_\lambda(u_0, v_0) \leq \lim_{n\to\infty} \inf_{(u,v)\in\mathcal{N}_\lambda^+} J_\lambda(u_n, v_n) = \inf_{(u,v)\in\mathcal{N}_\lambda^+} J_\lambda(u, v) < 0. \tag{4.2}$$

□

Claim 4.1. $u_0(x), v_0(x) > 0$ a.e. in Ω.

Proof. Firstly, for $(u_n, v_n) \in \mathcal{N}_\lambda^+$, we have

$$(p+\gamma-1)Q_n - \lambda r(r+\gamma-1)S_n > 0, \tag{4.3}$$

and it is equivalent to

$$(p+\gamma-1)R_n - \lambda r(r-p)S_n > 0. \tag{4.4}$$

By Vitali's convergence Theorem, we have

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \int_\Omega H(x, u_n, v_n)dx = \int_\Omega H(x, u_0, v_0)dx = S_0. \tag{4.5}$$

next, using Hölder's inequality, we obtain, as $n \to \infty$

$$\begin{aligned}
& \int_\Omega a(x)\, u_n^{1-\gamma} dx \\
\leq \; & \int_\Omega a(x)\, u_0^{1-\gamma} dx + \|a\|_\infty \int_\Omega |u_n - u_0|^{1-\gamma} dx \\
\leq \; & \int_\Omega a(x)\, u_0^{1-\gamma} dx + |\Omega|^{\frac{p+\gamma-1}{p}} \|a\|_\infty \|u_n - u_0\|_{L^p(\Omega)}^{1-\gamma} \\
= \; & \int_\Omega a(x)\, u_0^{1-\gamma} dx + o(1),
\end{aligned}$$

and

$$\begin{aligned}
& \int_\Omega a(x)\, u_0^{1-\gamma} dx \\
\leq \; & \int_\Omega a(x)\, u_n^{1-\gamma} dx + \|a\|_\infty \int_\Omega |u_n - u_0|^{1-\gamma} dx \\
\leq \; & \int_\Omega a(x)\, u_n^{1-\gamma} dx + |\Omega|^{\frac{p+\gamma-1}{p}} \|a\|_\infty \|u_n - u_0\|_{L^p(\Omega)}^{1-\gamma} \\
= \; & \int_\Omega a(x)\, u_n^{1-\gamma} dx + o(1).
\end{aligned}$$

Thus

$$\int_\Omega a(x)\, u_n^{1-\gamma} dx = \int_\Omega a(x)\, u_0^{1-\gamma} dx + o(1).$$

Similarly, we have

$$\int_\Omega b(x)\, v_n^{1-\gamma} dx = \int_\Omega b(x)\, v_0^{1-\gamma} dx + o(1),$$

and

$$R_n = R_0 + o(1). \tag{4.6}$$

Moreover, from the Brézis-Lieb Lemma [9] , we obtain

$$\|(u_n, v_n)\|_E^p = \|(u_0, v_0)\|_E^p + \|u_n - u_0\|_{X_0}^p + \|v_n - v_0\|_{X_0}^p + o(1),$$

and

$$Q_n = Q_0 + o(1). \tag{4.7}$$

Therefore, it follows from (4.5) and (4.6) that

$$\begin{aligned}
&\lim_{n\to\infty} [(p+\gamma-1)Q_n - \lambda r(r+\gamma-1)S_n] \\
&= \lim_{n\to\infty} [(p+\gamma-1)R_n - \lambda(r-p)S_n] \\
&= (p-1+\gamma)R_0 - \lambda r(r-p)S_0 \geq 0.
\end{aligned}$$

Let

$$(p+\gamma-1)R_0 - \lambda r(r-p)S_0 = 0. \tag{4.8}$$

Hence, by (4.5)-(4.8), and using the weakly lower semi continuity of the norm, we obtain

$$0 \geq Q_0 - R_0 - \lambda r S_0 = Q_0 - \lambda r \left(\frac{r+\gamma-1}{p+\gamma-1}\right) S_0.$$

From (4.3), one has a contradiction, and

$$(p+\gamma-1)R_0 - \lambda r(r-p)S_0 > 0. \tag{4.9}$$

Furthermore, let us consider the functions $0 < z, w \in E$. From Lemma 3.5 , there exits a sequence of continuous functions $(g_n)_{n\in\mathbb{N}}$ such that

$$g_n(s)(u_n + sz, v_n + sw) \in \mathcal{N}_\lambda^+, g_n(0) = 1,$$

and

$$\begin{aligned}
&Q(g_n(s)(u_n + sz, v_n + sw)) - R(g_n(s)(u_n + sz, v_n + sw)) \\
&\quad -\lambda r S(g_n(s)(u_n + sz, v_n + sw)) = 0.
\end{aligned}$$

Since

$$Q_n - R_n - \lambda r S_n = 0, \tag{4.10}$$

then, for s small enough, it follows that

$$\begin{aligned}
0 &= \left(g_n^p(s)-1\right)Q\left(u_n+sz,v_n+sw\right)+\left(Q\left(u_n+sz,v_n+sw\right)-Q_n\right)\\
&\quad-\left(g_n^{1-\gamma}(s)-1\right)R\left(u_n+sz,v_n+sw\right)-\left(R\left(u_n+sz,v_n+sw\right)-R_n\right)\\
&\quad-\lambda r\left(g_n^r(s)-1\right)S\left(u_n+sz,v_n+sw\right)-\lambda r\left(S\left(u_n+sz,v_n+sw\right)-S_n\right)\\
&\le \left(g_n^p(s)-1\right)Q\left(u_n+sz,v_n+sw\right)+\left(Q\left(u_n+sz,v_n+sw\right)-Q_n\right)\\
&\quad-\left(g_n^{1-\gamma}(s)-1\right)R\left(u_n+sz,v_n+sw\right)\\
&\quad-\lambda r\left(g_n^r(s)-1\right)S\left(u_n+sz,v_n+sw\right)-\lambda r\left(S\left(u_n+sz,v_n+sw\right)-S_n\right).
\end{aligned}$$

Next by $s>0$ and passing to the limit as $s\to 0$, we get

$$\begin{aligned}
0 &\le g_n'(0)\left(pQ_n-(1-\gamma)R_n-\lambda r^2S_n\right)+p\left(T(u_n,z)+T(v_n,w)\right)\\
&\quad-\lambda r\int_\Omega\left(zf(x,u_n,v_n)+wg(x,u_n,v_n)\right)dx\\
&\le g_n'(0)\left((p-r)Q_n+(r+\gamma-1)R_n\right)+p\left(T(u_n,z)+T(v_n,w)\right).
\end{aligned}$$

Then, by (4.3) and (4.10) we obtain

$$g_n'(0)\ge-\frac{p\left(T(u_n,z)+T(v_n,w)\right)}{(p-r)Q_n+(r+\gamma-1)R_n}, \tag{4.11}$$

where $g_n'(0)\in[-\infty,+\infty]$ denotes the right derivative of $g_n(s)$ at zero, and since $(u_n,v_n)\in\mathcal{N}_\lambda^+$, $g_n'(0)\neq-\infty$.

For simplicity, we assume that the right derivative of g_n at $s=0$ exists.

Moreover, from (4.11) we obtain that $g_n'(0)$ is uniformly bounded from below. Now, using condition (ii), we obtain

$$\begin{aligned}
&\left|g_n(s)-1\right|\frac{\|(u_n,v_n)\|}{n}+sg_n(s)\frac{\|(z,w)\|}{n}\\
\ge\ & J_\lambda(u_n,v_n)-J_\lambda\left(g_n(s)\left(u_n+sz,v_n+sw\right)\right)\\
=\ & -\frac{p+\gamma-1}{p(1-\gamma)}Q_n+\lambda\left(\frac{r+\gamma-1}{1-\gamma}\right)S_n\\
&+\frac{p+\gamma-1}{p(1-\gamma)}g_n^p(s)Q\left(u_n+sz,v_n+sw\right)\\
&-\lambda\left(\frac{r+\gamma-1}{1-\gamma}\right)g_n^r(s)S\left(u_n+sz,v_n+sw\right)\\
=\ & \frac{p+\gamma-1}{p(1-\gamma)}\left(Q\left(u_n+sz,v_n+sw\right)-Q_n\right)\\
&+\frac{p+\gamma-1}{p(1-\gamma)}\left(g_n^p(s)-1\right)Q\left(u_n+sz,v_n+sw\right)\\
&-\lambda\frac{r+\gamma-1}{1-\gamma}\left(S\left(u_n+sz,v_n+sw\right)-S_n\right)\\
&-\lambda\frac{r+\gamma-1}{1-\gamma}\left(g_n^r(s)-1\right)S\left(u_n+sz,v_n+sw\right).
\end{aligned}$$

Then, dividing again by $s > 0$ and passing to the limit as $s \to 0$, we obtain

$$\begin{aligned}
& \frac{1}{n}\left(\left|g_n'(0)\right| \|(u_n, v_n)\| + \|(z, w)\|\right) \\
\geq\ & \frac{g_n'(0)}{1-\gamma}\left((p-r) Q_n + (r+\gamma-1) R_n\right) \\
& + \frac{p(p+\gamma-1)}{1-\gamma}\left(T(u_n, z) + T(v_n, w)\right) \\
& - \lambda\left(\frac{r+\gamma-1}{1-\gamma}\right) \int_\Omega \left(z f(x, u_n, v_n) + w g(x, u_n, v_n)\right) dx,
\end{aligned}$$

which implies that

$$\begin{aligned}
& \left|g_n'(0)\right| \left(-\frac{\|(u_n, v_n)\|}{n} - \left((r-p) Q_n - (r+\gamma-1) R_n\right)\right) \\
\leq\ & \frac{1}{n}\|(z, w)\| - \frac{p(p+\gamma-1)}{1-\gamma}\left(T(u_n, z) + T(v_n, w)\right) \\
& + \lambda\left(\frac{r+\gamma-1}{1-\gamma}\right) \int_\Omega \left(f(x, u_n, v_n) z + g(x, u_n, v_n) w\right) dx.
\end{aligned}$$

Hence, there exists a positive constant L such that

$$-\frac{\|(u_n, v_n)\|}{n} - \left((p-r) Q_n + (r+\gamma-1) R_n\right) \geq L > 0,$$

then

$$\begin{aligned}
(4.12) \qquad & \left|g_n'(0)\right| \\
\leq\ & L^{-1}\left(\frac{1}{n}\|(z, w)\| + \frac{p(p+\gamma-1)}{1-\gamma}\left(T(u_n, z) + T(v_n, w)\right)\right. \\
& \left. + \lambda\left(\frac{r+\gamma-1}{1-\gamma}\right) \int_\Omega \left(f(x, u_n, v_n) z + g(x, u_n, v_n) w\right) dx\right),
\end{aligned}$$

Thus, according to (4.12), $g_n'(0)$ is uniformly bounded from above. Consequently,

$$(4.13) \qquad g_n'(0) \text{ is uniformly bounded for } n \text{ large enough.}$$

Moreover, from condition (ii), it follows that for $t > 0$ small enough, one has

$$
\begin{aligned}
& \frac{1}{n}\left(\left|g_n(s)-1\right|\left\|\left(u_n, v_n\right)\right\|+s g_n(s)\|(z, w)\|\right) \\
\geq\ & \frac{1}{n}\left(\left\|g_n(s)\left(u_n+s z, v_n+s w\right)-\left(u_n, v_n\right)\right\|\right) \\
\geq\ & J_\lambda\left(u_n, v_n\right)-J_\lambda\left(g_n(s)\left(u_n+s z, v_n+s w\right)\right) \\
& =-\frac{g_n^p(s)-1}{p} Q_n+\frac{g_n^{1-\gamma}(s)-1}{1-\gamma} R_n+\lambda\left(g_n^r(s)-1\right) S_n \\
& +\frac{g_n^p(s)}{p}\left(Q_n-Q\left(u_n+s z, v_n+s w\right)\right) \\
& +\frac{g_n^{1-\gamma}(s)}{1-\gamma}\left(R\left(u_n+s z, v_n+s w\right)-R_n\right) \\
& +\lambda g_n^r(s)\left(S\left(u_n+s z, v_n+s w\right)-S_n\right) .
\end{aligned}
$$

Dividing by $s>0$ and passing to the limit as $s \rightarrow 0$, we obtain

$$
\begin{aligned}
& \frac{1}{n}\left(\left|g_n^{\prime}(0)\right|\left\|\left(u_n, v_n\right)\right\|+\|(z, w)\|\right) \\
\geq\ & -g_n^{\prime}(0)\left(Q_n-R_n-\lambda r S_n\right)-\left(T(u_n, z)+T(v_n, w)\right) \\
& +\lambda \int_{\Omega}\left(f(x, u_n, v_n) z+g(x, u_n, v_n) w\right) d x \\
& +\frac{1}{1-\gamma} \liminf_{s \rightarrow 0^{+}}\left(\frac{R\left(u_n+s z, v_n+s w\right)-R_n}{s}\right) .
\end{aligned} \tag{4.14}
$$

From (4.13), we deduce that

$$
\begin{aligned}
& \frac{1}{1-\gamma} \liminf_{s \rightarrow 0^{+}}\left(\frac{R\left(u_n+s z, v_n+s w\right)-R_n}{s}\right) \\
\leq\ & T(u_n, z)+T(v_n, w)-\lambda \int_{\Omega}\left(f(x, u_n, v_n) z+g(x, u_n, v_n) w\right) d x \\
& +\frac{\left|g_n^{\prime}(0)\right|\left\|\left(u_n, v_n\right)\right\|+\|(z, w)\|}{n},
\end{aligned} \tag{4.15}
$$

where we have used the fact that

$$
R\left(u_n+s z, v_n+s w\right)-R_n \geq 0 \text{ for all } t>0 \text{ and all } x \in \Omega .
$$

Using Fatou's Lemma, we get

$$
\begin{aligned}
& \int_{\Omega}\left(a(x) u_n^{-\gamma} z+b(x) v_n^{-\gamma} w\right) d x \\
\leq\ & \frac{1}{1-\gamma} \liminf_{s \rightarrow 0^{+}}\left(\frac{R\left(u_n+s z, v_n+s w\right)-R_n}{s}\right) .
\end{aligned}
$$

Then, from (4.15) it follows for n large enough

$$\begin{aligned}
&\int_\Omega \left(a\left(x\right)u_n^{-\gamma}z+b\left(x\right)v_n^{-\gamma}w\right)dx\\
\leq\ & T(u_n,z)+T(v_n,w)-\lambda\int_\Omega\left(f(x,u_n,v_n)z+g(x,u_n,v_n)w\right)dx\\
&+\frac{\left|g_n'\left(0\right)\right|\left\|(u_n,v_n)\right\|+\left\|(z,w)\right\|}{n},
\end{aligned}$$

By (4.14) and Fatou's Lemma again, we conclude that

$$\begin{aligned}
&T(u_0,z)+T(v_0,w)\\
\geq\ &\int_\Omega\left(a\left(x\right)u_0^{-\gamma}z+b\left(x\right)v_0^{-\gamma}w\right)dx\\
&+\lambda\int_\Omega\left(f(x,u_0,v_0)z+g(x,u_0,v_0)w\right)dx,
\end{aligned}\tag{4.16}$$

for all $(z,w)\in X$, with $z,w\geq 0$. Then, the maximum principle theorem implies that $u_0\left(x\right),v_0\left(x\right)>0$ a.e. in Ω.
Now, we prove that $(u_0,v_0)\in\mathcal{N}_\lambda^+$ for all $\lambda\in(0,\Gamma)$. Choosing $(z,w)=(u_0,v_0)$ in (4.16), we obtain

$$Q_0-R_0-\lambda rS_0\geq 0.$$

On the other hand, from the weakly lower semi continuity of the norm, we get

$$Q_0-R_0-\lambda rS_0\leq 0,$$

and

$$Q_0-R_0-\lambda rS_0=0,\tag{4.17}$$

this implies that $(u_0,v_0)\in\mathcal{N}_\lambda$.

Hence, combining (4.9) and (4.17), we have $(u_0,v_0)\in\mathcal{N}_\lambda^+$.

Next, we show that (u_0,v_0) is a positive solution of system (P_λ) for all $\lambda\in(0,\Gamma)$. Let $(z,w)\in X$ and $\varepsilon>0$. We define $(\omega_1,\omega_2)\in X$ with $(\omega_1,\omega_2)=(u_0+\varepsilon z,v_0+\varepsilon w),\omega_1^+=\max\{\omega_1,0\}$ and $\omega_2^+=\max\{\omega_2,0\}$. Let

$$\begin{aligned}
\Omega_\varepsilon^+ &=\left\{(u_0+\varepsilon z,v_0+\varepsilon w):u_0+\varepsilon z>0\text{ and }v_0+\varepsilon w>0\right\},\\
\Omega_\varepsilon^- &=\left\{(u_0+\varepsilon z,v_0+\varepsilon w):u_0+\varepsilon z\leq 0\text{ and }v_0+\varepsilon w\leq 0\right\},
\end{aligned}$$

and

$$\begin{aligned}
&T(u_0,\omega_1^+)\\
&=\int\limits_Q \frac{|u_0(x)-u_0(y)|^{p-2}(u_0(x)-u_0(y))(\omega_1^+(x)-\omega_1^+(y))}{|x-y|^{N+ps}}dxdy\\
&=\int\limits_Q \frac{|u_0(x)-u_0(y)|^{p-2}(u_0(x)-u_0(y))((u_0+\varepsilon z)(x)-(u_0+\varepsilon z)(y))}{|x-y|^{N+ps}}dxdy\\
&-\int\limits_{\Omega_\varepsilon^-\times\Omega_\varepsilon^-} \frac{|u_0(x)-u_0(y)|^{p-2}(u_0(x)-u_0(y))((u_0+\varepsilon z)(x)-(u_0+\varepsilon z)(y))}{|x-y|^{N+ps}}dxdy\\
&=T(u_0,u_0)+\varepsilon T(u_0,z)\\
&-\int\limits_{\Omega_\varepsilon^-\times\Omega_\varepsilon^-} \frac{|u_0(x)-u_0(y)|^{p-2}(u_0(x)-u_0(y))((u_0+\varepsilon z)(x)-(u_0+\varepsilon z)(y))}{|x-y|^{N+ps}}dxdy.
\end{aligned}$$

By Claim 4.1, the measure of the domain of integration $\Omega_\varepsilon^-\times\Omega_\varepsilon^-$ tends to 0 as $\varepsilon\to0^+$ that

$$\int\limits_{\Omega_\varepsilon^-\times\Omega_\varepsilon^-} \frac{|u_0(x)-u_0(y)|^{p-2}(u_0(x)-u_0(y))((u_0+\varepsilon z)(x)-(u_0+\varepsilon z)(y))}{|x-y|^{N+ps}}dxdy \underset{\varepsilon\to0^+}{\to} 0,$$

then

$$T(u_0,\omega_1^+) \underset{\varepsilon\to0^+}{\to} T(u_0,u_0). \tag{4.18}$$

Similarly, we can obtain

$$T(v_0,\omega_2^+) \underset{\varepsilon\to0^+}{\to} T(v_0,v_0). \tag{4.19}$$

On the other hand, we have

$$\begin{aligned}
&\int_\Omega a(x)\,u_0^{-\gamma}\omega_1^+dx\\
&=\int_{\Omega_\varepsilon^+} a(x)\,u_0^{-\gamma}\omega_1 dx\\
&=\int_\Omega a(x)\,u_0^{-\gamma}(u_0+\varepsilon z)\,dx-\int_{\Omega_\varepsilon^-} a(x)\,u_0^{-\gamma}(u_0+\varepsilon z)\,dx\\
&=\int_\Omega a(x)\,u_0^{1-\gamma}dx+\varepsilon\int_\Omega a(x)\,u_0^{-\gamma}zdx-\int_{\Omega_\varepsilon^-} a(x)\,u_0^{-\gamma}(u_0+\varepsilon z)\,dx\\
&\geq \int_\Omega a(x)\,u_0^{1-\gamma}dx+\varepsilon\int_\Omega a(x)\,u_0^{-\gamma}zdx,
\end{aligned} \tag{4.20}$$

and

$$\int_\Omega b(x)\,v_0^{-\gamma}\omega_2^+dx \geq \int_\Omega b(x)\,u_0^{1-\gamma}dx+\varepsilon\int_\Omega b(x)\,v_0^{-\gamma}zdx. \tag{4.21}$$

Now, we can write

$$
\begin{aligned}
& \int_\Omega f(x,u_0,v_0)\omega_1^+ dx \\
& = \int_{\Omega_\varepsilon^+} f(x,u_0,v_0)\omega_1 dx \\
& = \int_\Omega f(x,u_0,v_0)\left(u_0+\varepsilon z\right) dx - \int_{\Omega_\varepsilon^-} f(x,u_0,v_0)\left(u_0+\varepsilon z\right) dx \\
& = \int_\Omega f(x,u_0,v_0)u_0 dx + \varepsilon \int_\Omega f(x,u_0,v_0) z dx \\
& - \int_{\Omega_\varepsilon^-} f(x,u_0,v_0)\left(u_0+\varepsilon z\right) dx
\end{aligned}
$$

$$
\geq \int_\Omega f(x,u_0,v_0)u_0 dx + \varepsilon \int_\Omega f(x,u_0,v_0) z dx, \tag{4.22}
$$

and

$$
\int_\Omega g(x,u_0,v_0)\omega_2^+ dx \geq \int_\Omega g(x,u_0,v_0)v_0 dx + \varepsilon \int_\Omega g(x,u_0,v_0) z dx, \tag{4.23}
$$

Combining (4.18)-(4.23), we arrive at

$$
\begin{aligned}
0 \leq\ & T(u_0,\omega_1^+) + T(v_0,\omega_2^+) - \int_\Omega \left(a\left(x\right)u_0^{-\gamma}\omega_1^+ + b\left(x\right)v_0^{-\gamma}\omega_2^+\right) dx \\
& -\lambda \int_\Omega \left(f(x,u_0,v_0)\omega_1^+ + g(x,u_0,v_0)\omega_2^+\right) dx \\
\leq\ & T(u_0,u_0) + T(v_0,v_0) - \int_\Omega \left(a\left(x\right)u_0^{1-\gamma} + b\left(x\right)u_0^{1-\gamma}\right) dx \\
& - \left(\int_\Omega f(x,u_0,v_0)u_0 + \int_\Omega g(x,u_0,v_0)v_0\right) dx \\
& + \varepsilon \left(T(u_0,z) + T(u_0,z) - \int_\Omega \left(a\left(x\right)u_0^{-\gamma} z + b\left(x\right)v_0^{-\gamma} w\right) dx \right. \\
& \left. - \int_\Omega \left(f(x,u_0,v_0)z + g(x,u_0,v_0)w\right) dx \right) \\
& = \varepsilon \left(T(u_0,z) + T(u_0,z) - \int_\Omega \left(a\left(x\right)u_0^{-\gamma} z + b\left(x\right)v_0^{-\gamma} w\right) dx \right. \\
& \left. - \int_\Omega \left(f(x,u_0,v_0)z + g(x,u_0,v_0)w\right) dx \right).
\end{aligned}
$$

Then

$$
\begin{aligned}
& T(u_0,z) + T(u_0,z) - \int_\Omega \left(a\left(x\right)u_0^{-\gamma} z + b\left(x\right)v_0^{-\gamma} w\right) dx \\
& \qquad - \int_\Omega \left(f(x,u_0,v_0)z + g(x,u_0,v_0)w\right) dx \geq 0.
\end{aligned}
$$

Since the equality holds if we replace (z,w) by $(-z,-w)$, then (u_0,u_0) is a positive week solution of the problem (P_λ). □

4.2. Positive solutions in $\mathcal{N}_\lambda^-$

Similarly to the arguments in $\mathcal{N}_\lambda^+$, applying Ekeland's variational principle to the minimization problem $\alpha^- = \inf_{(u,v)\in\mathcal{N}_\lambda^-} J_\lambda(u,v)$.

Then, there exists a minimizing sequence $\{(u_n,v_n)\} \subset \mathcal{N}_\lambda^-$ such that

i) $J_\lambda(u_n,v_n) < \alpha^- + \frac{1}{n}$,

ii) $J_\lambda(u,v) \geq J_\lambda(u_n,v_n) - \frac{1}{n}\left\|(u-u_n, v-v_n)\right\|_E$ for all $(u,v)\in\mathcal{N}_\lambda^+$.

Firstly, let $u_n, v_n \geq 0$. Clearly, $\{(u_n,v_n)\}$ is a bounded sequence in E. So, there exist subsequences denoted by $\{(u_n,v_n)\}$ and $\tilde{u}_0,\tilde{v}_0 \geq 0$ such that $(u_n,v_n) \rightharpoonup (\tilde{u}_0,\tilde{v}_0)$ weakly in E, strongly in $L^{1-\gamma}(\Omega)$ and $(u_n(x),v_n(x)) \to (\tilde{u}_0(x),\tilde{v}_0(x))$ in a.e Ω, as $n\to\infty$. Moreover, using the weak lower semi continuity of norm, we obtain that

$$J_\lambda(\tilde{u}_0,\tilde{v}_0) \leq \lim_{n\to\infty} \inf_{(u,v)\in\mathcal{N}_\lambda^-} J_\lambda(u_n,v_n) < 0, \tag{4.24}$$

$(\tilde{u}_0,\tilde{v}_0) \neq (0,0)$ in a.e Ω.

Now, we prove that $\tilde{u}_0,\tilde{v}_0 > 0$ in a.e Ω. Similarly to the argument, for $(u_n,v_n)\in\mathcal{N}_\lambda^-$, one has

$$(p+\gamma-1)Q_n - \lambda r(r+\gamma-1)S_n < 0, \tag{4.25}$$

equivalent to

$$(p+\gamma-1)R_n - \lambda r(r-p)S_n < 0. \tag{4.26}$$

Next, from (4.25) and (4.26), we obtain that

$$\begin{aligned}
&\lim_{n\to\infty} \left[(p+\gamma-1)Q_n - \lambda r(r+\gamma-1)S_n\right] \\
&= \lim_{n\to\infty} \left[(p+\gamma-1)R_n - \lambda(r-p)S_n\right] \\
&= (p-1+\gamma)R_0 - \lambda r(r-p)S_0 \leq 0.
\end{aligned}$$

Moreover, repeating the same argument as in Claim 4.1, we obtain, for all $\lambda\in(0,\Gamma)$

$$(p+\gamma-1)R_0 - \lambda r(r-p)S_0 < 0. \tag{4.27}$$

Let us consider the functions $0 < z,w \in E$. Then there exits a sequence of continuous functions $(g_n)_{n\in\mathbb{N}}$ such that $g_n(s)(u_n+sz, v_n+sw)\in\mathcal{N}_\lambda^-$ and $g_n(0)=1$.

Therefore, repeating the same arguments as in Claim 4.1, we have $g_n'(0)$ is uniformly bounded for n large enough.

We conclude that $\tilde{u}_0(x), \tilde{v}_0(x) > 0$ a.e. in Ω and

$$\begin{aligned}
T(\tilde{u}_0,z) + T(\tilde{v}_0,w) \quad \leq \quad & -\int_\Omega \left(a(x)\,\tilde{u}_0^{-\gamma} z + b(x)\,\tilde{v}_0^{-\gamma} w\right) dx \\
& -\lambda \int_\Omega \left(f(x,\tilde{u}_0,\tilde{v}_0)z + g(x,\tilde{u}_0,\tilde{v}_0)w\right) dx,
\end{aligned} \tag{4.28}$$

for all $(z,w)\in E$. Finally, we obtain that $(\tilde{u}_0,\tilde{v}_0)$ isa positive nontrivial solution of system (P_λ). The proof of Theorem 1.1 is completed.

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