Novi Sad J. Math. Vol. 51, No. 2, 2021, 1-16
https://doi.org/10.30755/NSJOM.10082

$\mathcal{I}$-statistical limit points and $\mathcal{I}$-statistical cluster points in probabilistic normed spaces

Samiran Das[1] **and Argha Ghosh**[2,3]

Abstract. In this paper, we introduce the notions of $\mathcal{I}$-statistical limit points and $\mathcal{I}$-statistical cluster points for a sequence in probabilistic normed spaces and study some basic properties of the sets of all $\mathcal{I}$-statistical limit points and $\mathcal{I}$-statistical cluster points of a sequence in probabilistic normed spaces including their interrelationship. Also, using the additive property of $\mathcal{I}$-asymptotic density zero sets, we establish $\mathcal{I}$-statistical analogue of some previous results.

AMS Mathematics Subject Classification (2010):40G99; 60F99

Key words and phrases: Continuous t-norm; probabilistic normed space; $\mathcal{I}$-statistical limit points; $\mathcal{I}$-statistical cluster points; $\mathcal{I}$-asymptotic density

1. Introduction

As a generalization of ordinary normed linear spaces, the notion of probabilistic normed spaces (PN spaces in short) was introduced by Šerstnev [27] considering the values of norms are probability distribution functions instead of non-negative real numbers. The theory of probabilistic normed spaces had gone through considerable developments before Alsina et al. [1] introduced a new, wider accepted definition of PN spaces. A detailed study in this direction can be seen from the book by Guillen and Harikrishnan [16] and many others.

On the other hand, the notion of statistical convergence for a sequence of real numbers was introduced by Fast [10] and Schoenberg [24] individually, which is a generalization of the notion of ordinary convergence for a sequence of real numbers. And after the seminal works of Sálat [26] and Fridy[11], over the years, a lot of studies have been done in this direction. In [12], Fridy introduced the notions of statistical limit points and statistical cluster points of a sequence of real numbers and studied the interrelation between them. And the notions of statistical convergence, statistical limit points and statistical cluster points for a sequence in a PN space was introduced and studied by Karakus [13]. For more studies on this convergence, see [2, 3, 21, 25] and many others.

Further, the idea of statistical convergence was extended to $\mathcal{I}$-convergence by Kostyrko et al. [15]. And using this notion of ideals in $\mathbb{N}$, the concepts of

[1]Department of Mathematics, The University of Burdwan, Golapbag, Burdwan-713104, West Bengal, India. e-mail: das91samiran@gmail.com

[2]Department of Mathematics, The University of Burdwan, Golapbag, Burdwan-713104, West Bengal, India. **Current Address:** Department of Mathematics, Brainware University, Barasat, Kolkata-700125, India. e-mail: buagbu@yahoo.co.in

[3]Corresponding author

statistical limit points and statistical cluster points were naturally extended to $\mathcal{I}$-limit points and $\mathcal{I}$-cluster points for a sequence of real numbers respectively by Kostyrko et al. in [14]. And later, the notions of $\mathcal{I}$-convergence, $\mathcal{I}$-limit points and $\mathcal{I}$-cluster points in PN spaces were introduced by Mursaleen et al. [18], which extend the concepts of statistical convergence, statistical limit points and statistical cluster points in PN spaces respectively. For more works on this convergence, see [8, 7, 14] and many others.

In 2011, Savas et al. [23] introduced the notion of $\mathcal{I}$-statistical convergence, and Mursaleen et al. [19] introduced the notion of $\mathcal{I}$-statistical cluster points for a sequence of real numbers. And in 2015, Savas et al. introduced and studied the notion of $\mathcal{I}$-statistical convergence for a sequence in PN spaces [22]. A few more works on this convergence can be found in [5, 4, 19, 17] and many others. The notion of $\mathcal{I}$-statistical limit points for a sequence of real numbers was introduced by Debnath et al. [6] and by Malik et al. [17] independently. The notion of $\mathcal{I}$-statistical limit points is a generalization of the concept of statistical limit points for a sequence of real numbers. One can find some established relationships between the notions of $\mathcal{I}$-statistical limit points and $\mathcal{I}$-statistical cluster points both in [6] and [17]. Also, the study of various methods of convergence (of sequences in PN spaces) has become engaging lately (see [1, 9, 22]). Therefore, it seems reasonable to introdice the notion of $\mathcal{I}$-statistical limit points and $\mathcal{I}$-cluster points in the theory of probabilistic normed spaces. And here we do that.

In Section 3 of this paper, we introduce the notion of $\mathcal{I}$-statistical limit points and $\mathcal{I}$-statistical cluster points for a sequence in probabilistic normed spaces. And we also study the sets of $\mathcal{I}$-statistical limit points and $\mathcal{I}$-statistical cluster points for a sequence in probabilistic normed spaces and establish some relationship between them. Moreover, in Section 4, using Condition APIO, we establish some theorems in probabilistic normed space analogue to the theorems of [12], as well as [13]. Consequently, our results generalize the results of [13].

2. Basic Definitions and Notations

In this section, we now recall some definitions and notations, which are needed forlater parts of the paper.

Definition 2.1. [1] A non-decreasing and left-continuous function $f : \mathbb{R} \to [0, \infty]$ is said to be a distribution function if $\inf_{t\in\mathbb{R}} f(t) = 0$, and $\sup_{t\in\mathbb{R}} f(t) = 1$.

We write $\mathcal{D}$ to denote the set of all distribution functions.

Definition 2.2. [1] A triangular norm (t-norm in short) is a continuous mapping $\circ : [0, 1] \times [0, 1] \to [0, 1]$ which satisfies the following conditions:

1. $a \circ 1 = a \ \forall a \in [0, 1]$;
2. $a \circ b = b \circ a \ \forall a, b \in [0, 1]$;
3. $c \circ d \geq a \circ b$ if $c \geq a$ and $d \geq b \ \forall a, b, c, d \in [0, 1]$;

4. $(a \circ b) \circ c = a \circ (b \circ c)$ $\forall a, b, c \in [0, 1]$.

Definition 2.3. [13] A probabilistic normed space (PN space in short) is a triplet $(X, \mathcal{N}, \circ)$, where X is a vector space over the field $\mathbb{R}$, $\mathcal{N}$ is a function from X into $\mathcal{D}$ and $\circ$ is a t-norm such that the following conditions hold:

1. $\mathcal{N}_x(0) = 0$;
2. $\mathcal{N}_x(t) = 1$ $\forall t > 0$ iff $x = 0$;
3. $\mathcal{N}_{\beta x}(t) = \mathcal{N}_x(\frac{t}{|\beta|})$ $\forall \beta \in \mathbb{R}/\{0\}$;
4. $\mathcal{N}_{x+y}(u + t) \geq \mathcal{N}_x(u) \circ \mathcal{N}_y(t)$ $\forall x, y \in X$ and $\forall u, t \in [0, \infty]$.

Here we write $\mathcal{N}_x$ to denote $\mathcal{N}(x)$ and $\mathcal{N}_x(t)$ to denote the value of $\mathcal{N}_x$ at $t \in \mathbb{R}$. And in this case, $\mathcal{N}$ is said to be the probabilistic norm of the PN space $(X, \mathcal{N}, \circ)$.

Now we give an example of a probabilistic normed space.

Example 2.4. Consider the linear space $\mathbb{R}$ of all real numbers. Let us define $a \circ b = min\{a, b\}$ for all $a, b \in \mathbb{R}$ and $\mathcal{N}_x = \epsilon_\infty$ for $x \neq 0$ and $\mathcal{N}_x = \epsilon_0$ for $x = 0$, where

$$\begin{aligned} \epsilon_0(t) &= 0, \quad -\infty \leq t \leq 0 \\ &= 1, \quad 0 < t \leq \infty; \end{aligned}$$

$$\begin{aligned} \epsilon_\infty(t) &= 0, \quad -\infty \leq t < \infty \\ &= 1, \quad t = \infty. \end{aligned}$$

We show that $(\mathbb{R}, \mathcal{N}, \circ)$ is a PN space. Let $x = 0$. Then $\mathcal{N}_x = \epsilon_0$, and so $\mathcal{N}_x(0) = \epsilon_0(0) = 0$. Again let $x \neq 0$. Then $\mathcal{N}_x = \epsilon_\infty$, and so $\mathcal{N}_x(0) = \epsilon_\infty(0) = 0$. Therefore for all $x \in \mathbb{R}$, we have $\mathcal{N}_x(0) = 0$. Now for $x = 0$, we have $\mathcal{N}_x = \epsilon_0$. Thus for all $t > 0$, $\mathcal{N}_0(t) = \epsilon_0(t) = 1$ (by the definition of ϵ_0). Let $\mathcal{N}_x(t) = 1$ for all $t > 0$. We will show that $x = 0$. If possible, let $x \neq 0$. Then $\mathcal{N}_x = \epsilon_\infty$. And $\epsilon_\infty(t) = 0$ whenever $0 < t < \infty$ (by the definition of ϵ_∞), which contradicts the fact that $\mathcal{N}_x(t) = 1$ for all $t > 0$. Thus $x = 0$. Hence $\mathcal{N}_x(t) = 1$ for all $t > 0$ if and only if $x = 0$. Clearly, for all $\beta \in \mathbb{R} \setminus \{0\}$, we have $\mathcal{N}_{\beta x}(t) = \mathcal{N}_x(\frac{t}{|\beta|})$. Also, $\mathcal{N}_{x+y}(u + t) \geq \mathcal{N}_x(u) \circ \mathcal{N}_y(t)$ $\forall x, y \in \mathbb{R}$ and $\forall u, t \in [0, \infty]$ (by using the definitions of ϵ_∞ and ϵ_0). Hence $(\mathbb{R}, \mathcal{N}, \circ)$ is a PN space.

Definition 2.5. [13] Let $(X, \mathcal{N}, \circ)$ be a PN space. Then a sequence $\{x_k\}_{k \in \mathbb{N}}$ in X is said to be convergent to $\xi \in X$ with respect to the probabilistic norm $\mathcal{N}$ if for every $\varepsilon > 0$ and $\lambda \in (0, 1)$ there exists $k_0 \in \mathbb{N}$ such that $\mathcal{N}_{x_k - \xi}(\varepsilon) > 1 - \lambda$ whenever $k \geq k_0$.

In this case, we write $\lim\limits_{k \to \infty} \mathcal{N}(x_k) = \xi$.

Definition 2.6. [13] Let $(X, \mathcal{N}, \circ)$ be a PN space. Then a sequence $\{x_k\}_{k\in\mathbb{N}}$ in X is said to be a Cauchy sequence with respect to the probabilistic norm $\mathcal{N}$ if for every $\varepsilon > 0$ and $\lambda \in (0,1)$ there exists $k_0 \in \mathbb{N}$ such that $\mathcal{N}_{x_j - x_k}(\varepsilon) > 1-\lambda$ whenever $j, k \geq k_0$.

Definition 2.7. [13] Let $(X, \mathcal{N}, \circ)$ be a PN space and x be an element in X. Then for $\varepsilon > 0$ the ball centered at x and having radius $\lambda \in (0,1)$ is denoted by $B(x, \lambda, \varepsilon)$ and is defined by $B(x, \lambda, \varepsilon) = \{u \in X : \mathcal{N}_{x-u}(\varepsilon) > 1 - \lambda\}$.

Definition 2.8. [13] Let $(X, \mathcal{N}, \circ)$ be a PN space. Then $\xi \in X$ is said to be a limit point of the sequence $\{x_k\}_{k\in\mathbb{N}}$ in X with respect to the probabilistic norm $\mathcal{N}$ if there is a subsequence of the sequence $\{x_k\}_{k\in\mathbb{N}}$ which converges to l with respect to the probabilistic norm $\mathcal{N}$.

We write $L_x^{\mathcal{N}}$ to denote the set of all such limit points of the sequence $\{x_k\}_{k\in\mathbb{N}}$.

Definition 2.9. [13] Let $(X, \mathcal{N}, \circ)$ be a PN space. Then a subset A of X is said to be closed in X if A contains all its limit points with respect to the probabilistic norm $\mathcal{N}$.

Definition 2.10. [20] A subset M of $\mathbb{N}$ is said to have natural density or asymptotic density $d(M)$ if

$$d(M) = \lim_{n\to\infty} \tfrac{|M(n)|}{n}$$

exists, where $M(n) = \{j \in M : j \leq n\}$ and $|M(n)|$ represents the number of elements in $M(n)$.

Definition 2.11. [12] A sequence $x = \{x_k\}_{k\in\mathbb{N}}$ of real numbers is said to be statistically convergent to l if for every $\varepsilon > 0$ $d(A_\varepsilon) = 0$, where $A_\varepsilon = \{k \in \mathbb{N} : |x_k - l| \geq \varepsilon\}$.

Let $(X, \mathcal{N}, \circ)$ be a PN space. If $\{x_{k_j}\}_{j\in\mathbb{N}}$ is a subsequence of a sequence $x = \{x_k\}_{k\in\mathbb{N}}$ in X and $A = \{k_j : j \in \mathbb{N}\}$, then we abbreviate $\{x_{k_j}\}_{j\in\mathbb{N}}$ by $\{x\}_A$. In the case $d(A) = 0$, $\{x\}_A$ is called a subsequence of natural density zero or a thin subsequence of x. On the other hand, $\{x\}_A$ is a non-thin subsequence of x if $d(A)$ does not have natural density zero i. e., either $d(A)$ is a positive number or A fails to have natural density.

Definition 2.12. [13] Let $(X, \mathcal{N}, \circ)$ be a PN space. A sequence $x = \{x_k\}_{k\in\mathbb{N}}$ in X is said to be statistically convergent to $\xi \in X$ with respect to the probabilistic norm $\mathcal{N}$ if for every $\varepsilon > 0$ and $\lambda \in (0,1)$

$$d(\{k \in \mathbb{N} : \mathcal{N}_{x_k - \xi}(\varepsilon) \leq 1 - \lambda\}) = 0.$$

In this case, we write $st\text{-}\lim_{k\to\infty} \mathcal{N}(x_k) = \xi$.

Definition 2.13. [13] Let $(X, \mathcal{N}, \circ)$ be a PN space. Then $\xi \in X$ is said to be a statistical limit point of a sequence $x = \{x_k\}_{k\in\mathbb{N}}$ in X with respect to the probabilistic norm $\mathcal{N}$, if there exists a nonthin subsequence of x that converges to ξ with respect to the probabilistic norm $\mathcal{N}$.

Similarly, $\xi \in X$ is an ordinary limit point of the sequence x with respect to the probabilistic norm $\mathcal{N}$ if there is a subsequence of x that converges to ξ. And the set of all ordinary limit points and statistical limit points of the sequence x with respect to the probabilistic norm $\mathcal{N}$ in X are denoted by $L_x^{\mathcal{N}}$ and $\Lambda_x^{\mathcal{N}}$, respectively. Clearly, $\Lambda_x^{\mathcal{N}} \subset L_x^{\mathcal{N}}$.

Definition 2.14. [13] Let $(X, \mathcal{N}, \circ)$ be a PN space. Then $\xi \in X$ is said to be a statistical cluster point of a sequence $x = \{x_k\}_{k\in\mathbb{N}}$ in X with respect to the probabilistic norm $\mathcal{N}$, if for every $\varepsilon > 0$ and $\lambda \in (0,1)$ the set $\{k \in \mathbb{N} : \mathcal{N}_{x_j-\xi}(\varepsilon) > 1-\lambda\}$ does not have natural density zero.

The set of all such statistical cluster points of x is denoted by $\Gamma_x^{\mathcal{N}}$. Clearly, $\Gamma_x^{\mathcal{N}} \subset L_x^{\mathcal{N}}$.

We now recall definitions of an ideal and an filter in a non-empty set X.

Definition 2.15. [15] Let $X \neq \emptyset$. A class $\mathcal{I}$ of subsets of X is said to be an ideal in X, provided $\mathcal{I}$ satisfies the conditions:
(i) $\emptyset \in \mathcal{I}$,
(ii) $A, B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$,
(iii) $A \in \mathcal{I}, B \subset A \Rightarrow B \in \mathcal{I}$.

An ideal $\mathcal{I}$ in a non-empty set X is said to be non-trivial if $X \notin \mathcal{I}$ and $\mathcal{I} \neq \{\emptyset\}$.

Definition 2.16. [15] Let $X \neq \emptyset$. A non-empty class $\mathcal{F}$ of subsets of X is said to be a filter in X, provided $\mathcal{F}$ satisfies the following conditions:
(i) $\emptyset \notin \mathcal{F}$,
(ii) $A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$,
(iii) $A \in \mathcal{F}, A \subset B \Rightarrow B \in \mathcal{F}$.

Definition 2.17. [15] Let $\mathcal{I}$ be a non-trivial ideal in a non-empty set X. Then the class $\mathcal{F}(\mathcal{I})= \{M \subset X : \exists N \in \mathcal{I} \textit{ such that } M = X \setminus N\}$ is a filter in X. This filter $\mathcal{F}(\mathcal{I})$ is called the filter associated with the ideal $\mathcal{I}$.

A non-trivial ideal $\mathcal{I}$ in $X(\neq \emptyset)$ is called admissible if $\{x\} \in \mathcal{I}$ for each $x \in X$.

From now on rest of the paper, we take $\mathcal{I}$ as a non-trivial admissible ideal in $\mathbb{N}$ unless otherwise mentioned.

Definition 2.18. [15] A sequence $x = \{x_k\}_{k\in\mathbb{N}}$ of real numbers is said to be $\mathcal{I}$-convergent to l if for any $\varepsilon > 0$

$$\{k \in \mathbb{N} : |x_k - l| \geq \varepsilon\} \in \mathcal{I}.$$

Definition 2.19. [4] A subset M of $\mathbb{N}$ is said to have $\mathcal{I}$-natural density $d^{\mathcal{I}}(M)$ if

$$d^{\mathcal{I}}(M) = \mathcal{I} - \lim_{n\to\infty} \frac{|M(n)|}{n}$$

exists, where $M(n) = \{j \in M : j \leq n\}$ and $|M(n)|$ represents the number of elements in $M(n)$.

Note 2.20. From Definition 2.19, for any nontrivial admissible ideal $\mathcal{I}$ in $\mathbb{N}$, $d(M) = r$ implies $d^{\mathcal{I}}(M) = r$.

Definition 2.21. [17] Let $x = \{x_k\}_{k\in\mathbb{N}}$ be a sequence of real numbers. Then in the case $d^{\mathcal{I}}(B) = 0$, $\{x\}_B$ is called a subsequence of $\mathcal{I}$-asymptotic density zero, or an $\mathcal{I}$-thin subsequence of x. On the other hand, $\{x\}_B$ is an $\mathcal{I}$-nonthin subsequence of x, if B does not have $\mathcal{I}$-asymptotic density zero, in other words, either $d^{\mathcal{I}}(B)$ is a positive number or B fails to have $\mathcal{I}$-asymptotic density.

Definition 2.22. [23] A sequence $x = \{x_k\}_{k\in\mathbb{N}}$ of real numbers is said to be $\mathcal{I}$-statistically convergent to l if for any $\varepsilon > 0$, $\delta > 0$

$$\{n \in \mathbb{N} : \tfrac{1}{n}|\{k \leq n : |x_k - l| \geq \varepsilon\}| \geq \delta\} \in \mathcal{I}.$$

Definition 2.23. [6] A real number l is said to be an $\mathcal{I}$-statistical limit point of a sequence $x = \{x_k\}_{k\in\mathbb{N}}$ of real numbers provided that for each $\varepsilon > 0$ there is a set $M = \{m_1 < m_2 < ...\} \subset \mathbb{N}$ such that $M \notin \mathcal{I}$ and $\{x_{m_k}\}_{k\in\mathbb{N}}$ is statistically convergent to l.

Definition 2.24. [17] A real number l is said to be an $\mathcal{I}$-statistical limit point of a sequence $x = \{x_k\}_{k\in\mathbb{N}}$ of real numbers if there exists an $\mathcal{I}$-nonthin subsequence of x that converges to l.

Remark 2.25. Even though both definitions (Definition 2.23 and Definition 2.24) are the definitions of $\mathcal{I}$-statistical limit points of a sequence of real numbers, we follow Definition 2.24 to introduce the definition of $\mathcal{I}$-statistical limit points for a sequence in a probabilistic normed space.

Definition 2.26. [19] A real number l is said to be an $\mathcal{I}$-statistical cluster point of a sequence $x = \{x_k\}_{k\in\mathbb{N}}$ of real numbers if for each $\varepsilon > 0$ the set $\{k \in \mathbb{N} : |x_k - l| < \varepsilon\}$ does not have $\mathcal{I}$-asymptotic density zero.

Definition 2.27. [22] Let $(X, \mathcal{N}, \circ)$ be a PN space. Then a sequence $x = \{x_k\}_{k\in\mathbb{N}}$ in X is said to be $\mathcal{I}$-statistically convergent to $\xi \in X$ with respect to probabilistic norm $\mathcal{N}$ if for every $\varepsilon > 0$, $\delta > 0$ and $\lambda \in (0, 1)$

$$\{n \in \mathbb{N} : \tfrac{1}{n}|\{k \leq n : \mathcal{N}_{x_j-\xi}(\varepsilon) \leq 1 - \lambda\}| \geq \delta\} \in \mathcal{I}.$$

In this case, we write $\mathcal{I}\text{-}st\text{-}\lim_{k\to\infty} \mathcal{N}(x_k) = \xi$.

3. $\mathcal{I}$-statistical limit points and $\mathcal{I}$-statistical cluster points in a PN space

In this section, following the line of Fridy [12] and Karakus [13], we introduce the notion of $\mathcal{I}$-statistical limit points and $\mathcal{I}$-statistical cluster points for a sequence in a PN space $(X, \mathcal{N}, \circ)$ with respect to the probabilistic norm $\mathcal{N}$, and we study an $\mathcal{I}$-statistical analogue of a few theorems in probabilistic normed spaces related to those papers.

Let $(X, \mathcal{N}, \circ)$ be a PN space and $x = \{x_k\}_{k\in\mathbb{N}}$ be a sequence in X. Then in the case $d^{\mathcal{I}}(B) = 0$, $\{x\}_B$ is called a subsequence of $\mathcal{I}$-asymptotic density zero, or an $\mathcal{I}$-thin subsequence of x. On the other hand, $\{x\}_B$ is an $\mathcal{I}$-nonthin subsequence of x, if B does not have $\mathcal{I}$-asymptotic density zero, in other words, either $d^{\mathcal{I}}(B)$ is a positive number or B fails to have $\mathcal{I}$-asymptotic density.

Definition 3.1. Let $(X, \mathcal{N}, \circ)$ be a PN space. Then $\xi \in X$ is said to be an $\mathcal{I}$-statistical limit point of a sequence $x = \{x_k\}_{k\in\mathbb{N}}$ in X with respect to the probabilistic norm $\mathcal{N}$, if there exists an $\mathcal{I}$-nonthin subsequence of x that converges to ξ.

We write $\Lambda_x^S(\mathcal{I})_{\mathcal{N}}$ to denote the set of all such $\mathcal{I}$-statistical limit points of the sequence x.

Remark 3.2. Equivalently, the definition of $\mathcal{I}$-statistical limit points of a sequence in a PN space can also be given in the following way.

Definition 3.3. Let $(X, \mathcal{N}, \circ)$ be a PN space. Then $\xi \in X$ is said to be an $\mathcal{I}$-statistical limit point of a sequence $x = \{x_k\}_{k\in\mathbb{N}}$ in X with respect to the probabilistic norm $\mathcal{N}$, if there exists an $\mathcal{I}$-nonthin subsequence of x that statistically converges to ξ.

Lemma 3.4. *[13] Let $(X, \mathcal{N}, \circ)$ be a PN space. If $x = \{x_k\}_{k\in\mathbb{N}}$ is a sequence in X such that $st\text{-}\lim_{k\to\infty} \mathcal{N}(x_k) = \xi$, then x has a nonthin subsequence $\{x_{m_k}\}_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} \mathcal{N}(x_{m_k}) = \xi$.*

Remark 3.5. We now show that both of the definitions (Definition 3.1 and Definition 3.3) are equivalent.

Theorem 3.6. *Definition 3.1 and Definition 3.3 are equivalent.*

Proof. We first show that Definition 3.1 implies Definition 3.3. Let $\xi \in X$ be an $\mathcal{I}$-statistical limit point of a sequence $x = \{x_k\}_{k\in\mathbb{N}}$ in X with respect to the probabilistic norm $\mathcal{N}$. Then there exists a set $M = \{m_1 < m_2 < ...\} \subset \mathbb{N}$ and $d^{\mathcal{I}}(M) \neq 0$ such that $\lim_{k\to\infty} \mathcal{N}(x_{m_k}) = \xi$. Now since every convergent sequence is statistically convergent to the same limit in X with respect to the probabilistic norm $\mathcal{N}$ (see [13]). Thus $st\text{-}\lim_{k\to\infty} \mathcal{N}(x_{m_k}) = \xi$. Hence, Definition 3.3 holds.

Conversely, we assume that Definition 3.3 holds, we show that Definition 3.1 holds. Let $\xi \in X$ be an $\mathcal{I}$-statistical limit point of a sequence $x = \{x_k\}_{k\in\mathbb{N}}$ in X with respect to the probabilistic norm $\mathcal{N}$. Then there exists a set $M = \{m_1 < m_2 < ...\} \subset \mathbb{N}$ and $d^{\mathcal{I}}(M) \neq 0$ such that $st\text{-}\lim_{k\to\infty} \mathcal{N}(x_{m_k}) = \xi$. Then by above Lemma 3.4 there exists a set $L = \{m_{n_1} < m_{n_2} < ...\} \subset M$ and $d(L) \neq 0$ such that $\lim_{k\to\infty} \mathcal{N}(x_{m_{n_k}}) = \xi$. Now since $\mathcal{I}$ is an admissible ideal and $d(L) \neq 0$, $d^{\mathcal{I}}(L) \neq 0$. Thus $\{x_{m_{n_k}}\}_{k\in\mathbb{N}}$ is an $\mathcal{I}$-nonthin subsequence of x. Also, $\lim_{k\to\infty} \mathcal{N}(x_{m_{n_k}}) = \xi$. Hence, Definition 3.1 holds.

$\square$

Now we give an example of an $\mathcal{I}$-statistical limit point in a PN space.

Example 3.7. Let $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$. Consider the PN space $(\mathbb{R}, \mathcal{N}, \circ)$ from Example 2.4. We define a sequence $x = \{x_k\}_{k\in\mathbb{N}}$ as follows:

$$\begin{aligned} x_k &= 1, \quad \text{if } k \text{ is odd} \\ &= 0, \quad \text{if } k \text{ is even.} \end{aligned}$$

Let B be the set of all odd numbers. Then $d(B) = \frac{1}{2}$. Since $\mathcal{I}$ is an admissible ideal, $d^{\mathcal{I}}(B) = \frac{1}{2}$. Thus $\{x\}_B$ is an $\mathcal{I}$-nonthin subsequence of x which converges to 1 with respect to the norm $\mathcal{N}$. Hence 1 is an $\mathcal{I}$-statistical limit point of x.

Definition 3.8. Let $(X, \mathcal{N}, \circ)$ be a PN space. Then $\xi \in X$ is said to be an $\mathcal{I}$-statistical cluster point of a sequence $x = \{x_k\}_{k\in\mathbb{N}}$ in X with respect to the probabilistic norm $\mathcal{N}$, if for every $\varepsilon > 0$ and $\lambda \in (0,1)$ the set $\{k \in \mathbb{N} : \mathcal{N}_{x_k - \xi}(\varepsilon) > 1 - \lambda\}$ does not have $\mathcal{I}$-asymptotic density zero.

We write $\Gamma_x^S(\mathcal{I})_{\mathcal{N}}$ to denote the set of all such $\mathcal{I}$-statistical cluster points of the sequence x.

Now we give an example of an $\mathcal{I}$-statistical cluster point in a PN space.

Example 3.9. Let $\mathcal{I}$ be an admissible in $\mathbb{N}$. Consider the PN space $(\mathbb{R}, \mathcal{N}, \circ)$ from Example 2.4. We define a sequence $x = \{x_k\}_{k\in\mathbb{N}}$ as follows:

$$\begin{aligned} x_k &= 1, \quad \text{if } k \text{ is a perfect square} \\ &= 0, \quad \text{otherwise.} \end{aligned}$$

Let B be the set of all perfect squares. Then $d(B) = 0$. Since $\mathcal{I}$ is an admissible ideal, thus $d^{\mathcal{I}}(B) = 0$. Now $\{k \in \mathbb{N} : \mathcal{N}_{x_k - 0}(\varepsilon) > 1 - \lambda\} = \mathbb{N} \setminus B$ for every $\varepsilon > 0$ and $\lambda \in (0,1)$. Since $d^{\mathcal{I}}(\mathbb{N} \setminus B) = 1$, thus $d^{\mathcal{I}}(\{k \in \mathbb{N} : \mathcal{N}_{x_k - 0}(\varepsilon) > 1 - \lambda\}) \neq 0$. Hence 0 is an $\mathcal{I}$-statistical cluster point of x.

Note 3.10. Let $(X, \mathcal{N}, \circ)$ be a PN space. If $\mathcal{I} = \mathcal{I}_{fin} = \{A \subset \mathbb{N} : |A| < \infty\}$ then the notions of $\mathcal{I}$-statistical limit points and $\mathcal{I}$-statistical cluster points in X with respect to the probabilistic norm $\mathcal{N}$ coincide with the notions of statistical limit points and statistical cluster points in X with respect to the probabilistic norm $\mathcal{N}$ respectively. Thus in a PN space, the notions of $\mathcal{I}$-statistical cluster points and $\mathcal{I}$-statistical limit points generalize the notions of statistical cluster points and statistical limit points respectively.

We now cite an example to show that: There exists an ideal $\mathcal{I}$ in $\mathbb{N}$ for which there exists a sequence $x = \{x_k\}_{k\in\mathbb{N}}$ having $\mathcal{I}$-statistical limit but no statistical limit in a PN space.

Example 3.11. Let $C_k = \{(2^{2^k})^n : n \in \mathbb{N}\}$, $k \in \mathbb{N}$. Let $\mathcal{I} = \{B \subset \mathbb{N} : |B \cap C_k| < \infty \text{ } for \text{ } some \text{ } k\}$. Then $\mathcal{I}$ is an ideal of $\mathbb{N}$. Let $C = \cup_{n=1}^{\infty}(2^{2^n}, 2.2^{2^n}]$. Also, we consider the linear space $\mathbb{R}$ of all real numbers. Let us define $a \circ b = min\{a, b\}$ for all $a, b \in \mathbb{R}$ and $\mathcal{N}_x = \epsilon_\infty$, for $x \neq 0$ and $\mathcal{N}_x = \epsilon_0$ for $x = 0$, where

$$\begin{aligned} \epsilon_0(t) &= 0, \quad -\infty \leq t \leq 0 \\ &= 1, \quad 0 < t \leq \infty; \end{aligned}$$

$$\begin{aligned}\epsilon_\infty(t) &= 0, \quad -\infty \leq t < \infty \\ &= 1, \quad t = \infty.\end{aligned}$$

Then $(\mathbb{R}, \mathcal{N}, \circ)$ is a PN space (see Example 2.4).

Now we define a sequence $x = \{x_k\}_{k\in\mathbb{N}}$ in $(\mathbb{R}, \mathcal{N}, \circ)$ as follows: $x_k = 1$ if $k \in C$ and $x_k = 0$, if $k \notin C$. Let $t > 0$ and $0 < \lambda < 1$ be given. Let $K = \{k \in \mathbb{N} : \mathcal{N}_{x_k}(t) \leq 1 - \lambda\}$. Since for $k \notin C$ we have $\mathcal{N}_0(t) = \epsilon_0(t) = 1$, thus $K \subset C$. And for $k \in C$, $\mathcal{N}_1(t) = \epsilon_\infty(t) = 0$ we have $C \subset K$. Hence $C = K$. Now for $k = 2.2^{2^n}$, $n \in \mathbb{N}$ we have $\frac{|C(k)|}{k} \geq \frac{1}{2}$ and for $k = 2^{2^n}$, $n \in \mathbb{N}$ we have $\frac{|(\mathbb{N}\setminus C)(k)|}{k} \geq \frac{1}{2}$. Thus $d(C)$ does not exists. Hence x is not statistically convergent in $(\mathbb{R}, \mathcal{N}, \circ)$.

Now we show that $d^{\mathcal{I}}(C) = 0$. Let $\delta > 0$ be given. Choose p be the smallest natural number such that $\frac{1}{2^{2^p - 1}} < \delta$. Now if $2^{2^n} \in C_k$ for some $k \in \mathbb{N}$ then $k \leq n$ and the smallest element in C_k greater than 2^{2^n} is $2^{2^n}.2^{2^k}$. Thus for $i = 2^{2^n}.2^{2^k}$ we have

$$\frac{|C(i)|}{i} \leq \frac{2.2^{2^n}}{2^{2^n}.2^{2^k}} = \frac{1}{2^{2^k - 1}}.$$

Therefore for every $i \in C_p$ we have $\frac{|C(i)|}{i} \leq \frac{1}{2^{2^p-1}} < \delta$. Thus $\mathbb{N} \setminus C_p \subset \{i \in \mathbb{N} : \frac{|C(i)|}{i} \geq \delta\}$. Since $\mathbb{N} \setminus C_p \in \mathcal{I}$, thus $\{i \in \mathbb{N} : \frac{|C(i)|}{i} \geq \delta\} \in \mathcal{I}$. Hence $d^{\mathcal{I}}(C) = 0$, and x is $\mathcal{I}$-statistically convergent to 0.

Theorem 3.12. *Let $(X, \mathcal{N}, \circ)$ be a PN space. Then for a sequence $x = \{x_k\}_{k\in\mathbb{N}}$ in X, we have $\Lambda^S_x(\mathcal{I})_{\mathcal{N}} \subset \Gamma^S_x(\mathcal{I})_{\mathcal{N}} \subset L^{\mathcal{N}}_x$.*

Proof. Let ς be a arbitrary element in $\Lambda^S_x(\mathcal{I})_{\mathcal{N}}$. Then there exists a subsequence $\{x_{k_j}\}_{j\in\mathbb{N}}$ of x such that $\lim_{j\to\infty} \mathcal{N}(x_{k_j}) = \varsigma$ and $d^{\mathcal{I}}(\{k_j : j \in \mathbb{N}\}) \neq 0$. Let $\varepsilon > 0$ and $\lambda \in (0, 1)$ be given. Since $\lim_{j\to\infty} \mathcal{N}(x_{k_j}) = \varsigma$, thus $E = \{k_j : \mathcal{N}_{x_{k_j}-\varsigma}(\varepsilon) \leq 1 - \lambda\}$ is a finite set. Also,

$$\{k \in \mathbb{N} : \mathcal{N}_{x_k-\varsigma}(\varepsilon) > 1 - \lambda\} \supset \{k_j : j \in \mathbb{N}\} \setminus E$$

$$\Rightarrow K = \{k_j : j \in \mathbb{N}\} \subset \{k \in \mathbb{N} : \mathcal{N}_{x_k-\varsigma}(\varepsilon) > 1 - \lambda\} \cup E.$$

Now if $d^{\mathcal{I}}(\{k \in \mathbb{N} : \mathcal{N}_{x_k-\varsigma}(\varepsilon) > 1 - \lambda\}) = 0$, then we have $d^{\mathcal{I}}(K) = 0$, which is a contradiction. Thus ς is an $\mathcal{I}$-statistical cluster point of x. Since $\varsigma \in \Lambda^S_x(\mathcal{I})_{\mathcal{N}}$ is arbitrary, $\Lambda^S_x(\mathcal{I})_{\mathcal{N}} \subset \Gamma^S_x(\mathcal{I})_{\mathcal{N}}$.

Now we show that $\Gamma^S_x(\mathcal{I})_{\mathcal{N}} \subset L^{\mathcal{N}}_x$. Let $\varsigma \in \Gamma^S_x(\mathcal{I})_{\mathcal{N}}$. Also, let $\varepsilon > 0$ and $\lambda \in (0, 1)$ be given. Then the set $J = \{j \in \mathbb{N} : \mathcal{N}_{x_j-\varsigma}(\varepsilon) > 1 - \lambda\}$ does not have $\mathcal{I}$-asymptotic density zero. Thus J is an infinite subset of $\mathbb{N}$, so we can write $J = \{j_k : j_1 < j_2 < ...\}$. And we have a subsequence $\{x\}_J$ of x which converges to ς with respect to the probabilistic norm $\mathcal{N}$. Hence $\varsigma \in L^{\mathcal{N}}_x$.

Therefore $\Lambda^S_x(\mathcal{I})_{\mathcal{N}} \subset \Gamma^S_x(\mathcal{I})_{\mathcal{N}} \subset L^{\mathcal{N}}_x$. □

Theorem 3.13. *Let $(X, \mathcal{N}, \circ)$ be a PN space. Let $x = \{x_k\}_{k\in\mathbb{N}}$ and $y = \{y_k\}_{k\in\mathbb{N}}$ be two sequences in X such that $d^{\mathcal{I}}(\{k : x_k \neq y_k\}) = 0$. Then $\Lambda^S_x(\mathcal{I})_{\mathcal{N}} = \Lambda^S_y(\mathcal{I})_{\mathcal{N}}$ and $\Gamma^S_x(\mathcal{I})_{\mathcal{N}} = \Gamma^S_y(\mathcal{I})_{\mathcal{N}}$.*

Proof. Let $\zeta \in \Gamma_x^S(\mathcal{I})_{\mathcal{N}}$. Also, let $\varepsilon > 0$ and $0 < \lambda < 1$ be given. Then $\{k \in \mathbb{N} : \mathcal{N}_{x_k-\zeta}(\varepsilon) > 1-\lambda\}$ does not have $\mathcal{I}$-asymptotic density zero. Let $B = \{k \in \mathbb{N} : x_k = y_k\}$. Then $d^{\mathcal{I}}(B) = 1$. Therefore $\{k \in \mathbb{N} : \mathcal{N}_{x_k-\zeta}(\varepsilon) > 1-\lambda\} \cap B$ does not have $\mathcal{I}$-asymptotic density zero. Consequently, $\zeta \in \Gamma_y^S(\mathcal{I})_{\mathcal{N}}$. Since $\zeta \in \Gamma_x^S(\mathcal{I})_{\mathcal{N}}$ is arbitrary, thus $\Gamma_x^S(\mathcal{I})_{\mathcal{N}} \subset \Gamma_y^S(\mathcal{I})_{\mathcal{N}}$. And by symmetry we have $\Gamma_y^S(\mathcal{I})_{\mathcal{N}} \subset \Gamma_x^S(\mathcal{I})_{\mathcal{N}}$. Hence $\Gamma_x^S(\mathcal{I})_{\mathcal{N}} = \Gamma_y^S(\mathcal{I})_{\mathcal{N}}$.

Now we prove that $\Lambda_x^S(\mathcal{I})_{\mathcal{N}} = \Lambda_y^S(\mathcal{I})_{\mathcal{N}}$. Let $\eta \in \Lambda_x^S(\mathcal{I})_{\mathcal{N}}$. Then x has an $\mathcal{I}$-nonthin subsequence $\{x_{k_j}\}_{j\in\mathbb{N}}$ that converges to η with respect to the probabilistic norm $\mathcal{N}$. Let $K = \{k_j : j \in \mathbb{N}\}$. Since $d^{\mathcal{I}}(\{k_j : x_{k_j} \neq y_{k_j}\}) = 0$, we have $d^{\mathcal{I}}(\{k_j : x_{k_j} = y_{k_j}\}) \neq 0$. Consequently, from the later set, we have an $\mathcal{I}$-nonthin subsequence $\{y\}_{K'}$ of $\{y\}_K$ that converges to η with respect to probabilistic norm $\mathcal{N}$. Thus $\eta \in \Lambda_y^S(\mathcal{I})_{\mathcal{N}}$. Since $\eta \in \Lambda_x^S(\mathcal{I})_{\mathcal{N}}$ is arbitrary, thus $\Lambda_x^S(\mathcal{I})_{\mathcal{N}} \subset \Lambda_y^S(\mathcal{I})_{\mathcal{N}}$. And by symmetry we have $\Lambda_x^S(\mathcal{I})_{\mathcal{N}} \supset \Lambda_y^S(\mathcal{I})_{\mathcal{N}}$. Hence $\Lambda_x^S(\mathcal{I})_{\mathcal{N}} = \Lambda_y^S(\mathcal{I})_{\mathcal{N}}$. $\square$

Example 3.14. Let $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$. Consider the PN space $(\mathbb{R}, \mathcal{N}, \circ)$ from Example 2.4. We define sequences $x = \{x_k\}_{k\in\mathbb{N}}$ and $y = \{y_k\}_{k\in\mathbb{N}}$ as follows:

$$\begin{aligned} x_k &= 1, \text{ if } k \text{ is a perfect square} \\ &= 0, \text{ otherwise;} \end{aligned}$$

and

$$\begin{aligned} y_k &= 2, \text{ if } k \text{ is a perfect square} \\ &= 0, \text{ otherwise.} \end{aligned}$$

Let B be the set of all perfect squares. Then $d(B) = 0$. Since $\mathcal{I}$ is an admissible ideal, $d^{\mathcal{I}}(B) = 0$. Thus $d^{\mathcal{I}}(\{k : x_k \neq y_k\}) = d^{\mathcal{I}}(B) = 0$.

Clearly, $\Gamma_x^S(\mathcal{I})_{\mathcal{N}} = \Gamma_y^S(\mathcal{I})_{\mathcal{N}} = \{0\}$ and $\Lambda_x^S(\mathcal{I})_{\mathcal{N}} = \Lambda_y^S(\mathcal{I})_{\mathcal{N}} = \{0\}$.

Theorem 3.15. *Let $(X, \mathcal{N}, \circ)$ be a PN space and $x = \{x_k\}_{k\in\mathbb{N}}$ be a sequence in X. Then $\Gamma_x^S(\mathcal{I})_{\mathcal{N}}$ is a closed subset of X.*

Proof. If $\Gamma_x^S(\mathcal{I})_{\mathcal{N}} = \emptyset$ then there is nothing to prove. We assume $\Gamma_x^S(\mathcal{I})_{\mathcal{N}} \neq \emptyset$. Clearly, it is sufficient to prove that $\Gamma_x^S(\mathcal{I})_{\mathcal{N}}$ contains all its limit points. Let ξ be a limit point of $\Gamma_x^S(\mathcal{I})_{\mathcal{N}}$. Also, let $\varepsilon > 0$ and $0 < \lambda < 1$ be given. Choose $0 < \sigma < 1$ such that $(1-\sigma)\circ(1-\sigma) > 1-\lambda$. Then $B(\xi, \sigma, \frac{\varepsilon}{2}) \cap (\Gamma_x^S(\mathcal{I})_{\mathcal{N}} \setminus \{\xi\}) \neq \emptyset$. Choose $\beta \in (\xi, \sigma, \frac{\varepsilon}{2}) \cap (\Gamma_x^S(\mathcal{I})_{\mathcal{N}} \setminus \{\xi\})$. Since $\beta \in \Gamma_x^S(\mathcal{I})_{\mathcal{N}}$,

$$d^{\mathcal{I}}(\{k \in \mathbb{N} : \mathcal{N}_{x_k-\beta}(\tfrac{\varepsilon}{2}) > 1-\sigma\}) \neq 0.$$

Now we show that

$$\{k \in \mathbb{N} : \mathcal{N}_{x_k-\beta}(\tfrac{\varepsilon}{2}) > 1-\sigma\} \subset \{k \in \mathbb{N} : \mathcal{N}_{x_k-\xi}(\varepsilon) > 1-\lambda\}.$$

Let $k \in \{k \in \mathbb{N} : \mathcal{N}_{x_k-\beta}(\frac{\varepsilon}{2}) > 1-\sigma\}$. Then $\mathcal{N}_{x_k-\beta}(\frac{\varepsilon}{2}) > 1-\sigma$. Since $\beta \in (\xi, \sigma, \frac{\varepsilon}{2})$, $\mathcal{N}_{\xi-\beta}(\frac{\varepsilon}{2}) > 1-\sigma$. Thus

$$\mathcal{N}_{x_k-\xi}(\varepsilon) \geq \mathcal{N}_{x_k-\beta}(\tfrac{\varepsilon}{2}) + \mathcal{N}_{\beta-\xi}(\tfrac{\varepsilon}{2}) > (1-\sigma)\circ(1-\sigma) = 1-\lambda.$$

Since $k \in \{k \in \mathbb{N} : \mathcal{N}_{x_k-\beta}(\frac{\varepsilon}{2}) > 1-\sigma\}$ is arbitrary, therefore

$$\{k \in \mathbb{N} : \mathcal{N}_{x_k-\beta}(\tfrac{\varepsilon}{2}) > 1-\sigma\} \subset \{k \in \mathbb{N} : \mathcal{N}_{x_k-\xi}(\varepsilon) > 1-\lambda\}.$$

Thus

$$d^{\mathcal{I}}(\{k : \mathcal{N}_{x_k-\xi}(\varepsilon) > 1-\lambda\}) \neq 0.$$

Hence $\xi \in \Gamma_x^S(\mathcal{I})_{\mathcal{N}})$. This completes the proof. $\square$

4. Condition APIO

In this section, using Condition APIO, we prove some theorems similar to that of [12] and [13].

Definition 4.1. (Additive property for $\mathcal{I}$-asymptotic density zero sets). [17] The $\mathcal{I}$-asymptotic density $d^{\mathcal{I}}$ is said to satisfy APIO if, given any countable collection of mutually disjoint sets $\{A_j\}_{j\in\mathbb{N}}$ in $\mathbb{N}$ with $d^{\mathcal{I}}(A_j) = 0$, for each $j \in \mathbb{N}$, there exists a collection of sets $\{B_j\}_{j\in\mathbb{N}}$ in $\mathbb{N}$ with the properties $|A_j \Delta B_j| < \infty$ for each $j \in \mathbb{N}$ and $d^{\mathcal{I}}(B = \bigcup\limits_{j=1}^{\infty} B_j) = 0$.

Theorem 4.2. *Let $(X, \mathcal{N}, \circ)$ be a PN space and $\mathcal{I}$ be an ideal in $\mathbb{N}$ such that $d^{\mathcal{I}}$ has Property APIO. Then a sequence $x = \{x_k\}_{k\in\mathbb{N}}$ in X is $\mathcal{I}$-statistically convergent to ξ if and only if there exists a subset B of $\mathbb{N}$ with $d^{\mathcal{I}}(B) = 1$ and $\lim\limits_{k\in B, k\to\infty} \mathcal{N}(x_k) = \xi$.*

Proof. Let $x = \{x_k\}_{k\in\mathbb{N}}$ be a sequence in X such that x is $\mathcal{I}$-statistically convergent to $\xi \in X$ with respect to probabilistic norm $\mathcal{N}$. Then for every $t > 0$ and $\lambda \in (0,1)$, the set $\{k \in \mathbb{N} : \mathcal{N}_{x_k-\xi}(t) \geq 1-\lambda\}$ has $\mathcal{I}$-asymptotic density zero. Set $A_1 = \{k \in \mathbb{N} : 0 \leq \mathcal{N}_{x_k-\xi}(t) < \frac{1}{2}\}$, $A_j = \{k \in \mathbb{N} : 1-\frac{1}{j} \leq \mathcal{N}_{x_k-\xi}(t) < 1-\frac{1}{j+1}\}$ for $j \geq 2, j \in \mathbb{N}$. Then $\{A_j\}_{j\in\mathbb{N}}$ is a countable sequence of mutually disjoint sets with $d^{\mathcal{I}}(A_j) = 0$ for all $j \in \mathbb{N}$. Then by assumption there exists a countable sequence of sets $\{B_j\}_{j\in\mathbb{N}}$ with $|A_j \Delta B_j| < \infty$ and $d^{\mathcal{I}}(B = \bigcup\limits_{j=1}^{\infty} B_j) = 0$. We claim that $\mathop{lim}\limits_{k\in\mathbb{N}\setminus B, k\to\infty} \mathcal{N}(x_k) = \xi$. To establish our claim, let $t > 0$ and $\delta \in (0,1)$ be given. Choose $i \in \mathbb{N}$ such that $\frac{1}{i+1} < \delta$. Then $\{k \in \mathbb{N} : \mathcal{N}_{x_k-\xi}(t) \leq 1-\delta\} \subset \bigcup\limits_{j=1}^{i+1} A_j$. Since $|A_j \Delta B_j| < \infty$ for all $j = 1, 2, \ldots, i+1$, there exists $n' \in \mathbb{N}$ such that $\bigcup\limits_{j=1}^{i+1} A_j \cap (n', \infty) = \bigcup\limits_{j=1}^{i+1} B_j \cap (n', \infty)$. Now if $k \notin B$, $k > n'$ then $k \notin \bigcup\limits_{j=1}^{i+1} B_j$. And consequently, $k \notin \bigcup\limits_{j=1}^{i+1} A_j$, which implies $\mathcal{N}_{x_k-\xi}(t) > 1-\delta$. This completes the proof of the necessity part.

Conversely, let there exists a subset B of the set of natural number $\mathbb{N}$ with $d^{\mathcal{I}}(B) = 1$ and $\lim\limits_{k\in B, k\to\infty} \mathcal{N}(x_k) = \xi$. We are to show that x is $\mathcal{I}$-statistically convergent to ξ with respect to the probabilistic norm $\mathcal{N}$. Since B is an infinite set we can write $B = \{k_j : k_1 < k_2 < \ldots\}$. Let $t > 0$ and $0 < \lambda < 1$ be given. Since $\lim_{k\in B,k\to\infty} \mathcal{N}(x_k) = \xi$ then there exists $j_0 \in \mathbb{N}$ such that for all $j > j_0$, $\mathcal{N}_{x_{k_j}-\xi}(t) > 1-\lambda$. Set $M = \{k \in \mathbb{N} : \mathcal{N}_{x_k-\xi}(t) \le 1-\lambda\}$. Then $M \subset N \setminus \{k_j : j > j_0\}$. Since $d^{\mathcal{I}}(B) = 1$, $d^{\mathcal{I}}(\mathbb{N} \setminus \{k_j : j > j_0\}) = 0$. Thus $d^{\mathcal{I}}(M) = 0$. Hence x is $\mathcal{I}$-statistically convergent to ξ with respect to the probabilistic norm $\mathcal{N}$. □

Corollary 4.3. *Let $(X, \mathcal{N}, \circ)$ be a PN space and $x = \{x_k\}_{k\in\mathbb{N}}$ be a sequence in X. Let $\mathcal{I}$ be an ideal in $\mathbb{N}$ such that $d^{\mathcal{I}}$ has Property APIO. If $\mathcal{I}$-st-$\lim_{k\to\infty} \mathcal{N}(x_k) = \alpha$ then $\alpha \in \Lambda^S_x(\mathcal{I})_{\mathcal{N}}$.*

Theorem 4.4. *Let $(X, \mathcal{N}, \circ)$ be a PN space and $x = \{x_k\}_{k\in\mathbb{N}}$ be a sequence in X. Let $\mathcal{I}$ be an ideal in $\mathbb{N}$ such that $d^{\mathcal{I}}$ has Property APIO. If $\mathcal{I}$-st-$\lim_{k\to\infty} \mathcal{N}(x_k) = \alpha$ then $\Lambda^S_x(\mathcal{I})_{\mathcal{N}} = \Gamma^S_x(\mathcal{I})_{\mathcal{N}} = \{\alpha\}$.*

Proof. From Corollary 4.3, we have $\alpha \in \Lambda^S_x(\mathcal{I})_{\mathcal{N}}$. Let $\beta \in \Lambda^S_x(\mathcal{I})_{\mathcal{N}}$ be such that $\alpha \ne \beta$. Then there exist two subsets $J = \{j_q : j_1 < j_2 < ...\}$ and $K = \{k_r : k_1 < k_2 < ...\}$ of $\mathbb{N}$ such that

$$d^{\mathcal{I}}(J) \ne 0, \lim_{q\to\infty} \mathcal{N}(x_{j_q}) = \alpha$$

and

$$d^{\mathcal{I}}(K) \ne 0, \lim_{r\to\infty} \mathcal{N}(x_{k_r}) = \beta.$$

Let $t > 0$ and $0 < \lambda < 1$ be given. Choose $0 < \sigma < 1$ such that $(1-\sigma)\circ(1-\sigma) > (1-\lambda)$. Since the subsequence $\{x\}_K$ of x converges to β with respect to the probabilistic norm $\mathcal{N}$, there exists $r > N_0$ such that $\mathcal{N}_{x_{k_r}-\beta}(\frac{t}{2}) > 1-\sigma$.

Therefore $A = \{k_r \in K : \mathcal{N}_{x_{k_r}-\beta}(\frac{t}{2}) \le 1-\sigma\} \subset \{k_r \in K : k_1 < k_2 < ... < k_{N_0}\}$. Let $B = \{k_r \in K : \mathcal{N}_{x_{k_r}-\beta}(\frac{t}{2}) > 1-\sigma\}$

Since $\mathcal{I}$ is an admissible ideal and $A \in \mathcal{I}$ we have

$$d^{\mathcal{I}}(B) \ne 0. \tag{4.1}$$

Again $(\mathcal{I}\text{-}st) - \lim_{k\to\infty} \mathcal{N}(x_k) = \alpha$, which implies $d^{\mathcal{I}}(E) = 0$, where $E = \{k \in \mathbb{N} : \mathcal{N}_{x_k-\alpha}(\frac{t}{2}) \le 1-\sigma\}$. And consequently, $d^{\mathcal{I}}(\mathbb{N} \setminus E) \ne 0$.

Since $\alpha \ne \beta$, we have $B \cap (\mathbb{N} \setminus E) = \emptyset$, otherwise, for $k \in B \cap (\mathbb{N} \setminus E)$

$$\mathcal{N}_{\alpha-\beta}(t) \ge \mathcal{N}_{x_k-\beta}(\tfrac{t}{2}) \circ \mathcal{N}_{x_k-\alpha}(\tfrac{t}{2}) > (1-\sigma)\circ(1-\sigma) > (1-\lambda);$$

since $\lambda > 0$ is arbitrary and $\mathcal{N}_{\alpha-\beta}(t) > 1-\lambda$ we have $\mathcal{N}_{\alpha-\beta}(t) = 1$, for all $t > 0$, which implies $\alpha = \beta$.

Therefore $B \subset E$. Since $d^{\mathcal{I}}(E) = 0$, we have $d^{\mathcal{I}}(B) = 0$, which contradicts (4.1). Hence $\Lambda^S_x(\mathcal{I})_{\mathcal{N}} = \{\alpha\}$.

Again let $\alpha, \beta \in \Gamma^S_x(\mathcal{I})_{\mathcal{N}}$ and $\alpha \ne \beta$. Let $t > 0$ and $0 < \lambda < 1$ be given. Choose $0 < \sigma < 1$ such that $(1-\sigma)\circ(1-\sigma) > (1-\lambda)$. Then we have

$$G = \{k \in \mathbb{N} : \mathcal{N}_{x_k-\alpha}(\frac{t}{2}) > 1-\sigma\}, d^{\mathcal{I}}(G) \ne 0 \tag{4.2}$$

and

$$H = \{k \in \mathbb{N} : \mathcal{N}_{x_k-\beta}(\frac{t}{2}) > 1-\sigma\}, d^{\mathcal{I}}(H) \neq 0. \tag{4.3}$$

Since $\alpha \neq \beta$ we have $G \cap H = \emptyset$, otherwise, for $k \in G \cap H$

$$\mathcal{N}_{\alpha-\beta}(t) \geq \mathcal{N}_{x_k-\alpha}(\tfrac{t}{2}) \circ \mathcal{N}_{x_k-\beta}(\tfrac{t}{2}) > (1-\sigma) \circ (1-\sigma) > (1-\lambda).$$

Since $\lambda > 0$ is arbitrary and $\mathcal{N}_{\alpha-\beta}(t) > 1-\lambda$, we have $\mathcal{N}_{\alpha-\beta}(t) = 1$, for all $t > 0$, which implies $\alpha = \beta$.

Therefore $H \subset (\mathbb{N} \setminus G)$ and $(\mathcal{I}\text{-}st) - \lim_{k\to\infty} \mathcal{N}(x_k) = \alpha$, which implies $d^{\mathcal{I}}(\mathbb{N} \setminus G) = 0$. Thus $d^{\mathcal{I}}(H) = 0$, which contradicts (4.3). Hence $\Gamma_x^S(\mathcal{I})_{\mathcal{N}} = \{\alpha\}$. □

Example 4.5. Consider the ideal $\mathcal{I} = \mathcal{I}_{fin}$ of all finite subsets of $\mathbb{N}$. Then $\mathcal{I}$ is an admissible ideal in $\mathbb{N}$ and $d^{\mathcal{I}}$ has Property APIO. Consider the PN space $(\mathbb{R}, \mathcal{N}, \circ)$ from the Example 2.4. We define a sequence $x = \{x_k\}_{k\in\mathbb{N}}$ as follows:

$$\begin{aligned} x_k &= 1, \quad \text{if } k \text{ is a prime number} \\ &= 0, \quad \text{otherwise.} \end{aligned}$$

Let P be the set of all prime numbers. Then $d(P) = 0$. Since $\mathcal{I} = \mathcal{I}_{fin}$ is an admissible ideal, $d^{\mathcal{I}}(P) = 0$. Clearly, $\mathcal{I}\text{-}st\text{-}\lim_{k\to\infty} \mathcal{N}(x_k) = 0$ and $\Lambda_x^S(\mathcal{I})_{\mathcal{N}} = \Gamma_x^S(\mathcal{I})_{\mathcal{N}} = \{0\}$.

Note 4.6. One can see that in the last example we have not used the APIO property, rather we use the admissible property of the ideal. Thus we can say that the converse of the Theorem 4.4 does not necessarily hold.

Let $(X, \mathcal{N}, \circ)$ be a probabilistic normed space. Now we define the norm topology on X as follows: A subset U of X is said to be open if for every $x \in U$ there exists $0 < r < 1$ such that for all $t > 0$, $B(x, r, t) \subset U$.

Definition 4.7. Let $(X, \mathcal{N}, \circ)$ be a PN space. Then X is said to be a second countable PN space if X has countable basis under its norm topology.

Lemma 4.8. *Let $(X, \mathcal{N}, \circ)$ be a second countable PN space. Then every subspace of X is second countable under subspace norm topology.*

Proof. Let $\mathcal{C}$ is a countable basis for X under its norm topology. Let Y be a subset of X. Then clearly $\{C \cap Y : C \in \mathcal{C}\}$ is a countable basis for the subspace Y of X under its subspace norm topology. □

Lemma 4.9. *Let $(X, \mathcal{N}, \circ)$ be a second countable PN space. Then every open covering of X contains a countable subcollection covering X.*

Proof. Since X is second countable PN space, X has a countable basis under norm topology, say, $\mathcal{C} = \{C_n\}_{n\in\mathbb{N}}$. Now let $\mathcal{B}$ be an open covering of X. Then for each $n \in \mathbb{N}$, we can choose an element B_n of $\mathcal{B}$ such that B_n contains C_n. We claim that the countable subcollection $\mathcal{B}' = \{B_n\}_{n\in\mathbb{N}}$ of $\mathcal{B}$ covers X. Indeed, for each $x \in X$, we can choose an element $B \in \mathcal{B}$ such that $x \in B$.

Since B is open, thus there exists $C_n \in \mathcal{C}$ such that $x \in C_n \subset B$. Now since $C_n \subset B$ and $n \in \mathbb{N}$, the subcollection is well-defined. Also, since $C_n \subset B_n$, $x \in B_n$. Hence $\mathcal{B}'$ covers X. This completes the proof. $\square$

Theorem 4.10. *Let $(X, \mathcal{N}, \circ)$ be a second countable PN space. Let $\mathcal{I}$ be an ideal such that $d^{\mathcal{I}}$ has Property APIO. Then for any sequence $x = \{x_k\}_{k\in\mathbb{N}}$ there exists a sequence $y = \{y_k\}_{k\in\mathbb{N}}$ in X such that $L_y^{\mathcal{N}} = \Gamma_x^S(\mathcal{I})_{\mathcal{N}}$ and $d^{\mathcal{I}}(\{k : x_k \neq y_k\}) = 0$.*

Proof. At first, we prove that $\Gamma_x^S(\mathcal{I})_{\mathcal{N}} \subset L_x^{\mathcal{N}}$. Let $\xi \in \Gamma_x^S(\mathcal{I})_{\mathcal{N}}$. Let $t > 0$ and $0 < \lambda < 1$ be given. Then we have $d^{\mathcal{I}}(\{k \in \mathbb{N} : \mathcal{N}_{x_k-\xi}(t) > 1-\lambda\}) \neq 0$. Now we claim that $d(\{k \in \mathbb{N} : \mathcal{N}_{x_k-\xi}(t) > 1-\lambda\}) \neq 0$. If possible, let $d(\{k \in \mathbb{N} : \mathcal{N}_{x_k-\xi}(t) > 1-\lambda\}) = 0$. Then

$$\lim_{n\to\infty} \tfrac{|S(n)|}{n} = 0,$$

where $S = \{k \in \mathbb{N} : \mathcal{N}_{x_k-\xi}(t) > 1-\lambda\}$. Since $\mathcal{I}$ is admissible, therefore

$$\mathcal{I} - \lim_{n\to\infty} \tfrac{|S(n)|}{n} = 0,$$

where $S = \{k \in \mathbb{N} : \mathcal{N}_{x_k-\xi}(t) > 1-\lambda\}$. Thus $d^{\mathcal{I}}(\{k \in \mathbb{N} : \mathcal{N}_{x_k-\xi}(t) > 1-\lambda\}) = 0$, which is a contradiction. Hence $d(\{k \in \mathbb{N} : \mathcal{N}_{x_k-\xi}(t) > 1-\lambda\}) \neq 0$. Thus ξ is a statistical cluster point of x, and hence a limit point of x. Thus $\Gamma_x^S(\mathcal{I})_{\mathcal{N}} \subset L_x^{\mathcal{N}}$. If $\Gamma_x^S(\mathcal{I})_{\mathcal{N}} = L_x^{\mathcal{N}}$, we will take $y = \{y_k\}_{k\in\mathbb{N}} = \{x_k\}_{k\in\mathbb{N}} = x$, and we are done. Let $\Gamma_x^S(\mathcal{I})_{\mathcal{N}}$ be a proper subset of $L_x^{\mathcal{N}}$. Let $\eta \in L_x^{\mathcal{N}} \setminus \Gamma_x^S(\mathcal{I})_{\mathcal{N}}$. Choose a ball $B(\eta, r_\eta, t) = \{u \in X : \mathcal{N}_{\eta-u}(t) > 1 - r_\eta\}$ with the center at η and radius $r_\eta \in (0,1)$ such that $d^{\mathcal{I}}(\{k \in \mathbb{N} : x_k \in B(\eta, r_\eta, t)\}) = 0$. Then the collection of all such $B(\eta, r_\eta, t)$'s is an open cover for $L_x^{\mathcal{N}} \setminus \Gamma_x^S(\mathcal{I})_{\mathcal{N}}$. Since $L_x^{\mathcal{N}} \setminus \Gamma_x^S(\mathcal{I})_{\mathcal{N}}$ is a subspace of a second countable space X, it is second countable (by Lemma4.9). Then there exists a countable subcover, say $\{B(\eta_j, r_j, t)\}_{j\in\mathbb{N}}$ of $\{B(\eta, r_\eta, t) : \eta \in L_x^{\mathcal{N}} \setminus \Gamma_x^S(I)_{\mathcal{N}}\}$ for $L_x^{\mathcal{N}} \setminus \Gamma_x^S(\mathcal{I})_{\mathcal{N}}$. Since each η_j is a limit point of x and $d^{\mathcal{I}}(\{k \in \mathbb{N} : x_k \in B(\eta_j, r_j, t)\}) = 0$, consequently each $B(\eta_j, r_j, t)$ contains an $\mathcal{I}$-thin subsequence of x. Let $I_1 = \{k \in \mathbb{N} : x_k \in B(\eta_1, r_1, t)\}, I_j = \{k \in \mathbb{N} : x_k \in B(\eta_j, r_j, t)\} \setminus (I_1 \cup I_2 ... \cup I_{j-1}), \forall j \geq 2, j \in \mathbb{N}$. Then $\{I_j\}_{j\in\mathbb{N}}$ is a countable collection of mutually disjoint sets with $d^{\mathcal{I}}(I_j) = 0, \forall j \in \mathbb{N}$. Since $d^{\mathcal{I}}$ has Property APIO, there exists a countable collection of sets $\{B_j\}_{j\in\mathbb{N}}$ such that $|I_j \Delta B_j| < \infty$ for each $j \in \mathbb{N}$ and $d^{\mathcal{I}}(B = \bigcup_{j=1}^{\infty} B_j) = 0$. Then $I_j \setminus B$ is a finite set and so $\{k \in \mathbb{N} : k \in I_{\eta_j}\} \setminus B$ is a finite set for each $j \in \mathbb{N}$. Let $\mathbb{N} \setminus B = \{j_i < j_2 < ...\}$ and we define a sequence $y = \{y_k\}_{k\in\mathbb{N}}$ as follows

$$y_k = \begin{cases} x_{j_k} & \text{if } k \in B, \\ x_k & \text{if } k \in \mathbb{N} \setminus B. \end{cases}$$

Obviously, the set $\{k : x_k \neq y_k\}(\subset B)$ has $\mathcal{I}$-asymptotic density zero and by Theorem 3.2 we have $\Gamma_x^S(\mathcal{I})_{\mathcal{N}} = \Gamma_y^S(\mathcal{I})_{\mathcal{N}}$. Now we show that $L_y^{\mathcal{N}} = \Gamma_y^S(\mathcal{I})_{\mathcal{N}}$. If possible, let $\Gamma_y^S(\mathcal{I})_{\mathcal{N}} \subsetneq L_y^{\mathcal{N}}$ and $l \in L_y^{\mathcal{N}} \setminus \Gamma_y^S(\mathcal{I})_{\mathcal{N}}$. Then there exists a subsequence of y converging to l. Note that the subsequence must be $\mathcal{I}$-thin but

$\{y\}_B$ has no limit point. Therefore no such l can exist. Hence $L_y^{\mathcal{N}} = \Gamma_y^S(\mathcal{I})_{\mathcal{N}}$. Consequently, $L_y^{\mathcal{N}} = \Gamma_x^S(\mathcal{I})_{\mathcal{N}}$. □

Acknowledgment

The first author is grateful to CSIR, New Delhi, India, for his fellowship, funding under CSIR-JRF scheme during the preparation of this paper and to Dr. Prasanta Malik, Assistant Professor, The University of Burdwan, Burdwan, India for his valuable guidance. The authors convey their gratitude to the learned referees for their most valuable suggestions.

References

[1] Alsina, C., Schweizer, B., and Sklar, A. On the definition of a probabilistic normed space. *Aequationes Math. 46*, 1-2 (1993), 91–98.

[2] Connor, J. R-type summability methods, Cauchy criteria, P-sets and statistical convergence. *Proc. Amer. Math. Soc. 115*, 2 (1992), 319–327.

[3] Connor, J. S. The statistical and strong p-Cesàro convergence of sequences. *Analysis 8*, 1-2 (1988), 47–63.

[4] Das, P., and Savas, E. On $\mathcal{I}$-statistically pre-Cauchy sequences. *Taiwanese J. Math. 18*, 1 (2014), 115–126.

[5] Das, P., Savas, E., and Ghosal, S. K. On generalizations of certain summability methods using ideals. *Appl. Math. Lett. 24*, 9 (2011), 1509–1514.

[6] Debnath, S., and Rakshit, D. On $\mathcal{I}$-statistical convergence. *Iran. J. Math. Sci. Inform. 13*, 2 (2018), 101–109.

[7] Demirci, K. $\mathcal{I}$-limit superior and limit inferior. *Math. Commun. 6*, 2 (2001), 165–172.

[8] Dems, K. On $\mathcal{I}$-Cauchy sequences. *Real Anal. Exchange 30*, 1 (2004/05), 123–128.

[9] Esi, A., and Savas, E. On lacunary statistically convergent triple sequences in probabilistic normed space. *Appl. Math. Inf. Sci. 9*, 5 (2015), 2529–2534.

[10] Fast, H. Sur la convergence statistique. *Colloq. Math. 2* (1951), 241–244 (1952).

[11] Fridy, J. A. On statistical convergence. *Analysis 5*, 4 (1985), 301–313.

[12] Fridy, J. A. Statistical limit points. *Proc. Amer. Math. Soc. 118*, 4 (1993), 1187–1192.

[13] Karakus, S. Statistical convergence on probabilistic normed spaces. *Math. Commun. 12*, 1 (2007), 11–23.

[14] Kostyrko, P., Máčaj, M., Šalát, T., and Sleziak, M. $\mathcal{I}$-convergence and extremal $\mathcal{I}$-limit points. *Mathematica slovaca 55*, 4 (2005), 443–464.

[15] Kostyrko, P., Šalát, T., and Wilczyński, W. a. a. $\mathcal{I}$-convergence. *Real Anal. Exchange 26*, 2 (2000/01), 669–685.

[16] Lafuerza Guillen, B., and Harikrishnan, P. *Probabilistic normed spaces.* Imperial College Press, London, 2014.

[17] Malik, P., Ghosh, A., and Das, S. $\mathcal{I}$-statistical limit points and $\mathcal{I}$-statistical cluster points. *Proyecciones 38*, 5 (2019), 1011–1026.

[18] Mursaleen, M., Debnath, S., and Rakshit, D. $\mathcal{I}$-statistical limit superior and $\mathcal{I}$-statistical limit inferior. *Filomat 31*, 7 (2017), 2103–2108.

[19] Mursaleen, M., Debnath, S., and Rakshit, D. $\mathcal{I}$-statistical limit superior and $\mathcal{I}$-statistical limit inferior. *Filomat 31*, 7 (2017), 2103–2108.

[20] Niven, I., and Zuckerman, H. S. *An introduction to the theory of numbers*, fourth ed. John Wiley & Sons, New York-Chichester-Brisbane, 1980.

[21] Pehlivan, S., Güncan, A., and Mamedov, M. A. Statistical cluster points of sequences in finite dimensional spaces. *Czechoslovak Math. J. 54(129)*, 1 (2004), 95–102.

[22] Savaş, E., and Gürdal, M. $\mathcal{I}$-statistical convergence in probabilistic normed spaces. *Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. 77*, 4 (2015), 195–204.

[23] Savas, E., and Das, P. A generalized statistical convergence via ideals. *Appl. Math. Lett. 24*, 6 (2011), 826–830.

[24] Schoenberg, I. J. The integrability of certain functions and related summability methods. *Amer. Math. Monthly 66* (1959), 361–375.

[25] Steinhaus, H. Sur la convergence ordinaire et la convergence asymptotique. *Colloq. Math 2*, 1 (1951), 73–74.

[26] Šalát, T. On statistically convergent sequences of real numbers. *Math. Slovaca 30*, 2 (1980), 139–150.

[27] Šerstnev, A. N. Random normed spaces. Questions of completeness. *Kazan. Gos. Univ. Učen. Zap. 122*, kn. 4 (1962), 3–20.

Received by the editors Septebner 28, 2019
First published online August 31, 2020