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On generalized weakly (Ricci) ϕ-symmetric Lorentzian Para Sasakian manifold

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Abstract. The present paper attempts to introduce the notion of generalized weakly ϕ-symmetric and generalized weakly Ricci ϕ-symmetric Lorentzian Para Sasakian manifold. Furthermore, we study generalized weakly ϕ-symmetric Lorentzian Para-Sasakian spacetimes. In addition, the existence of a generalized weakly ϕ-symmetric Lorentzian Para Sasakian manifold is ensured by a suitable example.

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1. Introduction

Throughout the paper, we shall denote the Levi-Civita connection, Riemannian curvature tensor, Ricci tensor and Ricci operator by the symbols ∇, R (or $\bar{R}$), and Q, respectively. As a mild version of local symmetry[5], Takahashi [28] started the studies on locally ϕ-symmetric manifold. Further research has been conducted to weaken such notion by many authors. For details, we refer the reader to ([10], [11], [16], [12], [7], [23], [24], [21], [27], [8], [17], [18], [19] and the references therein).

Recently, in [2], the author has introduced the concept of a generalized weakly symmetric manifold which is defined as follows:

A Riemannian (or semi-Riemannian) manifold of dimension n is said to be generalized weakly symmetric if it admits the equation

$$\begin{aligned}(\nabla_X\bar{R})(Y,U,V,Z) &= A(X)\bar{R}(Y,U,V,Z)+B(Y)\bar{R}(X,U,V,Z)\\ &+B(U)\bar{R}(Y,X,V,Z)+D(V)\bar{R}(Y,U,X,Z)+D(Z)\bar{R}(Y,U,V,X)\\ &+\alpha(X)\bar{G}(Y,U,V,Z)+\beta(Y)\bar{G}(X,U,V,Z)+\beta(U)\ \bar{G}(Y,X,V,Z)\\ &+\gamma(V)\ \bar{G}(Y,U,X,Z)+\ \gamma(Z)\bar{G}(Y,U,V,X),\end{aligned} \tag{1.1}$$

where

$$\bar{G}(Y,U,V,\ W)=g(U,V)g(Y,W)-g(Y,\ V)g(U,W) \tag{1.2}$$

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for any vector fields X, Y and U and the 1-forms $A(X) = g(X, \pi_1)$, $B(X) = g(X, \pi_2)$, $D(X) = g(X, \varrho)$, $\alpha(X) = g(X, \delta_1)$, $\beta(X) = g(X, \delta_2)$ and $\gamma(X) = g(X, \sigma)$. The relation (1.1) can also be written as

$$\begin{aligned} & (\nabla_X R)(Y,U)V \\ = \; & A(X)R(Y,U)V + B(Y)R(X,U)V + B(U)R(Y,X)V + D(V)R(Y,U)X \\ & +g(R(Y,U)V,\ X)\varrho + \alpha(X)G(Y,U)V + \beta(Y)G(X,U)V \\ & +\beta(U)\ G(Y,X)V + \gamma(V)\ G(Y,U)X +\ g(G(Y,U)V, X\ )\sigma. \end{aligned} \tag{1.3}$$

Analogously, a semi-Riemannian (or Riemannian) manifold (M^n, g) is said to be generalized weakly Ricci-symmetric, if it satisfies the condition

$$\begin{aligned} (\nabla_X S)(U,V) \;=\; & A^*(X)S(U,V) + B^*(U)S(V,X) + D^*(V)S(U,X) \\ & +\alpha^*(X)g(U,V) + \beta^*(U)g(V,X) + \gamma^*(V)g(U,X) \end{aligned} \tag{1.4}$$

which can also be expressed as

$$\begin{aligned} (\nabla_X Q)(U) \;=\; & A^*(X)QU + B^*(U)QX + S(U,X)\varrho^* \\ & +\alpha^*(X)U + \beta^*(U)X + g(U,X)\sigma^* \end{aligned} \tag{1.5}$$

for any vector fields X, U and V and the 1-forms $A^*(X) = g(X, \pi_1^*)$, $B^*(X) = g(X, \pi_2^*)$, $D^*(X) = g(X, \varrho^*)$, $\alpha^*(X) = g(X, \delta_1^*)$, $\beta^*(X) = g(X, \delta_2^*)$ and $\gamma(X) = g(X, \sigma^*)$.

Recently, Hui [10] studied ϕ-pseudo symmetric and ϕ-pseudo Ricci symmetric Kenmotsu manifolds. In tune with [10], in this paper we would like to introduce the notion of a generalized weakly ϕ-symmetric manifold and a generalized weakly Ricci ϕ-symmetric manifold.

A Lorentzian Para-Sasakian manifold is said to be generalized weakly ϕ-symmetric if R admits the equation

$$\begin{aligned} & \phi^2\left((\nabla_X R)(Y,U)V\right) \\ = \; & A(X)R(Y,U)V + B(Y)R(X,U)V + B(U)\bar{R}(Y,X)V + D(V)\bar{R}(Y,U)X \\ & +g(R(Y,U)V,\ X)\varrho + \alpha(X)G(Y,U)V + \beta(Y)G(X,U)V \\ & +\beta(U)\ G(Y,X)V + \gamma(V)\ G(Y,U)X +\ g(G(Y,U)V, X\ )\sigma. \end{aligned} \tag{1.6}$$

The beauty of such generalized weakly ϕ-symmetric manifold is that it has the flavour of a

(i) locally ϕ-symmetric space [28] (for $A = B = D = 0 = \alpha = \beta = \gamma,$),

(ii) locally ϕ-recurrent space [8] (for $A \neq 0$, $B = D = \alpha = \beta = \gamma = 0$),

(iii) generalized ϕ-recurrent space in the sense of [9] (for $A \neq 0$, $\alpha \neq 0, B = D = \beta = \gamma = 0$),

(iv) quasi ϕ-recurrent space in the sense of [20] ($A \neq 0$, $B{=}D = 0$, $\alpha \neq 0, \beta = \gamma = (\beta^* - \gamma^*)\alpha$),

(v) pseudo ϕ-symmetric space in the sense of [10] (for $\frac{A}{2} = B = D = H \neq 0$, $\alpha = \beta = \gamma = 0$),

(vi) generalized pseudo ϕ-symmetric space in the sense of [1] (for $\frac{A}{2} = B = D = H_1 \neq 0, \frac{\alpha}{2} = \beta = \gamma = H_2 \neq 0$),

(vii) semi-pseudo ϕ-symmetric space in the sense of [30] ($A = \alpha = \beta = \gamma = 0, B = D \neq 0$),

(viii) generalized semi-pseudo ϕ-symmetric space in the sense of [3] ($A = 0 = \alpha$, $B = D \neq 0$, $\beta = \gamma \neq 0$),

(ix) almost pseudo ϕ-symmetric space in the sense of [6] (for $A = H_1 + K_1, B = D = H_1 \neq 0$ and $\alpha = \beta = \gamma = 0$),

(x) almost generalized pseudo ϕ-symmetric space in the sense of [3] ($A = H_1 + K_1, B = D = H_1 \neq 0$, $\alpha = H_2 + K_2$, $\beta = \gamma = H_2 \neq 0$),

(xi) weakly ϕ-symmetric space in the sense of [29] (for A, B, $D \neq 0$, $\alpha = \beta = \gamma = 0$).

A Lorentzian Para-Sasakian manifold is said to be generalized weakly Ricci ϕ-symmetric if Q admits the following

$$\begin{aligned} &\phi^2\left((\nabla_X Q)(U)\right) \\ = \; &A^*(X)QU + B^*(U)QX + S(U,X)\varrho^* \\ &+\alpha^*(X)U + \beta^*(U)X + g(U,X)\sigma^*. \end{aligned} \tag{1.7}$$

In 1989, Matsumoto [13] started the studies of Lorentzian Para-Sasakian manifolds which were also independently defined by Mihai and Rosca [15]. Matsumoto, Mihai and Rosaca [14] gave a five dimensional example of Lorentzian Para-Sasakian manifold.

We represent our paper as follows: Section 2, is concerned with some known results on Lorentzian Para Sasakian manifolds. In Section 3, we study generalized weakly ϕ-symmetric Lorentzian Para Sasakian manifolds. We observe that a generalized weakly ϕ-symmetric Lorentzian Para Sasakian manifold is η-Einstein whereas Lorentzian Para Sasakian manifold can not be η-Einstein for each of the curvature restrictions: (i) ϕ-symmetric , (ii) ϕ-recurrent, (iii) generalized ϕ-recurrent, (iv) pseudo ϕ-symmetric, (v) almost pseudo ϕ-symmetric, (vi) generalized pseudo ϕ-symmetric and (vii) generalized almost pseudo ϕ-symmetric. It is also found that there does not exist a Lorentzian Para-Sasakian manifold which is (i) ϕ-recurrent, (ii) generalized ϕ-recurrent provided the 1-forms are collinear, (iii) pseudo ϕ-symmetric, (iv) generalized semi-pseudo ϕ-symmetric provided the 1-forms are collinear, (v) generalized almost pseudo ϕ-symmetric provided the 1-forms are collinear. Section 4 deals with generalized weakly Ricci ϕ-symmetric Lorentzian Para Sasakian manifolds. Keeping in tune with Shaikh, Yoon and Hui [25], in Section 5 we study a generalized weakly ϕ-symmetric Lorentzian Para-Sasakian spacetime and obtain some interesting results. Finally, we have constructed an example of a generalized weakly ϕ-symmetric Lorentzian Para Sasakian manifold.

2. Properties of a Lorentzian Para Sasakian manifold

Let M be an n-dimensional differential manifold endowed with a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η, and a Lorentzian metric g of type $(0,2)$

such that for each point $a \in M$, the tensor $g_a : T_aM \times T_aM \to \mathbb{R}$ is a non-degenerate inner product of signature $(-, +, +, ..., +)$, where T_aM denotes the tangent space of M at a and $\mathbb{R}$ is the real number space which satisfies

$$\phi^2 = I + \eta \otimes \xi, \tag{2.1}$$

$$\eta(\xi) = -1, \tag{2.2}$$

$$g(X, \xi) = \eta(X), \tag{2.3}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.4}$$

for all vector fields X, Y on M^n. Then, the structure (ϕ, ξ, η, g) is called Lorentzian almost para contact structure and the manifold with the structure (ϕ, ξ, η, g) is called a Lorentzian almost para contact manifold. In a Lorentzian almost para contact manifold M, the following relations hold [13]:

$$\phi\xi = 0, \ \ \eta \circ \phi = 0, \tag{2.5}$$

$$g(\phi X, Y) = g(X, \phi Y). \tag{2.6}$$

If we put

$$\Omega(X, Y) = g(\phi X, Y) = g(X, \phi Y) \tag{2.7}$$

for any vector fields X and Y then the tensor field $\Omega(X, Y)$ is a symmetric $(0, 2)$ tensor field.

A Lorentzian almost para contact manifold M endowed with the structure (ϕ, ξ, η, g) is called a Lorentzian Para-Sasakian manifold if

$$(\nabla_X \phi)Y - g(\phi X, \phi Y)\xi = \eta(Y)\phi^2 X, \tag{2.8}$$

where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g. In a Lorentzian Para-Sasakian manifold M with the structure (ϕ, ξ, η, g), it is easily seen that ([13], [4], [22])

$$\nabla_X \xi = \phi X, \tag{2.9}$$

$$(\nabla_X \eta)Y = g(X, \phi Y) = \Omega(X, Y) = (\nabla_Y \eta)X, \tag{2.10}$$

$$S(X, \xi) = (n - 1)\eta(X), \ \ Q\xi = (n - 1)\xi, \tag{2.11}$$

$$R(\xi, X)Y + \eta(Y)X = g(X, Y)\xi, \tag{2.12}$$

$$R(Y,U)\xi + \eta(Y)U = \eta(U)Y, \tag{2.13}$$

$$\begin{aligned} R(Y,U)\phi X &= \phi R(Y,U)X + g(U,X)\phi Y - g(Y,X)\phi U + g(\phi Y,X)U \\ &\quad -g(\phi U,X)Y + 2[g(\phi Y,X)\eta(U) - g(\phi U,X)\eta(Y)]\xi \\ &\quad +2[\eta(U)\phi Y - \eta(Y)\phi U]\eta(X), \end{aligned} \tag{2.14}$$

$$\begin{aligned} R(X,Y)Z &= \phi R(X,Y)\phi Z + g(X,Z)Y - g(Y,Z)X \\ &\quad +\Omega(Y,Z)\phi X - \Omega(X,Z)\phi Y + 2[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)]\xi \\ &\quad +2[\eta(X)Y - \eta(Y)X]\eta(Z), \end{aligned} \tag{2.15}$$

$$\begin{aligned} (\nabla_X R)(Y,U)\xi &= 2[g(\phi U,X)Y - g(\phi Y,X)U] - \phi R(Y,U)X \\ &\quad +g(Y,X)\phi U - g(U,X)\phi Y - 2[g(\phi Y,X)\eta(U) \\ &\quad -g(\phi U,X)\eta(Y)]\xi - 2[\eta(U)\phi Y - \eta(Y)\phi U]\eta(X). \end{aligned} \tag{2.16}$$

for all vector fields X, Y, U and Z on M^n.

3. Generalized weakly ϕ-symmetric Lorentzian Para-Sasakian manifolds

In this section we consider a generalized weakly ϕ-symmetric Lorentzian Para-Sasakian manifold. Then by the virtue of (2.1), the equation (1.6) takes the following form

$$\begin{aligned} &(\nabla_X R)(Y,U)V + \eta((\nabla_X R)(Y,U)V)\xi \\ &\quad = A(X)R(Y,U)V + B(Y)R(X,U)V \\ &\qquad +B(U)R(Y,X)V + D(V)R(Y,U)X \\ &\qquad +g(R(Y,U)V,\ X)\varrho + \alpha(X)G(Y,U)V \\ &\qquad +\beta(Y)G(X,U)V + \beta(U)G(Y,X)V \\ &\qquad +\gamma(V)G(Y,U)X + g(G(Y,U)V,\ X)\sigma. \end{aligned} \tag{3.1}$$

From which we get

$$\begin{aligned} &g((\nabla_X R)(Y,U)V,W) + \eta((\nabla_X R)(Y,U)V)\eta(W) \\ &\quad = A(X)g(R(Y,U)V,W) + B(Y)g(R(X,U)V,W) \\ &\qquad +B(U)g(R(Y,X)V,W) + D(V)g(R(Y,U)X,W) \\ &\qquad +D(W)g(R(Y,U)V,\ X) + \alpha(X)g(G(Y,U)V,W) \\ &\qquad +\beta(Y)g(G(X,U)V,W) + \beta(U)g(G(Y,X)V,W) \\ &\qquad +\gamma(V)g(G(Y,U)X,W) + \gamma(W)g(G(Y,U)V,\ X). \end{aligned} \tag{3.2}$$

Now, taking an orthonormal frame field and then contracting (3.2) over Y and W, we obtain

$$\begin{aligned}(\nabla_X S)(U,V)+&(\nabla_X \bar{R})(\xi,U,V,\xi)\\ =\ & A(X)S(U,V)+B(R(X,U)V)+B(U)S(X,V)+D(V)S(U,X)\\ &+D(R(X,V)U)+(n-1)\alpha(X)g(U,V)+\beta(G(X,U)V)\\ &+(n-1)\beta(U)g(X,V)+(n-1)\gamma(V)g(U,X)+\gamma(G(X,V)U).\end{aligned} \tag{3.3}$$

Using (2.12), (2.5) and (2.9), we obtain

$$(\nabla_X R)(\xi,U,V,\xi)=0. \tag{3.4}$$

In view of (3.3) and (3.4), we have

$$\begin{aligned}(\nabla_X S)(U,V)&\\ =\ & A(X)S(U,V)+B(U)S(X,V)+D(V)S(U,X)\\ &+(n-1)\{\alpha(X)g(U,V)+\beta(U)g(X,V)+\gamma(V)g(U,X)\}\\ &+B(R(X,U)V)+D(R(X,V)U)+\beta(G(X,U)V)+\gamma(G(X,V)U).\end{aligned} \tag{3.5}$$

Theorem 3.1. *Every generalized weakly ϕ-symmetric Lorentzian Para Sasakian manifold is generalized weakly Ricci-symmetric, provided that $D(\xi)+\gamma(\xi)=0$.*

Proof. From, (3.5) it is obvious that a generalized weakly ϕ-symmetric Lorentzian Para Sasakian manifold is generalized weakly Ricci-symmetric if

$$B(R(X,U)V)+D(R(X,V)U)+\beta(G(X,U)V)+\gamma(G(X,V)U)=0. \tag{3.6}$$

From, (3.6), one can easily deduce that $D(\xi)+\gamma(\xi)=0$. □

Now, setting $V=\xi$ in (3.5) we get

$$\begin{aligned}(\nabla_X S)(U,\xi)&\\ =\ & (n-1)A(X)\eta(U)+(n-2)B(U)\eta(X)+D(\xi)S(U,X)\\ &+(n-1)\alpha(X)\eta(U)+(n-2)\beta(U)\eta(X)+(n-1)\gamma(\xi)g(U,X)\\ &+B(X)\eta(U)+D(X)\eta(U)-D(\xi)g(U,X)\\ &+\beta(X)\eta(U)+\gamma(X)\eta(U)-\gamma(\xi)g(U,X).\end{aligned} \tag{3.7}$$

Again, from the relation

$$(\nabla_X S)(U,V)=\nabla_X S(U,V)-S(\nabla_X U,V)-S(U,\nabla_X V)$$

we get

$$(\nabla_X S)(U,\xi)=(n-1)g(\phi X,U)-S(\phi X,U). \tag{3.8}$$

Using (3.8) in (3.7) we have

$$\begin{aligned}(n-1)g(\phi X,U)-S(\phi X,U)&\\ =\ & (n-1)A(X)\eta(U)+(n-2)B(U)\eta(X)+D(\xi)S(U,X)\\ &+(n-1)\alpha(X)\eta(U)+(n-2)\beta(U)\eta(X)+(n-1)\gamma(\xi)g(U,X)\\ &+B(X)\eta(U)+D(X)\eta(U)-D(\xi)g(U,X)\\ &+\beta(X)\eta(U)+\gamma(X)\eta(U)-\gamma(\xi)g(U,X).\end{aligned} \tag{3.9}$$

Plugging $U=\xi$, $X=\xi$ and $U=X=\xi$ in succession, we obtain from (3.9) that

$$\begin{aligned}(n-1)\{A(X)+\alpha(X)\}+B(X)+D(X)+\beta(X)+\gamma(X)&\\ =\ &(n-2)\eta(X)[B(\xi)+D(\xi)+\beta(\xi)+\gamma(\xi)],\end{aligned} \tag{3.10}$$

$$\begin{aligned}(n-2)[B(U)+\beta(U)]&\\ =\ &\eta(U)[(n-1)\{A(\xi)+\alpha(\xi)+D(\xi)+\gamma(\xi)\}\\ &+B(\xi)+\beta(\xi)],\end{aligned} \tag{3.11}$$

and

$$A(\xi)+B(\xi)+D(\xi)+\alpha(\xi)+\beta(\xi)+\gamma(\xi)=0, \tag{3.12}$$

respectively. From, (3.12) we infer that

Theorem 3.2. *In a generalized pseudo ϕ-symmetric Lorentzian Para-Sasakian manifold the vector field associate to the 1-form is perpendicular to the characteristic vector field ξ.*

Replacing U by X in (3.11), we get

$$\begin{aligned}(n-2)[B(X)+\beta(X)]&\\ =\ &\eta(X)[(n-1)\{A(\xi)+\alpha(\xi)+D(\xi)+\gamma(\xi)\}+B(\xi)+\beta(\xi)].\end{aligned} \tag{3.13}$$

Now, by the virtue of (3.10), (3.13) and (3.12) we obtain

$$\begin{aligned}(n-1)\{A(X)+B(X)+\alpha(X)+\beta(X)\}+D(X)+\gamma(X)&\\ =\ &(n-2)[D(\xi)+\gamma(\xi)]\eta(X).\end{aligned} \tag{3.14}$$

Now using (3.10), (3.11) and (3.12) in (3.9) we get

$$\begin{aligned}(n-1)g(\phi X,U)-S(\phi X,U)&\\ =\ &D(\xi)S(U,X)+\{(n-2)-D(\xi)\}g(U,X)\\ &+(n-2)[D(\xi)+\gamma(\xi)]\eta(U)\eta(X).\end{aligned} \tag{3.15}$$

Replacing X by ϕX in (3.15) we have

$$\begin{aligned}(n-1)g(X,U)-S(X,U)&\\ =\ &D(\xi)S(\phi X,U)+\{(n-2)-D(\xi)\}g(\phi X,U).\end{aligned} \tag{3.16}$$

Using (3.16) in (3.15) we obtain

$$\begin{aligned}
&[(n-1)+D(\xi)\{(n-2)-D(\xi)\}]g(X,U) \\
&\quad = [1-D(\xi)D(\xi)]S(X,U)+\{(n-2)+(n-2)D(\xi)\}g(\phi X,U) \\
&\qquad -(n-2)D(\xi)[D(\xi)+\gamma(\xi)]\eta(U)\eta(X).
\end{aligned} \tag{3.17}$$

This leads to the following:

Theorem 3.3. *In a generalized weakly ϕ-symmetric Lorentzian Para-Sasakian manifold the Ricci tensor S satisfies the relation (3.17).*

Again, setting $U=\xi$ in (3.5) we get

$$\begin{aligned}
&(\nabla_X S)(V,\xi) \\
&\quad = (n-1)A(X)\eta(V)+B(\xi)S(V,X)+(n-2)D(V)\eta(X) \\
&\qquad +(n-1)\alpha(X)\eta(V)+(n-2)\gamma(V)\eta(X)+(n-1)\beta(\xi)g(V,X) \\
&\qquad +D(X)\eta(V)+B(X)\eta(V)-B(\xi)g(V,X) \\
&\qquad +\gamma(X)\eta(V)+\beta(X)\eta(V)-\beta(\xi)g(V,X).
\end{aligned} \tag{3.18}$$

Replacing U by V in (3.7) then (3.18) yields

$$\begin{aligned}
&[B(\xi)-D(\xi)]S(V,X) \\
&\quad = [(n-2)\gamma(\xi)-D(\xi)-(n-2)\beta(\xi)+B(\xi)]g(V,X) \\
&\qquad +(n-2)[B(V)+\beta(V)-D(V)-\gamma(V)]\eta(X).
\end{aligned} \tag{3.19}$$

Now, putting $X=\xi$ in (3.19) we get

$$\begin{aligned}
&B(V)+\beta(V)-D(V)-\gamma(V) \\
&\quad = [D(\xi)-B(\xi)+\gamma(\xi)-\beta(\xi)]\eta(V).
\end{aligned} \tag{3.20}$$

Using (3.20) in (3.19) we get

$$S(V,X)=L_1 g(V,X)+L_2\eta(V)\eta(X), \tag{3.21}$$

where $L_1=\left(\frac{(n-2)\{\gamma(\xi)-\beta(\xi)\}-D(\xi)+B(\xi)}{B(\xi)-D(\xi)}\right)$ and $L_2=\left(\frac{(n-2)[D(\xi)-B(\xi)+\gamma(\xi)-\beta(\xi)]}{B(\xi)-D(\xi)}\right)$.
This leads to the following:

Theorem 3.4. *Every generalized weakly ϕ-symmetric Lorentzian Para-Sasakian manifold is η-Einstein, provided that $B(\xi)\neq D(\xi)$.*

Theorem 3.5. *Let (M^n,g) be a Lorentzian Para-Sasakian manifold. Then M can not be η-Einstein for each of the curvature restrictions: (i) ϕ-symmetric, (ii) ϕ-recurrent, (iii) generalized ϕ-recurrent, (iv) pseudo ϕ-symmetric, (v) almost pseudo ϕ-symmetric, (vi) generalized pseudo ϕ-symmetric and (vii) generalized almost pseudo ϕ-symmetric.*

Using (3.8) in (3.18) we get

$$\begin{aligned}
&(n-1)g(\phi X,V)-S(\phi X,V)\\
&= (n-1)A(X)\eta(V)+B(\xi)S(V,X)+(n-2)D(V)\eta(X)\\
&\quad +(n-1)\alpha(X)\eta(V)+(n-2)\gamma(V)\eta(X)+(n-1)\beta(\xi)g(V,X)\\
&\quad +D(X)\eta(V)+B(X)\eta(V)-B(\xi)g(V,X)\\
&\quad +\gamma(X)\eta(V)+\beta(X)\eta(V)-\beta(\xi)g(V,X). \qquad (3.22)
\end{aligned}$$

Now, setting $X=\xi$ in (3.22) and using (3.12), we get

$$[D(\xi)+\gamma(\xi)]\eta(V)=-[D(V)+\gamma(V)]. \qquad (3.23)$$

Using (3.21) in (3.14), we obtain

$$A(X)+B(X)+D(X)+\alpha(X)+\beta(X)+\gamma(X)=0 \qquad (3.24)$$

Next, in view of $\alpha=\beta=\gamma=0$, the relation (3.24) yields

$$A(X)+B(X)+D(X)=0. \qquad (3.25)$$

This motivates us to state

Theorem 3.6. *In a weakly ϕ-symmetric Lorentzian Para-Sasakian manifold $(M^n,g)(n>2)$, the sum of the associated 1-forms is given by (3.25).*

Theorem 3.7. *There does not exist an Lorentzian Para-Sasakian manifold which is*

(i) ϕ-recurrent,
(ii) generalized ϕ-recurrent, provided the 1-forms are collinear,
(iii) pseudo ϕ-symmetric,
(iv) generalized semi-pseudo ϕ-symmetric, provided the 1-forms are collinear,
(v) generalized almost pseudo ϕ-symmetric, provided the 1-forms are collinear.

4. Generalized weakly Ricci ϕ-symmetric Lorentzian Para-Sasakian manifolds

In this section we consider a generalized weakly Ricci ϕ-symmetric Lorentzian Para-Sasakian manifold. Then by the virtue of (2.1), (1.7) yields

$$\begin{aligned}
&(\nabla_X Q)(U)+\eta((\nabla_X Q)(U))\xi\\
&= A^*(X)QU+B^*(U)QX+S(U,X)\varrho^*\\
&\quad +\alpha^*(X)U+\beta^*(U)X+g(U,X)\sigma^*. \qquad (4.1)
\end{aligned}$$

from which it follows that

$$\begin{aligned}
&g(\nabla_X Q(U),V)-S(\nabla_X U,V)+\eta((\nabla_X Q)(U))\eta(V)\\
&= A^*(X)S(U,V)+B^*(U)S(V,X)+D^*(V)S(U,X)\\
&\quad +\alpha^*(X)g(U,V)+\beta^*(U)g(V,X)+\gamma^*(V)g(U,X). \qquad (4.2)
\end{aligned}$$

Putting $U = \xi$ in (4.2) and using (2.9),(2.11) we get

$$\begin{aligned} & (n-1)g(\phi X, V) - S(\phi X, V) \\ = \quad & A^*(X)(n-1)\eta(V) + B^*(\xi)S(V, X) + D^*(V)(n-1)\eta(X) \\ & +\alpha^*(X)\eta(V) + \beta^*(\xi)g(V, X) + \gamma^*(V)\eta(X). \end{aligned} \tag{4.3}$$

Setting $X = V = \xi$, $X = \xi$ and $V = \xi$ successively in (4.3), we get

$$(n-1)\{A^*(\xi) + B^*(\xi) + D^*(\xi)\} + \alpha^*(\xi) + \beta^*(\xi) + \gamma^*(\xi) = 0, \tag{4.4}$$

$$(n-1)D^*(V) + \gamma^*(V) = (n-1)\{A^*(\xi) + B^*(\xi)\} + \alpha^*(\xi) + \beta^*(\xi) \tag{4.5}$$

and

$$(n-1)A^*(X) + \alpha^*(X) = (n-1)\{B^*(\xi) + D^*(\xi)\} + \beta^*(\xi) + \gamma^*(\xi), \tag{4.6}$$

respectively.

Next using (4.4), (4.5) and (4.6) in (4.3), we get

$$\begin{aligned} & (n-1)g(\phi X, V) - S(\phi X, V) \\ = \quad & B^*(\xi)S(V, X) + \beta^*(\xi)g(V, X) + [(n-1)B^*(\xi) + \beta^*(\xi)]\eta(X)\eta(V). \end{aligned} \tag{4.7}$$

Replacing X by ϕX in (4.7) and then using (4.7) we obtain

$$\begin{aligned} & \{B^*(\xi)B^*(\xi) - 1\}S(X, V) \\ = \quad & \{(n-1)B^*(\xi) + \beta^*(\xi)\}g(\phi X, V) - \{B^*(\xi)\beta^*(\xi) + 1\}g(X, V) \\ & -B^*(\xi)[(n-1)B^*(\xi) + \beta^*(\xi)]\eta(X)\eta(V). \end{aligned} \tag{4.8}$$

Thus we infer

Theorem 4.1. *In a generalized weakly Ricci ϕ-symmetric Lorentzian Para-Sasakian manifold, the Ricci tensor S takes the form (4.8).*

If we put $(n-1)B^*(\xi) = -\beta^*(\xi)$ in (4.8), then we get

$$S(X, V) = \left(\frac{B^*(\xi)\beta^*(\xi) + 1}{1 - B^*(\xi)B^*(\xi)} \right) g(X, V),$$

which implies that the manifold under consideration is Einstein.

Corollary 4.2. *Every generalized weakly Ricci ϕ-symmetric Lorentzian Para-Sasakian manifold is an Einstein manifold, provided $(n-1)B^*(\xi) = -\beta^*(\xi)$.*

5. Generalized weakly ϕ-symmetric Lorentzian Para-Sasakian spacetime

We now assume that the matter distribution of a generalized weakly ϕ-symmetric Lorentzian Para-Sasakian spacetime is a perfect fluid. Then the Einstein's field equation without cosmological constant is given by

$$S(X,Y) - \frac{r}{2}g(X,Y) = kT(X,Y)$$

for all vector fields X, Y, where S is the Ricci tensor of type $(0,2)$, r is the scalar curvature, k is the gravitational constant and T is the energy-momentum tensor of type $(0,2)$. In a perfect fluid spacetime, the energy-momentum tensor admits the following

$$T(X,Y) = pg(x,y) + (\sigma + p)\eta(X)\eta(Y);$$

where σ and ρ are, respectively, the energy density and the isotropic pressure, and ξ denotes the flow vector field of the fluid.

Definition 5.1. Ricci tensor S of a Lorentzian Para-Sasakian manifold (M^n, g) is named cyclic parallel if

$$(\nabla_X S)(U,V) + (\nabla_U S)(V,X) + (\nabla_V S)(X,U) = 0$$

Analogously with the work of [26] and the equation (3.21) we infer

Theorem 5.2. *([26], Theorem 2.1. Page-310) In a generalized weakly ϕ-symmetric Lorentzian Para-Sasakian spacetime with cyclic parallel Ricci tensor, the associated scalars L_1 and L_2 are constants.*

Theorem 5.3. *([26], Theorem 2.2. Page-311) In a generalized weakly ϕ-symmetric Lorentzian Para-Sasakian spacetime with cyclic parallel Ricci tensor, the energy-momentum tensor is cyclic parallel.*

Theorem 5.4. *([26], Theorem 2.3. Page-311) In a generalized weakly ϕ-symmetric Lorentzian Para-Sasakian spacetime with constant associated scalars L_1 and L_2, if the energy-momentum tensor is cyclic parallel, then the Ricci tensor is cyclic parallel.*

Theorem 5.5. *([26], Theorem 3.1. Page-314) If a perfect fluid generalized weakly ϕ-symmetric Lorentzian Para-Sasakian spacetime obeys Einstein equation without cosmological constant and the square of the length of the Ricci operator is $\frac{1}{3}\left(\frac{1}{r^2}\right)$, then the spacetime can not contain pure matter. Also in such a spacetime without pure matter the pressure of the fluid is positive or negative according as $L_1 < \frac{L_2}{4}$ or $L_1 < \frac{L_2}{4}$.*

Definition 5.6. The Ricci tensor of a Lorentzian Para-Sasakian spacetime is said to be of Codazzi type if

$$(\nabla_X S)(U,V) = (\nabla_U S)(X,V).$$

Theorem 5.7. *([26], Theorem 4.1., Page-315) In a generalized weakly ϕ-symmetric Lorentzian Para-Sasakian spacetime, the energy-momentum tensor is of Codazzi type if and only if its Ricci tensor is of Codazzi type.*

Theorem 5.8. *([26], Theorem 4.2., Page-315) If the energy-momentum tensor of a perfect fluid in a generalized weakly ϕ-symmetric Lorentzian Para-Sasakian spacetime is of Codazzi type, then the integral curves of the flow vector field are geodesics.*

Theorem 5.9. *([26], Theorem 4.3., Page-318) If the energy-momentum tensor of a perfect fluid generalized weakly ϕ-symmetric Lorentzian Para-Sasakian spacetime is of Codazzi type, then both the energy density and isotropic pressure of the fluid are constants over a hypersurface orthogonal to ξ.*

Theorem 5.10. *([26], Theorem 4.4., Page-318) In a perfect fluid generalized weakly ϕ-symmetric Lorentzian Para-Sasakian spacetime with Codazzi type energy-momentum tensor, the fluid has vanishing vorticity and vanishing shear.*

Theorem 5.11. *([26], Theorem 4.5., Page-319) If the energy-momentum tensor of a perfect fluid generalized weakly ϕ-symmetric Lorentzian Para-Sasakian spacetime is of Codazzi type, then the possible local cosmological structure of the spacetime are of Petrov type I, D or O.*

Theorem 5.12. *([26], Theorem 4.6., Page-319) If a perfect fluid generalized weakly ϕ-symmetric Lorentzian Para-Sasakian spacetime with Codazzi type energy-momentum tensor admits a conformal Killing vector field, then the spacetime is either conformally flat or of Petrov type N.*

6. Example of a generalized weakly ϕ-symmetric Lorentzian Para-Sasakian manifold

(see [4], p-286-287) Let $M^3(\phi, \xi, \eta, g)$ be a Lorentzian Para-Sasakian manifold with a ϕ-basis

$$e_1 = e^z \frac{\partial}{\partial x}, \; e_2 = \phi e_1 = e^{z-\alpha x} \frac{\partial}{\partial y}, \; e_3 = \xi = \frac{\partial}{\partial z},$$

where α is non-zero constant. Then from Koszul's formula for Lorentzian metric g, we can obtain the Levi-Civita connection as follows

$$\begin{array}{lll} \nabla_{e_1} e_3 = e_2, & \nabla_{e_1} e_2 = 0, & \nabla_{e_1} e_1 = -e_3, \\ \nabla_{e_2} e_3 = e_1, & \nabla_{e_2} e_2 = \alpha e^z e_3, & \nabla_{e_2} e_1 = \alpha e^z e_2, \\ \nabla_{e_3} e_3 = 0, & \nabla_{e_3} e_2 = 0, & \nabla_{e_3} e_1 = 0. \end{array}$$

Using the above relations, one can easily calculate the non-vanishing components of the curvature tensor R (up to symmetry and skew-symmetry)

$$\begin{aligned} R(e_1, e_2)e_1 &= -(1 - \alpha^2 e^{2z})e_2, \\ R(e_1, e_2)e_2 &= (1 - \alpha^2 e^{2z})e_1, \\ R(e_1, e_3)e_1 &= -e_3, R(e_1, e_3)e_3 = e_1, \\ R(e_2, e_3)e_2 &= -e_3, R(e_2, e_3)e_3 = e_2. \end{aligned}$$

Since $\{e_1,\ e_2, e_3\}$ forms a basis, any vector field $X,\ Y,\ U,\ V \in \chi(M)$ can be written as

$$X = \sum_1^3 a_i e_i,\ Y = \sum_1^3 b_i e_i,\ Z = \sum_1^3 c_i e_i.$$

This implies that

$$R(X,Y)Z = le_1 + me_2 + ne_3,$$

where

$$\begin{aligned} l &= (a_1b_3 - a_3b_1)c_3 + c_2\ (a_1b_2 - a_2b_1\ )(1-\alpha^2e^{2z}), \\ m &= (a_2b_3 - a_3b_2)c_3 - c_1\ (a_1b_2 - a_2b_1\ )(1-\alpha^2e^{2z}), \\ n &= -(a_1b_3 - a_3b_1)c_1 - c_2\ (a_2b_3 - a_3b_2). \end{aligned}$$

Also,

$$\begin{aligned} R(e_1,Y)Z &= \{(1-\alpha^2e^{2z})b_2c_2 + b_3c_3\}e_1 - b_2\ c_1(1-\alpha^2e^{2z})e_2 - b_3c_1e_3, \\ R(e_2,Y)Z &= -b_1c_2(1-\alpha^2e^{2z})e_1 + \{\ (1-\alpha^2e^{2z})b_1c_1 - b_3\ c_3\}e_2 - b_3\ c_2e_3, \\ R(e_3,Y)Z &= -b_1c_3e_1 - b_2c_3e_2 + (b_1c_1 + b_2c_2)e_3, \\ R(X,e_1)Z &= -\{a_3c_3 + (1-\alpha^2e^{2z})a_2c_2\}e_1 + a_2\ c_1(1-\alpha^2e^{2z})e_2 + a_3c_1e_3, \\ R(X,e_2)Z &= a_1c_2(1-\alpha^2e^{2z})e_1 + \{a_3c_3 - (1-\alpha^2e^{2z})a_1c_1\}e_2 + a_3c_2e_3, \\ R(X,e_3)Z &= a_1c_3e_1 + a_2c_3e_2 - (a_1c_1 + a_2c_2)e_3, \\ R(X,Y)e_1 &= -\ (a_1b_2 - a_2b_1\ )(1-\alpha^2e^{2z})e_2 - (a_1b_3 - a_3b_1)e_3, \\ R(X,Y)e_2 &= (a_1b_2 - a_2b_1\ )(1-\alpha^2e^{2z})e_1 - (a_2b_3 - a_3b_2)e_3, \\ R(X,Y)e_3 &= (a_3b_1 - a_1b_3\ )e_1 + (a_3b_2 - a_2b_3)e_2, \end{aligned}$$

Again,

$$G(X,Y)Z = pe_1 + qe_2 + re_3,$$

where

$$\begin{aligned} p &= (a_1b_2 - a_2b_1)c_2 + (a_3b_1 - a_1b_3)c_3, \\ q &= (a_2b_1 - a_1b_2)c_1 + (a_3b_2 - a_2b_3)c_3, \\ r &= (a_3b_1 - a_1b_3)c_1 + (a_3b_2 - a_2b_3)c_2. \end{aligned}$$

Also, we have

$$\begin{aligned} G(e_1,Y)Z &= (b_2c_2 - b_3c_3)e_1 - b_2c_1e_2 - b_3c_1e_3, \\ G(e_2,Y)Z &= -b_1c_2e_1 + (b_1c_1 - b_3c_3)e_2 - b_3c_2e_3, \\ G(e_3,Y)Z &= b_1c_3e_1 + b_2c_3e_2 - (b_1c_1 + b_2c_2)e_3, \\ G(X,e_1)Z &= -(a_2c_2 - a_3c_3)e_1 + a_2c_1e_2 + a_3c_1e_3, \\ G(X,e_2)Z &= a_1c_2e_1 - (a_1c_1 - a_3c_3)e_2 + a_3c_2e_3, \\ G(X,e_3)Z &= a_1c_3e_1 + a_2c_3e_2 - (a_1c_1 + a_2c_2)e_3, \\ G(X,Y)e_1 &= -(a_1b_2 - a_2b_1\ )e_2 + (a_3b_1 - a_1b_3)e_3, \\ G(X,Y)e_2 &= (a_1b_2 - a_2b_1\ )e_1 + (a_3b_2 - a_2b_3)e_3, \\ G(X,Y)e_3 &= (a_3b_1 - a_1b_3\ )e_1 + (a_3b_2 - a_2b_3)e_2 \end{aligned}$$

and the components which can be obtained from these by the symmetry properties. Now, we calculate the covariant derivatives of the non-vanishing components of the curvature tensor as follows

$$\begin{aligned}
& (\nabla_{e_1} R)(X,Y)Z = -le_3 + ne_2 + a_1 R(e_3,Y)Z - a_3 R(e_2,Y)Z \\
& +b_1 R(X,e_3)Z - b_3 R(X,e_2)Z + c_1 R(X,Y)e_3 - c_3 R(X,Y)e_2 \\
= & -le_3 + ne_2 + a_1\{b_1c_3e_1 + b_2c_3e_2 - (b_1c_1 + b_2c_2)e_3\} \\
& -a_3\{-b_1c_2(1-\alpha^2e^{2z})e_1 + \{ (1-\alpha^2e^{2z})b_1c_1 - b_3\ c_3\}e_2 - b_3\ c_2e_3\} \\
& +b_1\{-a_1c_3e_1 - a_2c_3e_2 + (a_1c_1 + a_2c_2)e_3\} - b_3\{a_1c_2(1-\alpha^2e^{2z})e_1 \\
& +\{a_3c_3 - (1-\alpha^2e^{2z})a_1c_1\}e_2 + a_3c_2e_3\} + c_1\{(a_3b_1 - a_1b_3\ )e_1 \\
& +(a_3b_2 - a_2b_3)e_2\} - c_3\{(a_1b_2 - a_2b_1\ )(1-\alpha^2e^{2z})e_1 - (a_2b_3 - a_3b_2)e_3\},
\end{aligned}$$

$$\begin{aligned}
& (\nabla_{e_2}\bar{R})(X,Y)Z \\
= & \alpha e^z(le_2 + me_3) + ne_1 - \alpha e^z\{a_1R(e_2,Y)Z + a_2R(e_3,Y)Z\} \\
& -a_3R(e_1,Y)Z - \alpha e^z\{b_1R(e_2,Y)Z + b_2R(e_3,Y)Z\} - b_3R(e_1,Y)Z \\
& -\alpha e^z\{c_1R(e_2,Y)Z + c_2R(e_3,Y)Z\} - c_3R(e_1,Y)Z, \\
= & \alpha e^z(le_2 + me_3) + ne_1 - \alpha e^z(a_1+b_1+c_1)\{-b_1c_2(1-\alpha^2e^{2z})e_1 \\
& +\{ (1-\alpha^2e^{2z})b_1c_1 - b_3\ c_3\}e_2 - b_3\ c_2e_3\} \\
& -\alpha e^z(a_2+b_2+c_2)\{b_1c_3e_1 + b_2c_3e_2 - (b_1c_1 + b_2c_2)e_3\} \\
& +\alpha e^z(a_3+b_3+c_3)\{\{(1-\alpha^2e^{2z})b_2c_2 - b_3c_3\}e_1 - b_2\ c_1(1-\alpha^2e^{2z})e_2 - b_3c_1e_3\} \\
= & [n + \alpha e^z b_1c_2(a_1+b_1+c_1)(1-\alpha^2e^{2z}) - \alpha e^z b_1c_3(a_2+b_2+c_2) \\
& +\alpha e^z(a_3+b_3+c_3)\{(1-\alpha^2e^{2z})b_2c_2 - b_3c_3\}]e_1 \\
& +[\alpha e^z l - \alpha e^z(a_1+b_1+c_1)\{ (1-\alpha^2e^{2z})b_1c_1 - b_3\ c_3\} \\
& -\alpha e^z b_2c_3(a_2+b_2+c_2) - b_2c_1\alpha e^z(a_3+b_3+c_3)(1-\alpha^2e^{2z})]e_2 \\
& +[\alpha e^z m + \alpha e^z b_3c_2(a_1+b_1+c_1) + \alpha e^z(a_2+b_2+c_2)(b_1c_1+b_2c_2) \\
& -\alpha e^z b_3c_1(a_3+b_3+c_3)]e_3,
\end{aligned}$$

$$\text{and } (\nabla_{e_3}\bar{R})(X,Y)Z = 0.$$

With the help of the above relations one can easily bring out the following

$$\begin{aligned}
& \phi^2\left((\nabla_{e_1}R)(X,Y)Z\right) \\
= & [(1-\alpha^2e^{2z})\{a_3b_1 - a_1b_3\}c_2 + c_1(a_3b_1 - a_1b_3) \\
& -c_3(a_1b_2 - a_2b_1\ )(1-\alpha^2e^{2z})]e_1 \\
& +[n + a_1b_2c_3 - (1-\alpha^2e^{2z})a_3b_1c_1 - a_2b_1c_3 \\
& +(1-\alpha^2e^{2z})a_1b_3c_1 + c_1(a_3b_2 - a_2b_3)]e_2,
\end{aligned}$$

$$\begin{aligned}
&\phi^2\left((\nabla_{e_2}\bar{R})(X,Y)Z\right)\\
&= [n+\alpha e^z b_1c_2(a_1+b_1+c_1)(1-\alpha^2e^{2z})-\alpha e^z b_1c_3(a_2\\
&\quad +b_2+c_2)+\alpha e^z(a_3+b_3+c_3)\{(1-\alpha^2e^{2z})b_2c_2\\
&\quad -b_3c_3\}]e_1+[\alpha e^z l-\alpha e^z(a_1+b_1+c_1)\{(1-\alpha^2e^{2z})b_1c_1\\
&\quad -b_3\ c_3\}-\alpha e^z b_2c_3(a_2+b_2+c_2)-b_2c_1\alpha e^z(a_3\\
&\quad +b_3+c_3)(1-\alpha^2e^{2z})]e_2,
\end{aligned}$$

$$\phi^2\left((\nabla_{e_3}\bar{R})(X,Y)Z\right)=0.$$

For the following choice of the 1-forms

$$\begin{aligned}
A(e_1) &= \frac{1}{b_1\ +c_1},\ A(e_2)=\frac{1}{b_2\ +c_2},\ A(e_3)=\frac{1}{b_3+c_3},\\
\alpha(e_1) &= -\frac{1}{b_1\ +c_1},\alpha(e_2)=-\frac{1}{b_2\ +c_2},\alpha(e_3)=\frac{1}{b_3+c_3},\\
B(e_1) &= \frac{1}{a_1\ +b_1},B(e_2)=\frac{1}{a_2\ +b_2},B(e_3)=\frac{1}{a_3+b_3},\\
\beta(e_1) &= -\frac{1}{a_1\ +b_1},\beta(e_2)=-\frac{1}{a_2\ +b_2},\beta(e_3)=-\frac{1}{a_3+b_3},\\
D(e_1) &= \gamma(e_1)=0=D(e_2)=\gamma(e_2)=D(e_3)=\gamma(e_3),
\end{aligned}$$

one can easily verify the relations

$$\begin{aligned}
\phi^2((\nabla_{e_i}R)(X,Y)Z) &= A(e_i)R(X,Y)Z+B(X)R(e_i,Y)Z\\
&\quad +B(Y)R(X,e_i)Z+D(Z)R(X,Y)e_i\\
&\quad +g(R(X,Y)Z,e_i)\rho+\alpha(e_i)G(X,Y)Z\\
&\quad +\beta(X)G(e_i,Y)Z+\beta(Y)G(X,e_iZ)\\
&\quad +\gamma(U)G(X,Y,e_i,V)+g(G(X,Y)Z,e_i)\sigma,
\end{aligned}$$

provided $a_i=kb_i=-2kc_i$ for $i=1,2,3$. From the above, we can state that

Theorem 6.1. *There exist a Lorentzian Para-Sasakian manifold (M^3,g) which is generalized weakly ϕ-symmetric.*

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