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# Oscillation results for second-order mixed neutral integro-dynamic equations with damping and a nonpositive neutral term on time scales

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**Abstract.** In this work, we are concerned with studying a new class of second-order mixed neutral integro-dynamic equation with damping and a nonpositive neutral term of the form:

(0.1)

$$(r(t)(z^{\Delta}(t))^{\gamma})^{\Delta}+p(t)(z^{\Delta}(t))^{\gamma}+g(t,x(\tau(t)))+\int_{0}^{t}a(t,s)f(s,x(s))\Delta s=0,$$

where

$$z(t)=x(t)-p_1(t)x(\eta_1(t))+p_2(t)x(\eta_2(t)), \tag{0.2}$$

on a time scale $\mathbb{T}$. The obtained results not only present some new criteria for such kind of neutral differential equations and neutral difference equations as special cases, but also extend some results obtained on time scales. An example is given to illustrate the importance of our work.

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*Key words and phrases:* oscillation; time scales; integro-dynamic equations.

## 1. Introduction

In this paper, we investigate the oscillatory behavior for the following second-order neutral integro-dynamic equation

$$(r(t)((x(t)-p_1(t)x(\eta_1(t))+p_2(t)x(\eta_2(t)))^{\Delta})^{\gamma})^{\Delta}+p(t)((x(t)-p_1(t)x(\eta_1(t))+ p_2(t)x(\eta_2(t)))^{\Delta})^{\gamma}+g(t,x(\tau(t)))+\int_{0}^{t}a(t,s)f(s,x(s))\Delta s=0,$$

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on a time scale $\mathbb{T} \subseteq \mathbb{R}$ with $0 \in \mathbb{T}$ and $sup\mathbb{T} = \infty$.
Defining the time scale interval $[t_0, \infty)_{\mathbb{T}}$ by $[t_0, \infty)_{\mathbb{T}} = [t_0, \infty) \bigcap \mathbb{T}$,
subject to the following hypotheses:

$(H_1)$ $\eta_1(t), \eta_2(t) : \mathbb{T} \to \mathbb{T}$ are rd-continuous functions such that $\eta_1(t) \le t \le \eta_2(t)$, $\lim_{t\to+\infty} \eta_1(t) = \infty$ and $\eta_2(t) : \mathbb{T} \to \mathbb{T}$ is an injective rd-continuous increasing function.

$(H_2)$ $p_1(t), p_2(t), p(t)$ are nonnegative rd-continuous functions on an arbitrary time scale $\mathbb{T}$, such that $0 \le p_1(t) \le p_1 < 1$, and $r(t)$ is a positive rd-continuous function with $\frac{-p}{r}(t) \in \Re^+$, and

$$\int_{t_0}^{\infty} [\frac{1}{r(s)} e_{\frac{-p}{r}}(s, t_0)]^{\frac{1}{\gamma}} \Delta s = \infty, \tag{1.1}$$

where $\gamma$ is a quotient of odd positive integers.

$(H_3)$ $a(t, s) : \mathbb{T} \times \mathbb{R} \to \mathbb{R}^+$ is an rd-continuous function.

$(H_4)$ $f$ and $g \in C(\mathbb{T} \times \mathbb{R}, \mathbb{R})$ such that $uf(t, u) \ge m(t)|u|^{\beta+1}$ and $ug(t, u) \ge q(t)|u|^{\beta+1}$ for all $u \ne 0$, where $m(t) : \mathbb{T} \to [0, +\infty)$ is a nonnegative increasing rd-continuous function, $q(t) : \mathbb{T} \to [0, +\infty)$ is a nonnegative rd-continuous function, which is not identically zero for all sufficiently large $t$, and $\beta$ is a quotient of odd positive integers.

In 1988, a new calculus was introduced by Hilger in his thesis, known as the time scale calculus. This calculus unifies the continuous and the discrete analysis. The main proposal of the time scales calculus is to prove results for dynamic equations, where the domain of the unknown functions is the so-called time scale $\mathbb{T}$, which is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. For the calculus on time scales, we refer to [7]. For advances in dynamic equations on time scales, we refer to [8] and [1].

By a solution of (0.1), we mean a nontrivial real valued function $x(t)$ which satisfies (0.1) for $t \in \mathbb{T}$. Our attention is restricted to those solutions of (0.1) which exist on the half-line $[t_y, \infty)$ and satisfy $sup\{|y(t)| : t > t_*\} > 0$ for any $t_* = t_y$.

**Definition 1.1** ([2])**.** A nontrivial solution $x(t)$ is said to be oscillatory if it has an infinite number of zeros, that is, there exists a sequence of zeros $\{t_n\}$ such that $x(t_n) = 0$ and $\lim_{n\to\infty} t_n = \infty$. Otherwise, $x$ is said to be nonoscillatory.

**Definition 1.2.** A nontrivial solution $x(t)$ is said to be almost oscillatory if either $x(t)$ or $x^{\Delta}(t)$ is oscillatory.

**Definition 1.3.** The neutral differential equation is called oscillatory if all its solutions are oscillatory.

**Definition 1.4.** [7] The set of all positively regressive elements of $\Re$ is denoted by $\Re^+$ or $\Re^+(\mathbb{T}, \mathbb{R})$ and is defined as:

$$\Re^+ = \Re^+(\mathbb{T}, \mathbb{R}) = \{p \in \Re : 1 + \mu(t)p(t) > 0 \; for \; all \; t \in \mathbb{T}\}.$$

The motivation starts from Erbe et al. [11], where the authors were concerned with the oscillatory behavior of solutions of the following second-order nonlinear functional dynamic equation with a nonpositive neutral term of the form

$$(r(t)((y(t)-p(t)y(\tau(t)))^{\Delta})^{\gamma})^{\Delta}+f(t,y(\eta(t)))=0, t\in\mathbb{T}, \tag{1.2}$$

where $\tau(t),\eta(t)\in C_{rd}(\mathbb{T},\mathbb{T})$, $\tau(t)\le t$, $\lim_{t\to\infty}\tau(t)=\infty$, $\lim_{t\to\infty}\eta(t)=\infty$ and either $\eta(t)\ge 0$ or $\eta(t)\le 0$ for all sufficiently large $t$.
In 2017, Agwa et al. [5], introduced new oscillation results for the second-order nonlinear mixed neutral dynamic equation of the form

$$(r(t)(y^{\Delta}(t))^{\gamma})^{\Delta}+f(t,x(\tau_1(t)))+g(t,x(\tau_2(t)))=0,$$

where $y(t)=x(t)-p_1(t)x(\eta_1(t))+p_2(t)x(\eta_2(t))$.
In 2010 Chen et al. [9] and in 2012, Şenel [10] studied the following second-order nonlinear dynamic equations with damping

$$((x^{\Delta}(t))^{\gamma})^{\Delta}+p(t)(x^{\Delta}(t))^{\gamma}+q(t)f(x^{\sigma}(t))=0, \tag{1.3}$$

and

$$(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta}+p(t)(x^{\Delta}(t))^{\gamma}+f(t,x(g(t)))=0, \tag{1.4}$$

on a time scale $\mathbb{T}\subset\mathbb{R}$ respectively.
In 2017, Agwa et al. [4] established new oscillation criteria for the following second-order mixed nonlinear neutral dynamic equation with damping on time scales

$$(r(t)\phi(z^{\Delta}(t)))^{\Delta}+p(t)\phi(z^{\Delta}(t))+f(t,x(\tau_1(t)))+g(t,x(\tau_2(t)))=0, \tag{1.5}$$

where

$$\phi(s)=|s|^{\gamma-1}s, \qquad z(t)=x(t)+p_1(t)x(\eta_1(t))+p_2(t)x(\eta_2(t)). \tag{1.6}$$

In 2013, Grace et al. [12] studied the asymptotic behavior of nonoscillatory solutions of the following second-order integro-dynamic equation

$$(r(t)x^{\Delta}(t))^{\Delta}+\int_0^t a(t,s)f(s,x(s))\Delta s=0. \tag{1.7}$$

Also, in 2014 Grace et al. [3] studied the oscillatory and asymptotic behavior of the following second-order integro-dynamic equation

$$(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta}+\int_0^t a(t,s)f(s,x(s))\Delta s=0. \tag{1.8}$$

It is useful to note that, the above mentioned equations are special cases of our equation (0.1), and so the obtained results in [[3], [5], [4], [9], [11], [12], [10]] fail to apply in (0.1), but according to our criteria we can study the oscillatory behavior of (0.1).

Now, we present some useful Lemmas that play an important role in the proofs of our main results.

**Lemma 1.5.** *[7] If $\theta \in \Re^+$, then the initial value problem $y^\Delta = \theta(t)y$, $y(t_0) = y_0 \in \mathbb{R}$ has the unique positive solution $e_\theta(.,t_0)$ on $[t_0,\infty)_\mathbb{T}$. This solution satisfies the semi group property*

$$e_\theta(a,b)e_\theta(b,c) = e_\theta(a,c).$$

**Lemma 1.6.** *[7] If $x$ is a delta differentiable function, then*

$$(x^\gamma)^\Delta = \gamma x^\Delta \int_0^1 [hx^\sigma + (1-h)x]^{\gamma-1} dh.$$

**Lemma 1.7.** *[13] If $X$ and $Y$ are nonnegative real numbers, then*

$$\lambda XY^{\lambda-1} - X^\lambda \le (\lambda-1)Y^\lambda, \qquad \text{for all } \lambda > 1,$$

*where the equality holds if and only if $X = Y$.*

**Lemma 1.8.** *Assume that (1.1), $H_1 - H_4$ hold, and $x(t)$ is a nonoscillatory solution of (0.1). Then $z(t)$ satisfies one of the following two cases:*

*$(C_1)$ $z(t) > 0, z^\Delta(t) > 0$ and $(r(t)(z^\Delta(t))^\gamma)^\Delta \le 0$;*

*$(C_2)$ $z(t) < 0, z^\Delta(t) > 0$ and $(r(t)(z^\Delta(t))^\gamma)^\Delta \le 0$,*

*for $t \ge t_1$, where $t_1 \ge t_0$ is sufficiently large.*

*Proof.* Let $x(t)$ be a nonoscillatory solution of (0.1). We may assume that there exists $t_1 \ge t_0$ such that $x(t) > 0$ for all $t \ge t_1$ and there exists $t_2 \ge t_1 + \eta_1(t_1)$, such that $x(\eta_i(t)) > 0$ for all $t \ge t_2$, $i = 1, 2$. Now from (0.1) we have

$$\begin{aligned}(1.9)\quad (r(t)(z^\Delta(t))^\gamma)^\Delta + p(t)(z^\Delta(t))^\gamma &= -\int_0^t a(t,s)f(s,x(s))\Delta s - g(t,x(\tau(t))) \\ &= -\int_0^{t_2} a(t,s)f(s,x(s))\Delta s - \int_{t_2}^t a(t,s)f(s,x(s))\Delta s - g(t,x(\tau(t))),\end{aligned}$$

for $t \in [t_2,\infty)_\mathbb{T}$. Choosing $t_3 > t_2$ sufficiently large, then from $H_4$ we can find $k \ge 0$ such that

$$k := \int_0^{t_2} a(t,s)f(s,x(s))\Delta s + \int_{t_2}^{t_3} a(t,s)m(s)x^\beta(s)\Delta s.$$

In view of this, (1.9) can be written as

$$(1.10)\quad (r(t)(z^{\Delta}(t))^{\gamma})^{\Delta}+p(t)(z^{\Delta}(t))^{\gamma}<-\int_{0}^{t_2}a(t,s)f(s,x(s))\Delta s-$$

$$\int_{t_2}^{t_3}a(t,s)m(s)x^{\beta}(s)\Delta s-\int_{t_3}^{t}a(t,s)f(s,x(s))\Delta s-g(t,x(\tau(t))),$$

$$=-k-\int_{t_3}^{t}a(t,s)f(s,x(s))\Delta s-g(t,x(\tau(t))),$$

$$<-\int_{t_3}^{t}a(t,s)f(s,x(s))\Delta s-g(t,x(\tau(t)))<0.$$

From Lemma 1.5 and (1.10), we obtain

$$(1.11)\qquad [\frac{r(t)(z^{\Delta}(t))^{\gamma}}{e_{\frac{-p}{r}(t)}(t,t_0)}]^{\Delta}=\frac{(r(t)(z^{\Delta}(t))^{\gamma})^{\Delta}+p(t)(z^{\Delta}(t))^{\gamma}}{e^{\sigma}_{\frac{-p}{r}(t)}(t,t_0)}<0,$$

which means that $\frac{r(t)(z^{\Delta}(t))^{\gamma}}{e_{\frac{-p}{r}(t)}(t,t_0)}$ is decreasing for $t\in[t_3,\infty)_{\mathbb{T}}$ and $z^{\Delta}(t)$ is either eventually positive or eventually negative. We claim that

$$(1.12)\qquad z^{\Delta}(t)>0,$$

and therefore we have $(C_1)$ or $(C_2)$. Indeed, assume that (1.12) is not satisfied. Then there exists $t_4\in[t_3,\infty)_{\mathbb{T}}$ such that $z^{\Delta}(t)<0$ for all $t\in[t_4,\infty)_{\mathbb{T}}$. Using (1.11) and Lemma 1.5, we obtain

$$\frac{r(t)(z^{\Delta}(t))^{\gamma}}{e_{\frac{-p}{r}}(t,t_0)}\le\frac{r(t_4)(z^{\Delta}(t_4))^{\gamma}}{e_{\frac{-p}{r}}(t_4,t_0)},\text{ for } t\in[t_4,\infty)_{\mathbb{T}},$$

i.e.,

$$z^{\Delta}(t)\le -M[\frac{1}{r(t)}e_{\frac{-p}{r}}(t,t_4)]^{\frac{1}{\gamma}},\qquad \text{for } t\in[t_4,\infty)_{\mathbb{T}},$$

where $M=r^{\frac{1}{\gamma}}(t_4)|z^{\Delta}(t_4)|>0$. Integrating both sides from $t_4$ to $t$, we have

$$(1.13)\qquad z(t)\le z(t_4)-M\int_{t_4}^{t}[\frac{1}{r(s)}e_{\frac{-p}{r}}(s,t_4)]^{\frac{1}{\gamma}}\Delta s.$$

Using (1.1), we get $\lim_{t\to\infty}z(t)=-\infty$. Then we have the following two posibilities:

**Case** $(a)$**:** If $x(t)$ is unbounded, then there exists a sequence $\{t_k\}$ such that $\lim_{k\to\infty}t_k=\infty$ and $\lim_{k\to\infty}x(t_k)=\infty$. Assume that

$$x(t_k)=\max\{x(s):t_0\le s\le t_k\}.$$

Since $\lim_{t\to\infty}\eta_1(t)=\infty$, $\eta_1(t_k)>t_0$ for all sufficiently large $k$ and $\eta_1(t)\le t$, then

(1.14)
$$x(\eta_1(t_k)) = max\{x(s): t_0\le s\le \eta_1(t_k)\} \le max\{x(s): t_0\le s\le t_k\} = x(t_k).$$

Combining (1.14) and (0.2), we have

$$\begin{aligned} z(t_k) &= x(t_k) - p_1(t_k)x(\eta_1(t_k)) + p_2(t_k)x(\eta_2(t_k)),\\ &\ge x(t_k) - p_1(t_k)x(\eta_1(t_k)),\\ &\ge x(t_k) - p_1x(t_k) = (1-p_1)x(t_k) > 0, \end{aligned}$$

for all large k, which contradicts $\lim_{t\to\infty} z(t) = -\infty$.

**Case** $(b)$**:** If $x(t)$ is bounded, then $z(t)$ is bounded. This contradicts $\lim_{t\to\infty} z(t) = -\infty$.

So from Case $(a)$ and Case $(b)$, we conclude that (1.12) holds. Now using $z^\Delta(t)>0$ in (1.10), we get $[r(t)(z^\Delta(t))^\gamma]^\Delta<0$. Hence, $z(t)$ satisfies one of the two cases $(C_1)$ or $(C_2)$.

If $x(t)$ is an eventually negative solution of (0.1), then we can see that the transformation $y(t)=-x(t), y(t)>0$ transforms (0.1) into

$$(r(t)(v^\Delta(t))^\gamma)^\Delta + p(t)(v^\Delta(t))^\gamma - \int_0^t a(t,s)f(s,-y(s))\Delta s - g(t,-y(\tau(t))) = 0,$$

where

$$v(t) = y(t) - p_1(t)y(\eta_1(t)) + p_2(t)y(\eta_2(t)).$$

Thus,

(1.15)
$$\begin{aligned} (r(t)(v^\Delta(t))^\gamma)^\Delta + p(t)(v^\Delta(t))^\gamma &= \int_0^t a(t,s)f(s,-y(s))\Delta s + g(t,-y(\tau(t))),\\ &= \int_0^{t_2} a(t,s)f(s,x(s))\Delta s + \int_{t_2}^t a(t,s)f(s,x(s))\Delta s + g(t,-y(\tau(t))). \end{aligned}$$

Choosing $t_4>t_2$ sufficiently large, then from $H_4$, we can find $k_1\le 0$ such that

$$k_1 := \int_0^{t_2} a(t,s)f(s,x(s))\Delta s - \int_{t_2}^{t_4} a(t,s)m(s)y^\beta(s)\Delta s.$$

Hence (1.15) can be written as

$$
\begin{aligned}
(r(t)(v^{\Delta}(t))^{\gamma})^{\Delta}+&p(t)(v^{\Delta}(t))^{\gamma}\\
&<\int_{0}^{t_2} a(t,s)f(s,x(s))\Delta s-\int_{t_2}^{t_4} a(t,s)m(s)y^{\beta}(s)\Delta s\\
&\quad+\int_{t_4}^{t} a(t,s)f(s,x(s))\Delta s+g(t,-y(\tau(t))),\\
&=k_1+\int_{t_4}^{t} a(t,s)f(s,x(s))\Delta s+g(t,-y(\tau(t))),\\
&<-\int_{t_4}^{t} a(t,s)m(s)y^{\beta}(s)\Delta s+g(t,-y(\tau(t)))<0.
\end{aligned}
$$

It follows in a similar manner that $(C_1)$ or $(C_2)$ holds for $v(t)$. This completes the proof. $\square$

**Lemma 1.9.** *Assume that $x(t)$ is a positive solution of (0.1) and $z(t)$ satisfies $(C_2)$. Then* $\lim_{t\to\infty} x(t)=0$.

*Proof.* By $z(t)<0$ and $z^{\Delta}(t)>0$, we deduce that

$$\lim_{t\to\infty} z(t)=l\le 0.$$

As in the proof of Case $(a)$ of Lemma 1.8, we see that $x(t)$ is bounded. Thus $\lim_{t\to\infty} x(t)=a\ge 0$.
Now, if $a>0$, then there exists $t_k\subseteq[t_2,\infty)_{\mathbb{T}}$ such that $\lim_{k\to\infty} t_k=\infty$, $\lim_{k\to\infty} x(t_k)=a>0$ and

$$x(t_k)=\max\{x(s):t_0\le s\le t_k\}.$$

Hence

$$z(t_k)\ge x(t_k)-p_1(t_k)x(\eta_1(t_k))\ge x(t_k)-p_1x(t_k)=(1-p_1)x(t_k),$$

which means that $0>\lim_{k\to\infty} z(t_k)>(1-p_1)a>0$. We are led to a contradiction. Therefore, $a=0$ and $\lim_{t\to\infty} x(t)=0$. $\square$

**Lemma 1.10.** *Assume that (1.1) and $H_1$-$H_4$ hold. Let $x(t)$ be a nonoscillatory solution of (0.1) on $[t_0,\infty)_{\mathbb{T}}$ and $z(t)$ satisfies $(C_1)$. Then there exist suitable constants $b_1>0$ and $b_2:=\frac{z(t_3)}{L(t_4,t_3)}+r^{\frac{1}{\gamma}}(t_3)z^{\Delta}(t_3)\ge 0$ such that*

$$b_1\le z(t)\le b_2L(t,t_3), \tag{1.16}$$

*where*

$$L(t,t_0):=\int_{t_0}^{t}\frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}.$$

*Proof.* Taking into account the fact that $v(t)$ is increasing, we have

$$z(t) > z(t_3) := b_1. \tag{1.17}$$

Integrating $z^{\Delta}(t)$ from $t_3$ to $t$ and using that $r(t)(z^{\Delta}(t))^{\gamma}$ is decreasing, we obtain

$$\begin{aligned} z(t) &= z(t_3) + \int_{t_3}^{t} \frac{[r(s)(z^{\Delta}(s))^{\gamma}]^{\frac{1}{\gamma}}}{r^{\frac{1}{\gamma}}(s)} \Delta s \\ &\leq z(t_3) + r^{\frac{1}{\gamma}}(t_3) z^{\Delta}(t_3) L(t, t_3), \end{aligned}$$

where $L(t,t_3) := \int_{t_3}^{t} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}$. Hence $L(t,t_3)$ is a positive increasing function. Choosing $t_4 \geq t_3$ sufficiently large, we can write

$$z(t) \leq b_2 L(t,t_3), \text{ for all } t \in [t_4, \infty)_{\mathbb{T}},$$

where $b_2 := \frac{z(t_3)}{L(t_4,t_3)} + r^{\frac{1}{\gamma}}(t_3) z^{\Delta}(t_3)$ . Combining the previous inequality and (1.17), we get

$$b_1 \leq z(t) \leq b_2 L(t,t_3), \qquad \text{for } t \in [t_4, \infty)_{\mathbb{T}},$$

which is the desired inequality. This completes the proof. □

## 2. Main results

**Theorem 2.1.** *Assume that (1.1), $H_1 - H_4$ hold, and $\eta_2(t) \geq \tau(t) \geq t$. Moreover, suppose that there exists a positive real-valued $\Delta$ -differentiable function $\delta(t)$ such that for all sufficiently large $T > t_1$, we have*

(2.1)
$$\limsup_{t\to\infty} \int_{T}^{t} [\delta(u)[A(u) + q(u)B^{\beta}(u)] - \frac{\gamma^{\gamma}}{\beta^{\gamma}(\gamma+1)^{\gamma+1}} \frac{r(u)(\overline{\delta(u)}_{+})^{\gamma+1}}{\delta^{\gamma}(u)D^{\gamma}(u)}]\Delta u = \infty,$$

*where*

$$D(t) := \begin{cases} b_1^{\frac{\beta}{\gamma}-1}, & \frac{\beta}{\gamma} \geq 1, \\ [\frac{1}{b_2 L(\sigma(t),t_3)}]^{1-\frac{\beta}{\gamma}}, & \frac{\beta}{\gamma} \leq 1, \end{cases} \tag{2.2}$$

$$A(t) := \min\{A_1(t), A_2(t)\}, \tag{2.3}$$

*with*

$$A_1(t) := \frac{n_1}{(b_2 L(t,T))^{\beta}} \int_{T}^{t} \frac{a(t,s)m(s)}{(1+p_2(s))^{\beta}} \Delta s,$$

$$A_2(t) := \frac{n_2}{(b_2 L(t,T))^{\beta}} \int_{\eta_2^{-1}(T)}^{\eta_2^{-1}(t)} \frac{a(t,\eta_2(\xi))m(\eta_2(\xi))(\eta_2(\xi))^{\Delta}}{(1+p_2(\xi))^{\beta}} \Delta\xi,$$

*given that* $b_1, b_2, n_1$ *and* $n_2$ *are positive constants. Also*

$$B(t) := \min\{B_1(t) := \frac{1}{1+p_2(\tau(t))}, B_2(t) := \frac{\psi(t,T)}{1+p_2(\eta_2^{-1}(\tau(t)))}\}, \tag{2.4}$$

*where*

$$\psi(t,T) := \frac{\int_T^{\eta_2^{-1}(\tau(t))} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}}{\int_T^{t} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}},$$

$\overline{\delta(t)}_+ := \max\{0, \overline{\delta(t)}\}$ *and* $\overline{\delta(t)} := \delta^\Delta(t) - \frac{p(t)\delta(t)}{r(t)}$. *Then, every solution of (0.1) is almost oscillatory on* $[t_0,\infty)_{\mathbb{T}}$ *or converges to zero as* $t \to \infty$.

*Proof.* Assume that $x(t)$ is not almost oscillatory solution of (0.1). Then we can assume that there exists $t_3 \geq t_0$ such that $x(t) > 0$ and $x(\eta_i(t)) > 0, i = 1,2$ on $[t_3,\infty)_{\mathbb{T}}$. (When $x(t)$ is negative, the proof is similar.) By Lemma 1.8, $z(t)$ satisfies either $(C_1)$ or $(C_2)$. Since $x(t)$ is not almost oscillatory, we have two possibilities:

$(I)$ $x^\Delta(t) < 0$, for $t \geq t_3$,

$(II)$ $x^\Delta(t) > 0$, for $t \geq t_3$,

**Case** 1. Suppose that $(C_1)$ holds and $x^\Delta(t) < 0$. Then we have

$$\begin{aligned} z(t) &< x(t) + p_2(t)x(\eta_2(t)), \\ &\leq (1+p_2(t))x(t), for\ t \geq t_3. \end{aligned} \tag{2.5}$$

Using $(H_4)$, (2.5) and $(C_1)$ in (1.10), we get

(2.6)

$$\begin{aligned} (r(t)(z^\Delta(t))^\gamma)^\Delta &< -p(t)(z^\Delta(t))^\gamma - \int_{t_3}^t \frac{a(t,s)m(s)}{(1+p_2(s))^\beta} z^\beta(s)\Delta s - q(t)x^\beta(\tau(t)) \\ &< -p(t)(z^\Delta(t))^\gamma - z^\beta(t_3)\int_{t_3}^t \frac{a(t,s)m(s)}{(1+p_2(s))^\beta}\Delta s - \frac{q(t)}{[1+p_2(\tau(t))]^\beta} z^\beta(\tau(t)). \end{aligned}$$

Defining the function $w(t)$ by

$$w(t) = \delta(t)\frac{r(t)(z^\Delta(t))^\gamma}{z^\beta(t)}. \tag{2.7}$$

Clearly, $w(t) > 0$ and

$$\begin{aligned} w^\Delta(t) =& (\frac{\delta(t)}{z^\beta(t)})(r(t)(z^\Delta(t))^\gamma)^\Delta + r(\sigma(t))(z^\Delta(\sigma(t))^\gamma(\frac{\delta(t)}{z^\beta(t)})^\Delta, \\ =& (\frac{\delta(t)}{z^\beta(t)})(r(t)(z^\Delta(t))^\gamma)^\Delta \\ &+ r(\sigma(t))(z^\Delta(\sigma(t)))^\gamma \frac{z^\beta(t)\delta^\Delta(t) - \delta(t)(z^\beta(t))^\Delta}{z^\beta(t)z^\beta(\sigma(t))}. \end{aligned} \tag{2.8}$$

Combining (2.6) and (2.8), we obtain

$$w^{\Delta}(t) \leq \frac{-p(t)\delta(t)}{z^{\beta}(t)}(z^{\Delta}(t))^{\gamma} - \frac{\delta(t)z^{\beta}(t_3)}{z^{\beta}(t)}\int_{t_3}^{t}\frac{a(t,s)m(s)}{(1+p_2(s))^{\beta}}\Delta s + \delta^{\Delta}(t)\frac{r(\sigma(t))(z^{\Delta}(\sigma(t)))^{\gamma}}{z^{\beta}(\sigma(t))} - \frac{\delta(t)r(\sigma(t))(z^{\Delta}(\sigma(t)))^{\gamma}(z^{\beta}(t))^{\Delta}}{z^{\beta}(t)z^{\beta}(\sigma(t))} - \frac{\delta(t)q(t)}{[1+p_2(\tau(t))]^{\beta}}[\frac{z(\tau(t))}{z(t)}]^{\beta}.$$

Using (1.16), (2.7) and $z^{\Delta}(t) > 0$ in the above inequality, we get

$$w^{\Delta}(t) \leq -\delta(t)[A_1(t) + q(t)B_1^{\beta}(t)] - \frac{p(t)}{r(t)}w(t) + \frac{\delta^{\Delta}(t)}{\delta(\sigma(t))}w(\sigma(t)) - \frac{\delta(t)r(\sigma(t))(z^{\Delta}(\sigma(t)))^{\gamma}(z^{\beta}(t))^{\Delta}}{z^{\beta}(t)z^{\beta}(\sigma(t))}, \tag{2.9}$$

where $A_1(t) := [\frac{z(t_3)}{b_2L(t,t_3)}]^{\beta}\int_{t_3}^{t}\frac{a(t,s)m(s)}{(1+p_2(s))^{\beta}}\Delta s$ and $B_1(t) := \frac{1}{1+p_2(\tau(t))}$. From $(C1)$ and (2.7), we obtain

$$\frac{w(t)}{\delta(t)} = \frac{r(t)(z^{\Delta}(t))^{\gamma}}{z^{\beta}(t)} \geq \frac{r(\sigma(t))(z^{\Delta}(\sigma(t)))^{\gamma}}{z(\sigma(t))^{\beta}} = \frac{w(\sigma(t))}{\delta(\sigma(t))},$$

i.e.,

$$w(t) > \frac{\delta(t)}{\delta(\sigma(t))}w(\sigma(t)).$$

In view of this, (2.9)gives

$$\begin{aligned} w^{\Delta}(t) &\leq -\delta(t)[A_1(t) + q(t)B_1^{\beta}(t)] + (\delta^{\Delta}(t) - \frac{p(t)\delta(t)}{r(t)})\frac{w(\sigma(t))}{\delta(\sigma(t))} - \frac{\delta(t)r(\sigma(t))(z^{\Delta}(\sigma(t)))^{\gamma}(z^{\beta}(t))^{\Delta}}{z^{\beta}(t)z^{\beta}(\sigma(t))}, \\ &\leq -\delta(t)[A_1(t) + q(t)B_1^{\beta}(t)] + \frac{\overline{\delta(t)}_{+}}{\delta(\sigma(t))}w(\sigma(t)) - \frac{\delta(t)r(\sigma(t))(z^{\Delta}(\sigma(t)))^{\gamma}(z^{\beta}(t))^{\Delta}}{z^{\beta}(t)z^{\beta}(\sigma(t))}, \end{aligned} \tag{2.10}$$

where $\overline{\delta(t)}_{+} := \max\{0, \overline{\delta(t)}\}$ and $\overline{\delta(t)} := \delta^{\Delta}(t) - \frac{p(t)\delta(t)}{r(t)}$.
By Lemma 1.6, we have

$$(z^{\beta})^{\Delta} \geq \begin{cases} \beta z^{\Delta} z^{\beta-1}, & \beta \geq 1, \\ \beta z^{\Delta}(z^{\sigma})^{\beta-1}, & \beta \leq 1. \end{cases} \tag{2.11}$$

Also, from $(C1)$, we have

$$z^{\Delta} > \frac{(r^{\sigma})^{\frac{1}{\gamma}}}{r^{\frac{1}{\gamma}}} z^{\Delta\sigma}. \tag{2.12}$$

Substituting from (2.11) and (2.12) in the last term of (2.10), we can write

$$\begin{aligned} w^{\Delta}(t) &\leq -\delta(t)[A_1(t) + q(t)B_1^{\beta}(t)] + \frac{\overline{\delta(t)}_+}{\delta(\sigma(t))} w(\sigma(t)) - \\ &\frac{\beta\delta(t)(r(\sigma(t))^{1+\frac{1}{\gamma}}(z^{\Delta}(\sigma(t)))^{\gamma+1}}{r^{\frac{1}{\gamma}}(t)(z(\sigma(t)))^{\beta+1}} \\ &= -\delta(t)[A_1(t) + q(t)B_1^{\beta}(t)] + \frac{\overline{\delta(t)}_+}{\delta(\sigma(t))} w(\sigma(t)) - \\ &\frac{\beta\delta(t)(r(\sigma(t)))^{1+\frac{1}{\gamma}}(z^{\Delta}(\sigma(t)))^{\gamma+1}}{r^{\frac{1}{\gamma}}(t)(z(\sigma(t)))^{\beta+\frac{\beta}{\gamma}}}(z(\sigma(t)))^{\frac{\beta}{\gamma}-1} \\ &\leq -\delta(t)[A_1(t) + q(t)B_1^{\beta}(t)] + \frac{\overline{\delta(t)}_+}{\delta(\sigma(t))} w(\sigma(t)) - \\ &\frac{\beta\delta(t)D(t)}{(\delta(\sigma(t)))^{\lambda} r^{\frac{1}{\gamma}}(t)}(w(\sigma(t)))^{\lambda} \end{aligned} \tag{2.13}$$

where

$$D(t) := \begin{cases} b_1^{\frac{\beta}{\gamma}-1}, & \frac{\beta}{\gamma} \geq 1, \\ [\frac{1}{b_2 L(\sigma(t),t_3)}]^{1-\frac{\beta}{\gamma}}, & \frac{\beta}{\gamma} \leq 1. \end{cases}$$

Taking $\lambda = \frac{\gamma+1}{\gamma}$ and using Lemma 1.7 with

$$X = [\frac{\beta\delta(t)D(t)}{(\delta(\sigma(t)))^{\lambda} r^{\frac{1}{\gamma}}(t)}]^{\frac{1}{\lambda}} w(\sigma(t)) \text{ and } Y = [\frac{\overline{\delta(t)}_+}{\lambda\delta(\sigma(t))}[\frac{\beta\delta(t)D(t)}{(\delta(\sigma(t)))^{\lambda} r^{\frac{1}{\gamma}}(t)}]^{\frac{-1}{\lambda}}]^{\frac{1}{\lambda-1}},$$

we have

$$\frac{\overline{\delta(t)}_+}{\delta(\sigma(t))} w(\sigma(t)) - \frac{\beta\delta(t)D(t)}{(\delta(\sigma(t)))^{\lambda} r^{\frac{1}{\gamma}}(t)}(w(\sigma(t)))^{\lambda} \leq \frac{\gamma^{\gamma}}{\beta^{\gamma}(\gamma+1)^{\gamma+1}} \frac{r(t)(\overline{\delta(t)}_+)^{\gamma+1}}{\delta^{\gamma}(t)D^{\gamma}(t)}. \tag{2.14}$$

Substituting from (2.14) into (2.13), we obtain

$$w^{\Delta}(t) \leq -\delta(t)[A_1(t) + q(t)B_1^{\beta}(t)] + \frac{\gamma^{\gamma}}{\beta^{\gamma}(\gamma+1)^{\gamma+1}} \frac{r(t)(\overline{\delta(t)}_+)^{\gamma+1}}{\delta^{\gamma}(t)D^{\gamma}(t)}. \tag{2.15}$$

Integrating the above inequality from $t_3$ to $t$, we get

$$\begin{aligned} &\int_{t_3}^{t} [\delta(u)[A_1(u) + q(u)B_1^{\beta}(u)] - \frac{\gamma^{\gamma}}{\beta^{\gamma}(\gamma+1)^{\gamma+1}} \frac{r(u)(\overline{\delta(u)}_+)^{\gamma+1}}{\delta^{\gamma}(u)D^{\gamma}(u)}]\Delta u < \\ &\qquad w(t_3) - w(t) < w(t_3), \end{aligned} \tag{2.16}$$

which contradicts (2.1).
**Case** 2. Suppose that $(C_1)$ holds and $x^{\Delta}(t) > 0$. Then we have

$$\begin{aligned} z(t) &< x(t) + p_2(t)x(\eta_2(t)), \\ &\leq (1 + p_2(t))x(\eta_2(t)), forall\ t \geq t_3. \end{aligned}$$

Choosing $t_4$ sufficiently large such that $t_4 > t_3$ and $\eta_2^{-1}(t) > t_3$ for all $t \geq t_4$, then

$$x(t) \geq \frac{1}{1 + p_2(\eta_2^{-1}(t))} z(\eta_2^{-1}(t)), t \geq t_4. \tag{2.17}$$

Using $H_4$, (2.17) and $(C_1)$ into (0.1), we get

(2.18)

$$\begin{aligned} (r(t)(z^{\Delta}(t))^{\gamma})^{\Delta} &< -p(t)(z^{\Delta}(t))^{\gamma} - \int_{t_4}^{t} \frac{a(t,s)m(s)}{(1 + p_2(\eta_2^{-1}(s)))^{\beta}} z^{\beta}(\eta_2^{-1}(s))\Delta s - \\ &\quad q(t)x^{\beta}(\tau(t)) \\ &= -p(t)(z^{\Delta}(t))^{\gamma} - \int_{\eta_2^{-1}(t_4)}^{\eta_2^{-1}(t)} \frac{a(t,\eta_2(\xi))m(\eta_2(\xi))(\eta_2(\xi))^{\Delta}}{(1 + p_2(\xi))^{\beta}} z^{\beta}(\xi)\Delta\xi - \\ &\quad q(t)\frac{z^{\beta}(\eta_2^{-1}(\tau(t)))}{[1 + p_2(\eta_2^{-1}(\tau(t)))]^{\beta}} \\ &< -p(t)(z^{\Delta}(t))^{\gamma} - z^{\beta}(\eta_2^{-1}(t_4)) \int_{\eta_2^{-1}(t_4)}^{\eta_2^{-1}(t)} \frac{a(t,\eta_2(\xi))m(\eta_2(\xi))(\eta_2(\xi))^{\Delta}}{(1 + p_2(\xi))^{\beta}} \Delta\xi - \\ &\quad q(t)\frac{z^{\beta}(\eta_2^{-1}(\tau(t)))}{[1 + p_2(\eta_2^{-1}(\tau(t)))]^{\beta}}. \end{aligned}$$

Substituting from (2.18) into (2.8), we obtain

$$\begin{aligned} w^{\Delta}(t) &\leq \frac{-p(t)\delta(t)}{z^{\beta}(t)}(z^{\Delta}(t))^{\gamma} - \frac{\delta(t)z^{\beta}(\eta_2^{-1}(t_4))}{z^{\beta}(t)} \int_{\eta_2^{-1}(t_4)}^{\eta_2^{-1}(t)} \frac{a(t,\eta_2(\xi))m(\eta_2(\xi))(\eta_2(\xi))^{\Delta}}{(1 + p_2(\xi))^{\beta}} \Delta\xi + \\ &\quad \delta^{\Delta}(t)\frac{r(\sigma(t))(z^{\Delta}(\sigma(t)))^{\gamma}}{z^{\beta}(\sigma(t))} - \frac{\delta(t)r(\sigma(t))(z^{\Delta}(\sigma(t)))^{\gamma}(z^{\beta}(t))^{\Delta}}{z^{\beta}(t)z^{\beta}(\sigma(t))} - \\ &\quad \frac{\delta(t)q(t)}{[1 + p_2(\eta_2^{-1}(\tau(t)))]^{\beta}}[\frac{z(\eta_2^{-1}(\tau(t)))}{z(t)}]^{\beta}. \end{aligned} \tag{2.19}$$

Choose $t_5 \geq t_4$ such that $\eta_2^{-1}(\tau(t)) > t_4$ for all $t \geq t_5$, integrating $z^{\Delta}(t)$ from $\eta_2^{-1}(\tau(t))$ to $t$ and using the fact that $r(t)(z^{\Delta}(t))^{\gamma}$ is decreasing, we have

$$\begin{aligned} z(t) - z(\eta_2^{-1}(\tau(t))) &= \int_{\eta_2^{-1}(\tau(t))}^{t} \frac{(r(s)(z^{\Delta}(s))^{\gamma})^{\frac{1}{\gamma}}}{r^{\frac{1}{\gamma}}(s)} \Delta s \\ &\leq r^{\frac{1}{\gamma}}(\eta_2^{-1}(\tau(t)))z^{\Delta}(\eta_2^{-1}(\tau(t))) \int_{\eta_2^{-1}(\tau(t))}^{t} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}. \end{aligned}$$

Hence,

$$\frac{z(\eta_2^{-1}(\tau(t)))}{z(t)} \geq [1 - \frac{r^{\frac{1}{\gamma}}(\eta_2^{-1}(\tau(t)))z^{\Delta}(\eta_2^{-1}(\tau(t)))}{z(t)} \int_{\eta_2^{-1}(\tau(t))}^{t} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}]$$

$$(2.20) \qquad = 1 - \frac{r^{\frac{1}{\gamma}}(\eta_2^{-1}(\tau(t)))z^{\Delta}(\eta_2^{-1}(\tau(t)))}{z(t)}[\int_{t_4}^{t} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)} - \int_{t_4}^{\eta_2^{-1}(\tau(t))} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}].$$

Integrating $z^{\Delta}(t)$ from $t_4$ to $\eta_2^{-1}(\tau(t))$, we get

$$z(\eta_2^{-1}(\tau(t))) \geq r^{\frac{1}{\gamma}}(\eta_2^{-1}(\tau(t)))z^{\Delta}(\eta_2^{-1}(\tau(t))) \int_{t_4}^{\eta_2^{-1}(\tau(t))} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)},$$

i.e.,

$$(2.21) \qquad r^{\frac{1}{\gamma}}(\eta_2^{-1}(\tau(t)))z^{\Delta}(\eta_2^{-1}(\tau(t))) \leq z(\eta_2^{-1}(\tau(t)))[\int_{t_4}^{\eta_2^{-1}(\tau(t))} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}]^{-1}.$$

Substituting from (2.21) into (2.20), we have

$$\frac{z(\eta_2^{-1}(\tau(t)))}{z(t)} \geq 1 - \frac{z(\eta_2^{-1}(\tau(t)))}{z(t)}[\int_{t_4}^{\eta_2^{-1}(\tau(t))} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}]^{-1}[\int_{t_4}^{t} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)} - \int_{t_4}^{\eta_2^{-1}(\tau(t))} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}],$$

i.e.,

$$(2.22) \qquad \frac{z(\eta_2^{-1}(\tau(t)))}{z(t)} \geq \frac{\int_{t_4}^{\eta_2^{-1}(\tau(t))} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}}{\int_{t_4}^{t} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}} := \psi(t, t_4).$$

Using (2.22), (2.19) can be written as

$$w^{\Delta}(t) \leq \frac{-p(t)\delta(t)}{z^{\beta}(t)}(z^{\Delta}(t))^{\gamma} - \delta(t)[A_2(t) + q(t)B_2^{\beta}(t)] +$$
$$\delta^{\Delta}(t)\frac{r(\sigma(t))(z^{\Delta}(\sigma(t)))^{\gamma}}{z^{\beta}(\sigma(t))} - \frac{\delta(t)r(\sigma(t))(z^{\Delta}(\sigma(t)))^{\gamma}(z^{\beta}(t))^{\Delta}}{z^{\beta}(t)z^{\beta}(\sigma(t))},$$

where

$$A_2(t) := [\frac{z(\eta_2^{-1}(t_4))}{b_2 L(t, t_4)}]^{\beta} \int_{\eta_2^{-1}(t_4)}^{\eta_2^{-1}(t)} \frac{a(t, \eta_2(\xi))m(\eta_2(\xi))(\eta_2(\xi))^{\Delta}}{(1 + p_2(\xi))^{\beta}} \Delta\xi,$$

and

$$B_2(t) := \frac{\psi(t, t_4)}{1 + p_2(\eta_2^{-1}(\tau(t)))}.$$

Using the same technique we used in Case 1, we obtain

$$\int_{t_4}^{t}[\delta(u)[A_2(u)+q(u)B_2^{\beta}(u)]-\frac{\gamma^{\gamma}}{\beta^{\gamma}(\gamma+1)^{\gamma+1}}\frac{r(u)(\overline{\delta(u)}_{+})^{\gamma+1}}{\delta^{\gamma}(u)D^{\gamma}(u)}]\Delta u<$$

$$w(t_4)-w(t)<w(t_4), \tag{2.23}$$

which contradicts (2.1).

Finally, suppose that $(C_2)$ holds. Then, by Lemma 1.9, we have $\lim_{t\to\infty}x(t)=0$. Thus every solution of (0.1) is almost oscillatory on $[t_0,\infty)_{\mathbb{T}}$ or converges to zero as $t\to\infty$. This completes the proof. □

**Theorem 2.2.** *Assume that (1.1), $H_1-H_4$ hold, and $\tau(t)\geq\eta_2(t)\geq t$ . Moreover, assume that there exists a positive real-valued $\Delta$ -differentiable function $\delta(t)$ such that for all sufficiently large $T>t_1$, we have*

$$\limsup_{t\to\infty}\int_{T}^{t}[\delta(u)[A(u)+q(u)C^{\beta}(u)]-\frac{\gamma^{\gamma}}{\beta^{\gamma}(\gamma+1)^{\gamma+1}}\frac{r(u)(\overline{\delta(u)}_{+})^{\gamma+1}}{\delta^{\gamma}(u)D^{\gamma}(u)}]\Delta u=\infty, \tag{2.24}$$

*where*

$$C(t):=\min\{B_1(t):=\frac{1}{1+p_2(\tau(t))},B_3(t):=\frac{1}{1+p_2(\eta_2^{-1}(\tau(t)))}\}, \tag{2.25}$$

*$A(t)$ and $D(t)$ are as defined in (2.3) and (2.2), respectively.*
*Then, every solution of (0.1) is almost oscillatory on $[t_0,\infty)_{\mathbb{T}}$ or converges to zero as $t\to\infty$.*

*Proof.* The proof is similar to that of Theorem 2.1, so it is omitted. □

**Theorem 2.3.** *Assume that (1.1), $H_1-H_4$ hold, and $t\geq\tau(t)$ . Moreover, assume that there exists a positive real-valued $\Delta$ -differentiable function $\delta(t)$ such that for all sufficiently large $T>t_1$, we have*

$$\limsup_{t\to\infty}\int_{T}^{t}[\delta(u)[A(u)+q(u)E^{\beta}(u)]-\frac{\gamma^{\gamma}}{\beta^{\gamma}(\gamma+1)^{\gamma+1}}\frac{r(u)(\overline{\delta(u)}_{+})^{\gamma+1}}{\delta^{\gamma}(u)D^{\gamma}(u)}]\Delta u=\infty, \tag{2.26}$$

*where*

$$E(t):=\min\{B_2(t),B_4(t):=\frac{\varsigma(t,T)}{1+p_2(\tau(t))}\}, \tag{2.27}$$

$$\varsigma(t,T):=\frac{\int\limits_{T}^{\tau(t)}\frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}}{\int\limits_{T}^{t}\frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}},$$

*$A(t)$, $D(t)$ and $B_2(t)$ are as defined in (2.3), (2.2) and (2.4), respectively.*
*Then, every solution of (0.1) is almost oscillatory on $[t_0,\infty)_{\mathbb{T}}$ or converges to zero as $t\to\infty$.*

*Proof.* The proof is similar to that of Theorem 2.1, so it is omitted. □

**Theorem 2.4.** *Assume that (1.1), $H_1 - H_4$ hold, and $\eta_2(t) \geq \tau(t) \geq t$. Moreover, assume that there exist functions $H, h$ such that for each fixed $t$, $H(t,s)$ and $h(t,s)$ are rd-continuous functions with respect to $s$ on $\mathbb{D} \equiv \{(t,s) : t \geq s \geq t_0\}$,*

$$H(t,t) = 0,\ t \geq t_0, H(t,s) > 0,\ t > s \geq t_0, \tag{2.28}$$

*and $H$ has a non-positive continuous $\Delta$-partial derivative $H^{\Delta_s}(t,s)$ satisfying*

$$-H^{\Delta_s}(t,s) = h(t,s)(H(t,s))^{\frac{\gamma}{\gamma+1}}. \tag{2.29}$$

*Furthermore, suppose that there exists a positive real-valued $\Delta$ -differentiable function $\delta(t)$ such that for all sufficiently large $T > t_1$, we have*

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \Big[H(t,s)\delta(s)[A(u) + q(u)B^{\beta}(u)] - \frac{\gamma^{\gamma}}{\beta^{\gamma}(\gamma+1)^{\gamma+1}} \frac{r(s)(G_+(t,s))^{\gamma+1}}{\delta^{\gamma}(s)D^{\gamma}(s)}\Big]\Delta s = \infty, \tag{2.30}$$

*where*

$$G(t,s) = \overline{\delta}(s)H^{1-\frac{1}{\lambda}}(t,s) - \delta(\sigma(s))h(t,s), \qquad G_+(t,s) = max\{0, G(t,s)\}.$$

*Then, every solution of (0.1) is almost oscillatory on $[t_0,\infty)_{\mathbb{T}}$ or converges to zero as $t \to \infty$.*

*Proof.* Assume that $x(t)$ is not an almost oscillatory solution of (0.1). We may assume that there exists $t_3 \geq t_0$ such that $x(t) > 0$ and $x(\eta_i(t)) > 0, i = 1,2$ on $[t_3,\infty)_{\mathbb{T}}$. (When $x(t)$ is negative, the proof is similar.) Then by Lemma 1.8, $z(t)$ satisfies either $(C_1)$ or $(C_2)$. Since $x(t)$ is not almost oscillatory, we have two possibilities:

(*I*) $x^{\Delta}(t) < 0$, for $t \geq t_3$,

(*II*) $x^{\Delta}(t) > 0$, for $t \geq t_3$.

**Case** 1. Suppose that $(C_1)$ holds and $x^{\Delta}(t) < 0$. Proceeding as in the proof of first part of Theorem 2.1 until we get (2.13), it follows that

$$w^{\Delta}(t) \leq -\delta(t)[A_1(t) + q(t)B_1^{\beta}(t)] + \frac{\overline{\delta(t)}_+}{\delta(\sigma(t))} w(\sigma(t)) - \frac{\beta\delta(t)D(t)}{(\delta(\sigma(t)))^{\lambda} r^{\frac{1}{\gamma}}(t)}(w(\sigma(t)))^{\lambda}. \tag{2.31}$$

Multiplying both sides of the previous inequality by $H(t,s)$, we get

$$H(t,s)\delta(t)[A_1(t)+q(t)B_1^{\beta}(t)] \leq -H(t,s)w^{\Delta}(t)+\frac{\overline{\delta}(t)H(t,s)}{\delta(\sigma(t))}w(\sigma(t))- \frac{\beta\delta(t)D(t)H(t,s)}{\delta^{\lambda}(\sigma(t))r^{\frac{1}{\gamma}}(t)}w^{\lambda}(\sigma(t)).$$

Integrating the above inequality from $t_5 \to t$ and using integration by parts, we get

$$\int_{t_5}^{t} H(t,s)\delta(s)[A_1(s)+q(s)B_1^{\beta}(s)]\Delta s \leq \tag{2.32}$$

$$H(t,t_5)w(t_5)-\int_{t_5}^{t}[-H^{\Delta_s}(t,s)]w(\sigma(s))\Delta s+\int_{t_5}^{t}\frac{\overline{\delta}(s)H(t,s)}{\delta(\sigma(s))}w(\sigma(s))\Delta s$$
$$-\int_{t_5}^{t}\frac{\beta\delta(s)D(s)H(t,s)}{\delta^{\lambda}(\sigma(s))r^{\frac{1}{\gamma}}(s)}w^{\lambda}(\sigma(s))\Delta s$$
$$=H(t,t_5)w(t_5)+\int_{t_5}^{t}\frac{\overline{\delta}(s)H(t,s)-\delta(\sigma(s))h(t,s)H^{\frac{1}{\lambda}}(t,s)}{\delta(\sigma(s))}w(\sigma(s))\Delta s-$$
$$\int_{t_5}^{t}\frac{\beta\delta(s)D(s)H(t,s)}{\delta^{\lambda}(\sigma(s))r^{\frac{1}{\gamma}}(s)}w^{\lambda}(\sigma(s))\Delta s$$
$$\leq H(t,t_5)w(t_5)+\int_{t_5}^{t}\frac{G_+(t,s)}{\delta(\sigma(s))}H^{\frac{1}{\lambda}}(t,s)w(\sigma(s))\Delta s-$$
$$\int_{t_5}^{t}\frac{\beta\delta(s)D(s)H(t,s)}{\delta^{\lambda}(\sigma(s))r^{\frac{1}{\gamma}}(s)}w^{\lambda}(\sigma(s))\Delta s,$$

where

$$G(t,s)=\overline{\delta}(s)H^{1-\frac{1}{\lambda}}(t,s)-\delta(\sigma(s))h(t,s) \text{ and } G_+(t,s)=max\{0,G(t,s)\}.$$

Using Lemma 1.7 with

$$X=[\frac{\beta\delta(s)C(s)H(t,s)}{\delta^{\lambda}(\sigma(s))r^{\frac{1}{\gamma}}(s)}]^{\frac{1}{\lambda}}w(\sigma(s)) \text{ and } Y=[\frac{G_+(t,s)}{\lambda}[\frac{\beta\delta(s)C(s)}{r^{\frac{1}{\gamma}}}]^{\frac{-1}{\lambda}}]^{\frac{1}{\lambda-1}},$$

we get

$$\frac{G_+(t,s)}{\delta(\sigma(s))}H^{\frac{1}{\lambda}}(t,s)w(\sigma(s))-\frac{\beta\delta(s)C(s)H(t,s)}{\delta^{\lambda}(\sigma(s))r^{\frac{1}{\gamma}}(s)}w^{\lambda}(\sigma(s)) \leq \frac{\gamma^{\gamma}}{\beta^{\gamma}(\gamma+1)^{\gamma+1}}\frac{r(s)(G_+(t,s))^{\gamma+1}}{\delta^{\gamma}(s)D^{\gamma}(s)}. \tag{2.33}$$

Substituting from (2.33) into (2.32), we get

$$\frac{1}{H(t,t_5)}\int_{t_5}^{t}\Big[H(t,s)\delta(s)[A_1(s)+q(s)B_1^{\beta}(s)]- \\ \frac{\gamma^{\gamma}}{\beta^{\gamma}(\gamma+1)^{\gamma+1}}\frac{r(s)(G_+(t,s))^{\gamma+1}}{\delta^{\gamma}(s)D^{\gamma}(s)}\Big]\Delta s\le w(t_5),$$

which contradicts (2.30).

**Case** 2. Suppose that $(C_1)$ holds and $x^{\Delta}(t)>0$. Similarly, (2.13) can be written as

$$w^{\Delta}(t)\le-\delta(t)[A_2(t)+q(t)B_2^{\beta}(t)]+\frac{\overline{\delta(t)}_+}{\delta(\sigma(t))}w(\sigma(t))- \\ \frac{\beta\delta(t)D(t)}{(\delta(\sigma(t)))^{\lambda}r^{\frac{1}{\gamma}}(t)}(w(\sigma(t)))^{\lambda}. \tag{2.34}$$

Using the same technique, we obtain

$$\frac{1}{H(t,t_5)}\int_{t_5}^{t}\Big[H(t,s)\delta(s)[A_2(s)+q(s)B_2^{\beta}(s)]- \\ \frac{\gamma^{\gamma}}{\beta^{\gamma}(\gamma+1)^{\gamma+1}}\frac{r(s)(G_+(t,s))^{\gamma+1}}{\delta^{\gamma}(s)D^{\gamma}(s)}\Big]\Delta s\le w(t_5),$$

which contradicts (2.30).

Finally, suppose that $(C_2)$ holds. Then, by Lemma 1.9, we have $\lim_{t\to\infty}x(t)=0$. Thus every solution of Eq. (0.1) is almost oscillatory on $[t_0,\infty)_{\mathbb{T}}$ or converges to zero as $t\to\infty$. This completes the proof. $\square$

**Example 2.5.** Take $\mathbb{T}=[t_4,\infty)_{\mathbb{R}}$ where $t_4\ge1$ and consider the equation

$$\Big[\frac{1}{t^2}(z^{\Delta}(t))^3\Big]^{\Delta}+\frac{1}{t^4}(z^{\Delta}(t))^3+q(t)x^5(t+1)+ \\ \int_0^t\frac{(b_2L(t,t_4))^5m(s)x^5(s)}{s^2}\Delta s=0, t\in[t_4,\infty)_{\mathbb{R}}, \tag{2.35}$$

where

$$z(t)=x(t)-\frac{1}{3}x(\frac{t}{2})+\frac{1}{2}x(t+1),$$

noting that, we take

$$a(t,s)=\frac{(b_2L(t,t_4))^5}{s^2},$$

$$f(s,x(s))=m(s)x^{\beta}(s)=s^2x^5(s),$$

and

$$g(t, x(\tau(t))) = q(t)x^5(\tau(t)) = c_1 x^\beta(t+1),$$

such that $c_1 > 0$ is a positive constant.
Here

$$r(t) = \frac{1}{t^2},\ p(t) = \frac{1}{t^4},\ \gamma = 3, and\ \beta = 5,$$

$\eta_2(t) = \tau(t) = t + 1$ and $p_2(t) = \frac{1}{2}$. Hence we obtain

$$\psi(t, t_4) = 1 \text{ and } B(t) = \frac{2}{3}.$$

Since $\beta > \gamma = 3$, we obtain

$$D(t) = b_1^{\frac{\beta}{3}-1} = b_1^{\frac{2}{3}}.$$

Also, we have

$$1 - \mu(t)\frac{p(t)}{r(t)} = 1 > 0, \qquad \text{for all } t \in [t_4, \infty)_{\mathbb{R}}.$$

Using Lemma 2 in [6], we obtain

$$e_{\frac{-p}{r}}(t, t_4) \geq 1 - \int\limits_{t_4}^{t} \frac{p(s)}{r(s)}\Delta s = 1 - \int_{t_4}^{t} \frac{1}{s^2}\Delta s > \frac{1}{t}, \qquad \text{for all } t \in [t_4, \infty)_{\mathbb{R}},$$

so

$$\int\limits_{t_4}^{t} \big[\frac{1}{r(s)} e_{\frac{-p}{r}}(s, t_4)\big]^{\frac{1}{\gamma}} \Delta s \geq \int\limits_{t_4}^{t} \big[s^2 \frac{1}{s}\big]^{\frac{1}{3}} \Delta s = \int\limits_{t_4}^{t} s^{\frac{1}{3}} \Delta s \to \infty \text{ as } t \to \infty.$$

Hence (1.1) holds. Taking $\delta(t) = t$, then $\overline{\delta(t)} = 1 - \frac{1}{t} > 0$, $\overline{\delta_+(t)} = \frac{t-1}{t}$. Moreover, we can easily obtain

$$A(t) = \min\{(\frac{10}{9})^5 \frac{n_1 t^5(t - t_4)}{(t^{\frac{5}{3}} - t_4^{\frac{5}{3}})^5}, (\frac{10}{9})^5 \frac{n_2 t^5(t - t_4)}{(t^{\frac{5}{3}} - t_4^{\frac{5}{3}})^5}\} = c_2 \frac{t^5(t - t_4)}{(t^{\frac{5}{3}} - t_4^{\frac{5}{3}})^5},$$

where, $c_2 > 0$ is a positive constant. Then (2.1) can be written as

$$\limsup_{t\to\infty} \int\limits_{t_4}^{t} \delta(u)[A(u) + q(u)B^\beta(u)] - \frac{\gamma^\gamma}{\beta^\gamma(\gamma+1)^{\gamma+1}} \frac{r(u)(\overline{\delta(u)}_+)^{\gamma+1}}{\delta^\gamma(u)D^\gamma(u)}\Delta u =$$

$$\limsup_{t\to\infty} \int\limits_{t_4}^{t} c_2 \frac{u^6(u - t_4)}{(u^{\frac{5}{3}} - t_4^{\frac{5}{3}})^5} + c_1(\frac{2}{3})^5 u - \frac{3^3}{\beta^3 b_1^{\frac{\beta}{3}-1}(4)^4} \frac{(u-1)^4}{u^9}\Delta u = \infty.$$

Using Theorem 2.1, we conclude that every solution of (2.35) is almost oscillatory or tends to zero.

*Remark* 2.6. The results of [3], [5], [4], [9] ,[11], [12] and [10] can't be applied to (2.35). But according to Theorem 2.1, we obtain that every solution of (2.35) is almost oscillatory or converges to zero as $t \to +\infty$.

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