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Some characterizations of regularity and intra-regularity of Γ-semigroups by means of quasi-ideals[1]

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Abstract. The concept of regularity in Γ-semigroups is not very easy to deal with even though it shares some analogy with its analogue in semigroup theory. In this paper we establish a mechanism which translates the regularity in a Γ-semigroup (S, Γ) as the usual von Neumann regularity in an ordinary semigroup Ω_{γ_0} that we construct in terms of (S, Γ). This enables us to characterize the regularity in Γ-semigroups by means of quasi-ideals. A similar characterization is proved for those Γ-semigroups which are regular and intra-regular.

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1. Introduction and preliminaries

The aim of this paper is to give an alternative way to that in [5] for projecting a Γ-semigroup onto a certain semigroup which inherits several properties of the Γ-semigroup. Pasku in [5] associated to any Γ-semigroup an ordinary semigroup Σ_{γ_0} where $\gamma_0 \in \Gamma$ is a fixed element, and showed that Green's theorem for ordinary semigroups implies an analogue for Γ-semigroups. Also he showed that if for this particular γ_0, the local semigroup $S_{\gamma_0} = (S, \circ)$ with multiplication $\circ$ defined by $a \circ b = a\gamma_0 b$, is completely simple, then so is every S_γ. This result generalizes a result of Sen and Saha in [7] (see also [6]). It is important to emphasize that Σ_{γ_0} is used in [5] as a pathway which connects the two theories, Γ-semigroups with ordinary semigroups, and it is this connection that enables one to produce results for Γ-semigroups that are analogues of results in semigroup theory with minimal costs. But Pasku's Σ_{γ_0} doesn't seem to be very helpful when it comes to regularity or intra-regularity of Γ-semigroups because such concepts differ significantly from their counterparts for ordinary semigroups. For this reason we had to consider a different version of Σ_{γ_0}, which we call here Ω_{γ_0}, and is a quotient of a free product of a group whose underlying set is Γ with the free semigroup on S. This new semigroup enables us to relate

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the regularity of a Γ-semigroup S to the set $\mathscr{Q}(S)$ of all quasi-ideals of (S,Γ) which turns out to be a Γ-semigroup and that encodes in full the regularity of (S,Γ). An attempt has been made in [1] to make such a connection, but the author does not consider there the set $\mathscr{Q}(S)$ as a Γ-semigroup, and therefore misses the importance of $\mathscr{Q}(S)$ and the analogy that exists with the theory of ordinary semigroups. We also consider intra-regularity and in particular those Γ-semigroups which are regular and intra-regular at the same time. Again, we prove that such Γ-semigroups can be characterized as those Γ-semigroups whose quasi-ideals are idempotent. We obtain this characterization as an implication of its well known analogue for ordinary semigroups. Other results on intra-regular Γ-semigroups can be found in [2].

Now we give some elementary notions from the theory that will be needed in the rest of the paper. If S and Γ are two non empty sets, then every map $\cdot : S \times \Gamma \times S \to S$ will be called a Γ-multiplication in S. The result of this multiplication for $a, b \in S$ and $\gamma \in \Gamma$ is denoted by $a\gamma b$. According to Sen and Saha [7], a Γ-semigroup S is an ordered pair (S,Γ) equipped with a Γ-multiplication $\cdot$ on S which satisfies the following property

$$\forall (a, b, c, \alpha, \beta) \in S^3 \times \Gamma^2, (a\alpha b)\beta c = a\alpha(b\beta c).$$

Let S be a Γ-semigroup and A, B subset of S. We define the set

$$A\Gamma B = \{a\gamma b | a \in A, b \in B \text{ and } \gamma \in \Gamma\}.$$

For simplicity we write $a\Gamma B$ instead of $\{a\}\Gamma B$ and similarly we write $A\Gamma b$, and write $A\gamma B$ in place of $A\{\gamma\}B$.

By analogy with the definition of quasi-ideals in plain semigroups [8] we give the following.

Definition 1.1. A quasi-ideal of a Γ-semigroup S is a non empty subset Q of S such that $Q\Gamma S \cap S\Gamma Q \subseteq Q$.

It is easy to see that the principal quasi-ideal $(a)_q$ generated by a in a Γ-semigroup S exists and is given by

$$(a)_q = a \cup (a\Gamma S \cap S\Gamma a).$$

Given a Γ-semigroup S it is obvious that for any fixed $\gamma \in \Gamma$ one can associate to S a semigroup $(S_\gamma, \circ)$ where $S_\gamma = S$ and $\circ$ is defined by setting $x \circ y = x\gamma y$ for every $x, y \in S$.

2. The adjoint semigroup Ω_{γ_0}

To define Ω_{γ_0} we will use the fact that we can always define a multiplication $\bullet$ on any non empty set Γ in such a way that $(\Gamma, \bullet)$ becomes a group. This, in fact is, equivalent to the axiom of choice (see [3]). Also we use the concept of the free product of two semigroups. Material related to this concept can be found in [4] pp 258-261. Further, let $(F, \cdot)$ be the free semigroup on S. Its elements are finite strings $(x_1, ..., x_n)$ where each $x_i \in S$ and the product $\cdot$ is the concatenation of words. Now we define Ω_{γ_0} as the quotient semigroup of the free product $F * \Gamma$ of $(F, \cdot)$ with $(\Gamma, \bullet)$ by the congruence generated from the set of relations

$$((x, y), x\gamma_0 y), ((x, \gamma, y), x\gamma y)$$

for all $x, y \in S, \gamma \in \Gamma$ and with $\gamma_0 \in \Gamma$ a fixed element. We can also regard the group $(\Gamma, \bullet)$ as given by a presentation with generators the elements of Γ, and relations arising from the multiplication table of the group. So a presentation of Ω_{γ_0} has now as a generating set $S \cup \Gamma$, and relations those mentioned above together with those arising from the multiplication table of $(\Gamma, \bullet)$.

Lemma 2.1. *Every element of* Ω_{γ_0} *can be represented by an irreducible word which has the form* $(\gamma, x, \gamma'), (\gamma, x), (x, \gamma), \gamma$ *or* x *where* $x \in S$ *and* $\gamma, \gamma' \in \Gamma$.

Proof. First we have to prove that the reduction system arising from the given presentation is Noetherian and confluent, and therefore any element of Ω_{γ_0} is given by a unique irreducible word from $S \cup \Gamma$. Secondly, we have to prove that the irreducible words have one of these five forms. So if ω is a word of the form $\omega = (u, x, \gamma, y, v)$ for $\gamma \in \Gamma, x, y \in S$ and u, v possibly empty words, then ω reduces to $\omega' = (u, x\gamma y, v)$. And if $\omega = (u, x, y, v)$, then it reduces to $\omega' = (u, x\gamma_0 y, v)$. In this way we obtain a reduction system which is length reducing and therefore it is Noetherian. To prove that this system is confluent, from Newman's lemma, it is sufficient to prove that it is locally confluent. For this we need to see only the overlapping pairs.

1. $(x, y, z) \to (x\gamma_0 y, z)$ and $(x, y, z) \to (x, y\gamma_0 z)$ which both reduce to $(x\gamma_0 y\gamma_0 z)$.
2. $(x, \gamma, y, z) \to (x\gamma y, z)$ and $(x, \gamma, y, z) \to (x, \gamma, y\gamma_0 z)$ which both reduce to $(x\gamma y\gamma_0 z)$.
3. $(x, y, \gamma, z) \to (x\gamma_0 y, \gamma, z)$ and $(x, y, \gamma, z) \to (x, y\gamma z)$ which both reduce to $(x\gamma_0 y\gamma z)$.
4. $(x, \gamma, y, \gamma', z) \to (x\gamma y, \gamma', z)$ and $(x, \gamma, y, \gamma', z) \to (x, \gamma, y\gamma' z)$ which both reduce to $(x\gamma y\gamma' z)$.
5. $(\gamma_1, \gamma_2, \gamma_3) \to (\gamma_1 \bullet \gamma_2, \gamma_3)$ and $(\gamma_1, \gamma_2, \gamma_3) \to (\gamma_1, \gamma_2 \bullet \gamma_3)$ which both reduce to $\gamma_1 \bullet \gamma_2 \bullet \gamma_3$.

To complete the proof we need to show that the irreducible word representing the element of Ω_{γ_0} has one of the five forms stated. If the word which has neither a prefix nor a suffix made entirely of letters from Γ, then it reduces to an element of S by performing the appropriate reductions. If the word has the form $(\alpha, \omega, \alpha')$, (α, ω), or (ω, α'), where ω is a word which has neither a prefix nor a suffix made entirely of letters from Γ, and α, α' have only letters from Γ, then it reduces to an element of one of the first three forms. □

Definition 2.2. An element a of a Γ-semigroup (S, Γ) is called regular if there are $\gamma_1, \gamma_2 \in \Gamma$ and $x \in S$ such that $a\gamma_1 x\gamma_2 a = a$. The element x is called the inverse of a with respect to γ_1 and γ_2. If every element of (S, Γ) is regular, then (S, Γ) is called a regular Γ-semigroup.

Proposition 2.3. *If* S *is a regular* Γ*-semigroup then* Ω_{γ_0} *is a von Neumann regular semigroup and conversely.*

Proof. Since S is a regular Γ-semigroup it means that for every $a \in S, \exists x \in S, \gamma_1, \gamma_2 \in \Gamma$ such that $a = a\gamma_1 x \gamma_2 a$. An immediate implication of this is that a has an inverse in Ω_{γ_0} which is $(\gamma_1 x \gamma_2)$. We show that the same happens with all the remaining types of elements of Ω_{γ_0}. Let $\alpha_1 a \alpha_2$ be another element of Ω_{γ_0}. As its inverse we take $\alpha_2^{-1}\gamma_1 x \gamma_2 \alpha_1^{-1} \in \Omega_{\gamma_0}$, because

$$(\alpha_1 a \alpha_2)(\alpha_2^{-1}\gamma_1 x \gamma_2 \alpha_1^{-1})(\alpha_1 a \alpha_2) = \alpha_1 a \gamma_1 x \gamma_2 a \alpha_2 = \alpha_1 a \alpha_2.$$

Also $\alpha a \in \Omega_{\gamma_0}$ is regular and as its inverse we take $\gamma_1 x \gamma_2 \alpha^{-1} \in \Omega_{\gamma_0}$, because

$$(\alpha a)(\gamma_1 x \gamma_2 \alpha^{-1})(\alpha a) = \alpha a \gamma_1 x \gamma_2 a = \alpha a.$$

The same holds true for $a\alpha \in \Omega_{\gamma_0}$ which is regular with inverse $\alpha^{-1}\gamma_1 x \gamma_2 \in \Omega_{\gamma_0}$, because $(a\alpha)(\alpha^{-1}\gamma_1 x \gamma_2)(a\alpha) = a\gamma_1 x \gamma_2 a \alpha = a\alpha$. And finally every $\alpha \in \Gamma$ has inverse α^{-1}, its inverse in $(\Gamma, \bullet)$.

For the converse, if Ω_{γ_0} is regular, then every $a \in S$ has an inverse in Ω_{γ_0}. We will show that every $a \in S$ has an inverse in (S, Γ). For this we distinguish between the following five cases. First, if the inverse of a in Ω_{γ_0} is of the form $\alpha x \beta$ where $x \in S$, then $a\alpha x \beta a = a$ which means that a is regular in (S, Γ). Second, if αx is the inverse of a in Ω_{γ_0}, then $a(\alpha x)a = a$, which can be written as $a\alpha x \gamma_0 a = a$ proving the regularity of a in (S, Γ). Third, the inverse of a in Ω_{γ_0} is some $x\alpha$. This case is dealt with similarly to the second case. Fourth, the inverse of a in Ω_{γ_0} is some $x \in S$. Then, $axa = a$, or equivalently, $a\gamma_0 x \gamma_0 a = a$, which again implies that a is regular in (S, Γ). Finally, the inverse a in Ω_{γ_0} is some $\alpha \in \Gamma$. In this case, $a\alpha a = a$, then $a\alpha a \alpha a = a$ and a is regular in (S, Γ). □

Remark 2.4. If there is some $\gamma_0 \in \Gamma$ such that $(S_{\gamma_0}, \circ)$ is von Neumann regular, then (S, Γ) is regular in the sense of Definition 2.2. Indeed, if $a \in S$, then $a \in S_{\gamma_0}$, which is von Neumann regular, so there is $x \in S_{\gamma_0} = S$ such that $a\gamma_0 x \gamma_0 a = a$, hence a is regular. We also emphasize here that Ω_{γ_0} defined for this particular γ_0 is von Neumann regular.

Lemma 2.5. *If Q is a quasi-ideal of a Γ-semigroup S, then Q is a quasi-ideal of Ω_{γ_0}.*

Proof. Let p be an element from the intersection $Q\Omega_{\gamma_0} \cap \Omega_{\gamma_0} Q$. The following cases are possible. First, $p = qx = yq'$ where $q, q' \in Q$ and $x, y \in S$. Thus, $p = q\gamma_0 x = y\gamma_0 q' \in Q\Gamma S \cap S\Gamma Q \subseteq Q$. Second, $p = q(\alpha x) = q'y$ where $x, y \in S$ and $\alpha \in \Gamma$. Again, $p = q\alpha x = q'\gamma_0 y \in Q\Gamma S \cap S\Gamma Q \subseteq Q$, The two remaining cases are $p = qx = (y\beta)q'$, where $x, y \in S$ and $\beta \in \Gamma$, and $p = q(\alpha x) = (y\beta)q'$, where $x, y \in S$ and $\alpha, \beta \in \Gamma$. The corresponding proofs are similar to the previous proofs. □

A partial converse of the above holds true.

Lemma 2.6. *If Q is a quasi-ideal of Ω_{γ_0} which consists only of elements of S, then Q is a quasi-ideal of the Γ-semigroup S.*

Proof. Let $p = q\alpha x = y\beta q' \in Q\Gamma S \cap S\Gamma Q$ with $x, y \in S$ and $\alpha, \beta \in \Gamma$, then $p = q(\alpha x) = (y\beta)q' \in Q\Omega_{\gamma_0} \cap \Omega_{\gamma_0} Q \subseteq Q$. Thus Q is a quasi-ideal of (S, Γ). □

Lemma 2.7. *Let* Q *be a quasi-ideal of* (S, Γ) *and* $\alpha \in \Gamma$*, then* αQ *is a quasi-ideal of* Ω_{γ_0}*.*

Proof. Let $p = (\alpha q)w = w'(\alpha q') \in (\alpha Q)\Omega_{\gamma_0} \cap \Omega_{\gamma_0}(\alpha Q)$, then necessarily w equals to some $x \in S$ or has the form βx where $\beta \in \Gamma$ and $x \in S$, and w' has the form $w = \alpha y$ or $w = \alpha y \gamma$ where $y \in S$ and $\gamma \in \Gamma$. We give below the proof when $w = \beta x$ and $w' = \alpha y \gamma$. The other cases are dealt with similarly. In this case, we have

$$p = \alpha q \beta x = \alpha y \gamma \alpha q' \in \alpha Q \Gamma S \cap \alpha S \Gamma Q = \alpha(Q\Gamma S \cap S\Gamma Q) \subseteq \alpha Q,$$

which shows that αQ is a quasi-ideal of Ω_{γ_0}. □

An analogue of Proposition 2.7 of [5] holds true. It relates the quasi-ideal $(a)_q^{\Omega_{\gamma_0}}$ in Ω_{γ_0}, generated by some $a \in S$, with the quasi-ideal $(a)_q^{\Gamma}$ in S generated by a. We leave the proof to the reader.

Proposition 2.8. *For every* $a \in S$*,* $(a)_q^{\Omega_{\gamma_0}} = (a)_q^{\Gamma}$*.*

Lemma 2.9. *Let* $\alpha, \beta \in \Gamma$ *and* $a \in S$*. Then* $(\alpha a \beta)_q^{\Omega_{\gamma_0}} = \alpha(a)_q^{\Omega_{\gamma_0}}\beta$*,* $(\alpha a)_q^{\Omega_{\gamma_0}} = \alpha(a)_q^{\Omega_{\gamma_0}}$ *and* $(a\beta)_q^{\Omega_{\gamma_0}} = (a)_q^{\Omega_{\gamma_0}}\beta$*.*

Proof. We will make the proof for $\alpha a \beta$ only. The other proofs are similar. In the following we use the fact that in Ω_{γ_0}, for all $\alpha, \beta \in \Gamma$, we have that $\beta\Gamma = \Gamma = \Gamma\alpha$.

$$\begin{aligned}
(\alpha a \beta)_q^{\Omega_{\gamma_0}} &= \alpha a \beta \cup ((\alpha a \beta)\Omega_{\gamma_0} \cap \Omega_{\gamma_0}(\alpha a \beta)) \\
&= \alpha a \beta \cup ((\alpha a \Gamma \cup \alpha a \Gamma S \cup \alpha a \Gamma S \Gamma) \cap (\Gamma a \beta \cup S \Gamma a \beta \cup \Gamma S \Gamma a \beta)) \\
&= \alpha a \beta \cup ((\alpha a \Gamma \cup \alpha a \Gamma S \Gamma) \cap (\Gamma a \beta \cup \Gamma S \Gamma a \beta)) \\
&= \alpha a \beta \cup (\alpha a \Gamma S \beta \cap \alpha S \Gamma a \beta) = \alpha(a \cup (a\Gamma S \cap S \Gamma a))\beta \\
&= \alpha(a)_q^{\Gamma}\beta = \alpha(a)_q^{\Omega_{\gamma_0}}\beta,
\end{aligned}$$

hence, $(\alpha a \beta)_q^{\Omega_{\gamma_0}} = \alpha(a)_q^{\Omega_{\gamma_0}}\beta$. □

Theorem 2.10. *A* Γ*-semigroup* (S, Γ) *is regular if and only if the set* $\mathcal{Q}(S)$ *of quasi-ideals of* S *forms a* Γ*-semigroup, where the* Γ*-multiplication is given by* $Q_1\gamma Q_2 = \{q_1\gamma q_2 | q_1 \in Q_1, q_2 \in Q_2\}$*, and has the property that for every quasi-ideal* $Q \in \mathcal{Q}(S)$ *there is a family of pairs* $(\alpha_i, \beta_i) \in \Gamma \times \Gamma$ *together with a family of quasi-ideals* $Q_i \in \mathcal{Q}(S)$ *such that* $Q = \cup_{i \in I} Q\alpha_i Q_i \beta_i Q$*.*

Proof. We first define a Γ-semigroup structure on the set $\mathcal{Q}(S)$ of all quasi-ideals of (S, Γ). Let $Q_1, Q_2 \in \mathcal{Q}(S)$ and let $\alpha \in \Gamma$. We define

$$Q_1 \alpha Q_2 = \{q_1 \alpha q_2 | q_1 \in Q_1, q_2 \in Q_2\}.$$

To see that $Q_1\alpha Q_2 \in \mathscr{Q}(S)$ we recall from Lemma 2.7 that $\alpha Q_2 \in \mathscr{Q}(\Omega_{\gamma_0})$, where $\mathscr{Q}(\Omega_{\gamma_0})$ is the set of quasi-ideals of Ω_{γ_0}. But (S,Γ) is regular and so is Ω_{γ_0} (Proposition 2.3), hence Theorem 9.3 of [8] tells that the product $Q_1(\alpha Q_2) \in \mathscr{Q}(\Omega_{\gamma_0})$. But any quasi-ideal of Ω_{γ_0} with elements entirely lying in S is a quasi-ideal of (S,Γ) (lLemma 2.6) hence $Q_1\alpha Q_2 \in \mathscr{Q}(S)$. Now the fact that (S,Γ) is a Γ-semigroup implies easily that $Q_1\alpha(Q_2\beta Q_3) = (Q_1\alpha Q_2)\beta Q_3$ for any $\alpha,\beta \in \Gamma$ and $Q_1, Q_2, Q_3 \in \mathscr{Q}(S)$, thus proving that $(\mathscr{Q}(S),\Gamma)$ is a Γ-semigroup. Now let $Q \in \mathscr{Q}(S)$. Then $Q \in \mathscr{Q}(\Omega_{\gamma_0})$ and since Ω_{γ_0} is von Neumann regular, then from Theorem 9.3 of [8] there is $Q' \in \mathscr{Q}(\Omega_{\gamma_0})$ such that $Q = QQ'Q$. We can express Q' as a union of principal quasi-ideals $(a)_q^{\Omega_{\gamma_0}}$, $(\alpha a)_q^{\Omega_{\gamma_0}}$, $(a\beta)_q^{\Omega_{\gamma_0}}$ or $(\alpha a\beta)_q^{\Omega_{\gamma_0}}$ for every $a \in S$, $\alpha a \in \Gamma S$, $a\beta \in S\Gamma$ or $\alpha a\beta \in \Gamma S\Gamma$ that may be an element of Q'. It follows from Lemma 2.9 that Q' is a union of quasi-ideals $(a)_q^{\Omega_{\gamma_0}}$, $\alpha(a)_q^{\Omega_{\gamma_0}}$, $(a)_q^{\Omega_{\gamma_0}}\beta$ or $\alpha(a)_q^{\Omega_{\gamma_0}}\beta$, and then Q is the union of quasi-ideals $Q(a)_q^{\Omega_{\gamma_0}}Q = Q\gamma_0(a)_q^{\Omega_{\gamma_0}}\gamma_0 Q$, $Q\alpha(a)_q^{\Omega_{\gamma_0}}Q = Q\alpha(a)_q^{\Omega_{\gamma_0}}\gamma_0 Q$, $Q(a)_q^{\Omega_{\gamma_0}}\beta Q = Q\gamma_0(a)_q^{\Omega_{\gamma_0}}\beta Q$ or $Q\alpha(a)_q^{\Omega_{\gamma_0}}\beta Q$. Now the result follows.

For the converse, let $a \in S$ and let $(a)_q^\Gamma$ be the quasi-ideal of (S,Γ) generated by a which has the form

$$(a)_q^\Gamma = a \cup (a\Gamma S \cap S\Gamma a).$$

Since $\mathscr{Q}(S)$ has the stated property, then $(a)_q^\Gamma$ is expressed as

$$(a)_q^\Gamma = \cup_{i\in I}(a)_q^\Gamma \alpha_i Q_i \beta_i (a)_q^\Gamma,$$

which implies in particular that there is $i \in I$ such that $a \in (a)_q^\Gamma \alpha_i Q_i \beta_i (a)_q^\Gamma$. It follows that there are $y, z \in (a)_q^\Gamma$ and $q \in Q_i$ such that $a = y\alpha_i q\beta_i z$. But each of the elements y, z can be either a or it is of the form $a\gamma_1 s = t\gamma_2 a$ if it is in the intersection $a\Gamma S \cap S\Gamma a$, where $\gamma_1, \gamma_2 \in \Gamma$ and $s, t \in S$. In either case it follows that there are $\delta_1, \delta_2 \in \Gamma$ and $u \in S$ such that $a = a\delta_1 u\delta_2 a$ which shows that (S,Γ) is regular. $\square$

Definition 2.11. A non empty subset B of a Γ-semigroup S is called a bi-ideal of S if $B\Gamma B \subseteq B$ and $B\Gamma S\Gamma B \subseteq B$.

One can easily prove that quasi-ideals are bi-ideals. In what follows we prove that for regular Γ-semigroups bi-ideals are quasi-ideals. This is true for ordinary semigroups where regularity is the usual von Neumann regularity. We derive the above result as a consequence of Corollary 9.6 of [8] for ordinary semigroups by utilizing Ω_{γ_0}.

Lemma 2.12. *If B is a bi-ideal of a Γ-semigroup S, then B is a bi-ideal of Ω_{γ_0}.*

Proof. For every $b_1, b_2 \in B$, we see that $b_1 b_2 = b_1\gamma_0 b_2 \in B\Gamma B \subseteq B$. Also, for every $b_1, b_2 \in B$, $\alpha, \beta \in \Gamma$ and $x \in S$, we have

$$\begin{aligned}
&b_1 \cdot \alpha x \cdot b_2 = b_1 \alpha x \gamma_0 b_2 \in B\Gamma S\Gamma B \subseteq B,\\
&b_1 \cdot x\beta \cdot b_2 = b_1 \gamma_0 x \beta b_2 \in B\Gamma S\Gamma B \subseteq B,\\
&b_1 \cdot \alpha x\beta \cdot b_2 = b_1 \alpha x \beta b_2 \in B\Gamma S\Gamma B \subseteq B,\\
&b_1 \cdot x \cdot b_2 = b_1 \gamma_0 x \gamma_0 b_2 \in B\Gamma S\Gamma B \subseteq B,\\
&b_1 \alpha b_2 \in B\Gamma B \subseteq B,
\end{aligned}$$

which prove that B is a bi-ideal of $(\Omega_{\gamma_0}, \cdot)$. □

Also, we note in passing here that a partial converse of the above also holds true. More precisely, if B is a bi-ideal of a Ω_{γ_0} consisting only of elements of S, then B is a bi-ideal of (S, Γ). Indeed, since $B\Omega_{\gamma_0}B \subseteq B$, then for every $b_1, b_2 \in B$ and every $\alpha \in \Gamma$, $b_1\alpha b_2 \in B\Omega_{\gamma_0}B \subseteq B$, which shows that $B\Gamma B \subseteq B$. To prove that $B\Gamma S\Gamma B \subseteq B$ we need to show that for every $b_1, b_2 \in B$, $\alpha, \beta \in \Gamma$ and $x \in S$, $b_1\alpha x\beta b_2 \in B$. Indeed,

$$b_1\alpha x\beta b_2 = b_1 \cdot (\alpha x\beta) \cdot b_2 \in B\Omega_{\gamma_0}B \subseteq B,$$

which proves the claim.

Proposition 2.13. *If S is a regular Γ-semigroup, then every bi-ideal of S is also a quasi-ideal.*

Proof. Let S be a regular Γ-semigroup and let B be a bi-ideal of (S, Γ) which is also a bi-ideal of Ω_{γ_0} (Lemma 2.12). From Proposition 2.3 we have that Ω_{γ_0} is von Neumann regular, hence from Corollary 9.6 of [8], B is a quasi-ideal of Ω_{γ_0}. Now Lemma 2.6 implies that B is a quasi-ideal of (S, Γ). □

Definition 2.14. We say that a Γ-semigroup (S, Γ) is intra-regular if for each $a \in S$, there are $x, y \in S$ and $\gamma, \gamma_1, \gamma_2 \in \Gamma$ such that $a = x\gamma_1 a\gamma a\gamma_2 y$.

Lemma 2.15. *If (S, Γ) is intra-regular, then for every γ_0 the semigroup Ω_{γ_0} is an intra-regular semigroup.*

Proof. The intra-regularity of the elements of S follows from the definition. Let us now check the remaining cases. If αa is an element of Ω_{γ_0}, where $\alpha \in \Gamma$, $a \in S$ and $a = x\gamma_1 a\gamma a\gamma_2 y$, then $\alpha a = \alpha x\gamma_1\alpha^{-1}\alpha a\gamma\alpha^{-1}\alpha a\gamma_2 y$. A similar proof is available when the element is of the form $a\beta$ with $a \in S$ and $\beta \in \Gamma$. For the case when the element is $\alpha a\beta$ with $\alpha, \beta \in \Gamma$ and $a \in S$, assuming that $a = x\gamma_1 a\gamma a\gamma_2 y$, then $\alpha a\beta = \alpha x\gamma_1\alpha^{-1}\alpha a\beta\beta^{-1}\gamma\alpha^{-1}\alpha a\beta\beta^{-1}\gamma_2 y\beta$. The last case when the element is some $\gamma \in \Gamma$ follows from the fact that $(\Gamma, \bullet)$ is a group. □

Theorem 2.16. *A Γ-semigroup (S, Γ) is regular and intra-regular if and only if for every $Q \in \mathscr{Q}(S)$, and every $\gamma \in \Gamma$, $Q\gamma Q = Q$.*

Proof. Assume that (S, Γ) is regular and intra-regular, and let $\gamma_0 \in \Gamma$ be a fixed element. The resulting semigroup Ω_{γ_0} is von Neumann regular and intra-regular as well, therefore from Corollary 9.10 of [8], every quasi-ideal there is idempotent. If now $Q \in \mathscr{Q}(S)$, then from Lemma 2.5 we can regard Q as an element of $\mathscr{Q}(\Omega_{\gamma_0})$, hence in Ω_{γ_0} we have $QQ = Q$, or in other words $Q\gamma_0 Q = Q$. Since γ_0 was chosen arbitrarily, then the claim follows.

Conversely, assume that for every $\gamma \in \Gamma$ and every $Q \in \mathscr{Q}(S)$ we have $Q\gamma Q = Q$. In particular, we have that $a \in (a)_q^\Gamma \gamma (a)_q^\Gamma$, which can be written as

$$a \in (a \cup (a\Gamma S \cap S\Gamma a))\gamma(a \cup (a\Gamma S \cap S\Gamma a)).$$

Distinguish between the following cases. First, $a = a\gamma a$, in which case we have that a is regular and intra-regular. Second, $a = a\gamma(a\alpha x)$, where $a\alpha x = y\beta a \in a\Gamma S \cap S\Gamma a$. The regularity of a is obvious if we replace in the given equality $a\alpha x$ by $y\beta a$. To prove intra-regularity, we replace the middle a by $a\gamma(a\alpha x)$, and obtain $a = a\gamma(a\gamma a)\alpha x\alpha x$ which proves intra-regularity. Third, $a = (a\alpha x)\gamma a$, where $a\alpha x = y\beta a \in a\Gamma S \cap S\Gamma a$. The proof in this case is dual to that of the second case. The last case is when $a = (a\alpha x)\gamma(a\mu x')$, where $a\alpha x = y\beta a \in a\Gamma S \cap S\Gamma a$, and $a\mu x' = y'\nu a \in a\Gamma S \cap S\Gamma a$. Replacing $a\mu x'$ by $y'\nu a$ in the given equality we get the regularity, and replacing $a\alpha x$ by $y\beta a$ we get the intra-regularity. $\square$

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