Novi Sad J. Math. Vol. 52, No. 1, 2022, 29-38
https://doi.org/10.30755/NSJOM.09557

Curves used in highway design and Bezier curves

Aslı Ayar[1] and Bayram Şahin[23]

Abstract. Certain curves have been widely used in highway design engineering. The most common and important of these curves is the Bloss curve and it is important to investigate the existence of new Bloss curves for computer-aided geometric design (CAGD). In this paper, we consider certain curves that are very well known in CAGD and check the suitability of such curves in highway design engineering.

AMS Mathematics Subject Classification (2010): 65D17; 53A04

Keywords and phrases: Bloss curve; Bezier curve; trigonometric Bezier curve; $\lambda\mu-$Bezier-like curve; transition curve; highway geometric design

1. Introduction

For engineering, highway design is a very complicated task. An important part of the highway design is a geometrical design. One of the most important elements of road geometric design is transition curves. The most important of the transition curves are the spiral curves. The main reasons for these curves are the following; (1) the rate of change or centrifugal acceleration is consistent (smooth), (2) radius changes are linear at any curve point. Spirals don't have cusps, loops or inflection points by definition. Consequently, the spirals are used in many applications such as satellite orbit, railway, highway or robotic design. One of the spiral curves is known as the Bloss curve, which is used extensively in road design. Bloss curve conditions do not allow the graphs of transition curves to generate a sudden curl.

Curves and surfaces are very important for technological and geometric design. These are used in computer-aided design (CAD) and computer-aided manufacturing (CAM). Therefore, research in this area and obtaining new curves have been very important for CAD and CAM. The most known and commonly used in CAD/CAM curves are Bezier curves. Bezier curves are well-known and widely used curve classes for computer-aided geometric design. Sometimes by using Bezier and B-spline curves, a designer faces some problems for obtaining the desired shape. Therefore considering curves with the shape parameters, researchers have started to modify the representation of curves. Various versions of these curves have been given in literature by many authors and their applications have been given. Thus, rational curves, trigonometric curves and Bezier-like curves with exponential functions are obtained [1], [5], [10].

[1]Department of Mathematics, Ege University, Turkey, e-mail: asliayar1@gmail.com
[2]Department of Mathematics, Ege University, Turkey, e-mail: bayram.sahin@ymail.com
[3]Corresponding author

On the other hand, Walton and Meek obtained a cubic spiral Bezier curve and then they constructed transition curves by using this spiral [8]. Later they have extended this result for a more general case [9]. A more general case is considered by Habib and Sakai in [2], [3] and [4].

In this paper, we consider a planar cubic Bezier curve and investigate the suitable conditions for a Bezier curve to be a Bloss curve. But, as an unexpected result, we have seen that in the case of a Bloss curve, a third-order planar Bezier curve is just a line. We also investigated the existence of a planar Bloss $\lambda\mu-$Bezier-like curve with two shape parameters and a planar cubic Bloss trigonometric Bezier curve with two shape parameters. We have seen that these curves are not Bloss curves.

2. Preliminaries

For any vector $\mathbf{p}$, its parametric form is $\mathbf{p} = (p_x, p_y)$. The norm of vector $\mathbf{p}$ is formulated as $|\mathbf{p}| = \sqrt{p_x^2 + p_y^2}$. Positive angles are measured anti-clockwise. The dot product of two vectors, $\mathbf{p}$ and $\mathbf{m}$ is $\mathbf{p}.\mathbf{m} = p_x m_x + p_y m_y$ and the cross product of these two vectors is defined as $\mathbf{p} \times \mathbf{m} = |\mathbf{p}|.|\mathbf{m}|.\sin\theta$ where the angle θ is positive angle. The derivative of the function $\mathbf{f}$ is denoted by $\mathbf{f}^{'}$.

The signed curvature of a Bezier-like curve $\mathbf{f}\,(t)$ is defined by [7]

$$\kappa\,(t) = \frac{\mathbf{f}'\,(t) \times \mathbf{f}''\,(t)}{|\mathbf{f}'\,(t)\,|^3}. \tag{2.1}$$

The radius of curvature r is given by $r = \frac{1}{\kappa}$. If $\kappa\,(t) = 0,\ t \in [0, 1]$ then the curve $\mathbf{f}\,(t)$ reduces to a straight line. If $\kappa'\,(t) = 0$, then the curve has a point of inflection. If $\mathbf{t}$ is the unit tangent vector to $\mathbf{f}\,(t)$ at t, then the orientation of the unit normal vector $\mathbf{n}$ to $\mathbf{f}\,(t)$ at t is such that the anti-clockwise angle from $\mathbf{t}$ to $\mathbf{n}$ is $\pi/2$. We now recall the notions of Bezier Curves and Bloss Curves.

Definition 2.1. [1] For $t \in [0, 1]$, $B_{i,n}(t)$ Bezier basis functions are described by Bernstein polynomials as follows:

$$B_{i,n}\,(t) = \binom{n}{i}\,(t)^i\,(1-t)^{n-i} \tag{2.2}$$

For a cubic Bezier curve, $n = 3$ and its basis functions $B_0\,(t)$, $B_1\,(t)$, $B_2\,(t)$, $B_3\,(t)$, a cubic Bezier curve is defined as

$$\begin{aligned} \mathbf{f}\,(t) &= \sum_{i=0}^{3} B_{i,3}\,(t)\,\mathbf{P}_i \\ &= B_0\,(t)\,\mathbf{P}_0 + B_1\,(t)\,\mathbf{P}_1 + B_2\,(t)\,\mathbf{P}_2 + B_3\,(t)\,\mathbf{P}_3 \\ & t \in [0, 1] \end{aligned} \tag{2.3}$$

where $\mathbf{P}_i$ $(i = 0, 1, 2, 3)$ in E^2 are control points. Additionally, we suppose that the beginning point of the curve is origin i.e. $\mathbf{P}_0 = (0, 0)$ and the other control points are

$$\begin{aligned} \mathbf{P}_1 &= \mathbf{P}_0 + a\mathbf{t}_0, \\ \mathbf{P}_2 &= \mathbf{P}_1 + b\cos\theta\mathbf{t}_0 + b\sin\theta\mathbf{n}_0, \\ \mathbf{P}_3 &= \mathbf{P}_2 + d\cos(\theta + \phi)\mathbf{t}_0 + d\sin(\theta + \phi)\mathbf{n}_0 \end{aligned} \tag{2.4}$$

where $|P_1 - P_0| = a$, $|P_2 - P_1| = b$ and $|P_3 - P_2| = d$. Here θ is a positive angle from $P_1 - P_0$ to $P_2 - P_1$ and ϕ is the positive angle from $P_2 - P_1$ to $P_3 - P_2$, $\mathbf{t}_0$ and $\mathbf{n}_0$ are respectively the unit tangent vector and the unit normal vector at the beginning point of Bezier curve, see also Figure 1.

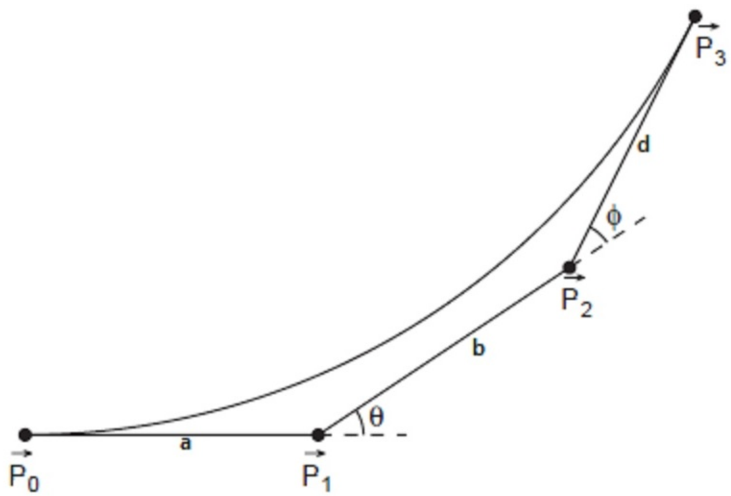

Figure 1: Bezier Curve and Its Control Points

Definition 2.2. [6] If a curve is a spiral then it yields the basic design conditions of the transition curve, i.e the curvature at the starting point should be zero and should rise to the maximum value at the endpoint. Also for $t \in (0, 1)$, its curvature varies monotonically with its arc length. The curve, which in addition to these conditions also satisfies the conditions

$$\kappa'(0) = 0 \tag{2.5}$$

and

$$\kappa'(1) = 0 \tag{2.6}$$

is called a Bloss Curve, where $\kappa(t)$ is the curvature of the curve and $\kappa'(t)$ is the derivative of $\kappa(t)$, i.e. the derivatives of the curvature at the beginning and ending points are zero.

Definition 2.3. [6] If a curve is a Bloss curve then it yields the basic design conditions of a transition curve, i.e the curvature at the starting point should be zero and should rise to the maximum value at the endpoint. Also for $t \in (0,1)$, its curvature varies monotonically with its arc length and

$$\kappa'(0) = 0 \tag{2.7}$$

and

$$\kappa'(1) = 0. \tag{2.8}$$

The curve, which in addition to these conditions also satisfies the conditions

$$\kappa''(0) = 0 \tag{2.9}$$

and

$$\kappa''(1) = 0 \tag{2.10}$$

is called a Grabowski Curve, where $\kappa(t)$ is the curvature of the curve and $\kappa''(t)$ is the second derivatives of $\kappa(t)$, i.e. the second derivative of the curvature at the beginning and ending points are zero.

In the sequel, we give notions of a planar $\lambda\mu-$ Bezier-like curve with two shape parameters and a planar cubic Trigonometric Bezier curve with two shape parameters.

Definition 2.4. [10] For $t \in [0,1]$, $\lambda\mu-$Bezier-like basis functions with two shape parameters $\lambda, \mu \in [0, \infty)$ are defined as

$$\begin{aligned} A_0(t;\lambda) &= (1-t)^3 e^{-\lambda t}, \\ A_1(t;\lambda) &= (1-t)^2 \left[1 + 2t - (1-t) e^{-\lambda t}\right], \\ A_2(t;\mu) &= t^2 \left[3 - 2t - te^{-\mu(1-t)}\right], \\ A_3(t;\mu) &= t^3 e^{-\mu(1-t)}. \end{aligned} \tag{2.11}$$

Given the control points $\mathbf{P}_i$ $(i = 0,1,2,3)$ in E^2, we define the $\lambda\mu-$Bezier-like curve with two shape parameters as

$$\begin{aligned} \mathbf{f}(t;\lambda,\mu) &= \sum_{i=0}^{3} A_i(t;\lambda,\mu)\,\mathbf{P}_i \\ &= A_0(t;\lambda)\,\mathbf{P}_0 + A_1(t;\lambda)\,\mathbf{P}_1 + A_2(t;\mu)\,\mathbf{P}_2 + A_3(t;\mu)\,\mathbf{P}_3 \\ & t \in [0,1], \quad \lambda,\mu \in [0,\infty). \end{aligned} \tag{2.12}$$

Definition 2.5. [5] For $t \in [0,1]$, cubic trigonometric Bezier like basis functions with two shape parameters $\lambda, \mu \in [-2,1]$ are defined as

$$\begin{aligned} C_0(t;\lambda) &= \left(1 - \sin\tfrac{\pi t}{2}\right)^2 \left(1 - \lambda \sin\tfrac{\pi}{2} t\right), \\ C_1(t;\lambda) &= \sin\tfrac{\pi t}{2}\left(1 - \sin\tfrac{\pi t}{2}\right)\left(2 + \lambda - \lambda \sin\tfrac{\pi t}{2}\right), \\ C_2(t;\mu) &= \cos\tfrac{\pi t}{2}\left(1 - \cos\tfrac{\pi t}{2}\right)\left(2 + \mu - \mu \cos\tfrac{\pi t}{2}\right), \\ C_3(t;\mu) &= \left(1 - \cos\tfrac{\pi t}{2}\right)^2 \left(1 - \mu \cos\tfrac{\pi t}{2}\right). \end{aligned} \tag{2.13}$$

Given the control points $\mathbf{P}_i$ $(i = 0, 1, 2, 3)$ in E^2, we define the cubic trigonometric Bezier like curve with two shape parameters as

$$\begin{aligned} \mathbf{f}(t; \lambda, \mu) &= \sum_{i=0}^{3} C_i(t; \lambda, \mu)\, \mathbf{P}_i \\ &= C_0(t; \lambda)\, \mathbf{P}_0 + C_1(t; \lambda)\, \mathbf{P}_1 + C_2(t; \mu)\, \mathbf{P}_2 + C_3(t; \mu)\, \mathbf{P}_3 \\ & \quad t \in [0, 1], \quad \lambda, \mu \in [-2, 1]. \end{aligned} \tag{2.14}$$

3. Examining conditions for Bezier curves and Bezier-like curves to be a Bloss curve

Planar cubic Bezier spirals have been studied by Walton and Meek and certain transition curves have been constructed. Here, we are going to show that there are some restrictions for such curves in highway design.

Theorem 3.1. *The cubic Bezier curve satisfying* (2.4) *is never a Bloss curve.*

Proof. If a Bezier curve is a Bloss curve, from Definition 2.2, its curvature must yield

$$\begin{aligned} \kappa(0) &= 0, \\ \kappa(1) &= c, \\ \kappa'(0) &= 0, \\ \kappa'(1) &= 0 \end{aligned} \tag{3.1}$$

where c is the ending curvature value and $c > 0$. Let $\mathbf{f}(t)$ be a planar cubic Bezier curve. From (2.1), the curvature of a Bezier curve $\mathbf{f}(t)$ is obtained as

$$\kappa(t) = \frac{u(t)}{v(t)}$$

where

$$\begin{aligned} u(t) &= 2\left(ab(t-1)^2 \sin(\theta) + bdt^2 \sin(\phi) - adt(t-1)\sin(\theta + \phi)\right), \\ v(t) &= \left(3\left(\left(a(t-1)^2 - 2b(t-1)t\cos(\theta) + dt^2\cos(\theta + \phi)\right)^2 \right.\right. \\ & \quad \left.\left. + t^2\left(-2b(t-1)\sin(\theta) + dt\sin(\theta + \phi)\right)^2\right)\right)^{3/2}, \end{aligned}$$

θ is the positive angle between $P_1 - P_0$ and $P_2 - P_1$, ϕ is the positive angle between $P_3 - P_2$ and $P_2 - P_1$, see also Figure 1. At the beginning point, the curvature is

$$\kappa(0) = \frac{2b\sin(\theta)}{3a^2}. \tag{3.2}$$

If $\kappa(0) = 0$, then θ must be zero. For $\theta = 0$ and $\kappa(1) = c$, we get

$$b = \frac{3cd^2}{2\sin(\phi)}. \tag{3.3}$$

The derivative of the curvature at the beginning point is

$$\kappa'(0) = \frac{2d\sin(\phi)}{3a^3}. \tag{3.4}$$

From the condition $\kappa'(0) = 0$ of a Bloss curve, ϕ must be zero. If θ and ϕ are both equal to zero, $\mathbf{f}(t)$ is not a curve, it is a line. Thus a Bezier Bloss curve is just a line. □

We now consider new curves other than Bezier curves and check the certain conditions for them to be a Bloss curve.

Theorem 3.2. *A planar $\lambda\mu-$ Bezier-like curve with two shape parameters given in* (2.12) *is not a Bloss curve.*

Proof. Let $\mathbf{f}(t)$ be $\lambda\mu-$ Bezier-like curve with two shape parameters. From (2.1) , the curvature of a Bezier curve $\mathbf{f}(t)$ is obtained as
(3.5)

$$\begin{aligned}\kappa(t) = \; & \Bigg\{ e^{-t\lambda-\mu}\Bigg(-t\Bigg(ae^{\mu}(t-1)\lambda(6-(t-1)\lambda(-6+(t-1)\lambda) \\ & -6be^{t\lambda+\mu}(2t-1)\cos(\theta) + de^{t(\lambda+\mu)}t(6+t\mu(6+t\mu))\cos(\theta+\phi)\Bigg) \\ & \Bigg(de^{(t-1)\mu}t(3+t\mu)\sin(\theta+\phi) - 6b(t-1)\sin(\theta)\Bigg) \\ & +\Bigg(e^{\mu}(t-1)(a(t-1)(3-(-1+t)\lambda) - 6be^{t\lambda}t\cos\theta) \\ & +de^{t(\lambda+\mu)}t^2(3+t\mu)\cos(\theta+\phi)\Bigg) \\ & \Bigg(6b(1-2t)\sin\theta + de^{(t-1)\mu}t(6+t\mu(6+t\mu))\sin(\theta+\phi)\Bigg)\Bigg)\Bigg\} \Bigg/ \\ & \Bigg\{ \Bigg(ae^{\mu}(t-1)^2((t-1)\lambda-3) + 6be^{t\lambda+\mu}(t-1)t\cos\theta - \\ & de^{t(\lambda+\mu)}t^2(3+t\mu)\cos(\theta+\phi)\Bigg)^2 e^{-2(t\lambda+\mu)} + t^2\Bigg(-6b(t-1)\sin\theta \\ & +de^{(t-1)\mu}t(3+t\mu)\sin(\theta+\phi)\Bigg)^2\Bigg\}^{3/2}\end{aligned}$$

where θ is the positive angle between $P_1 - P_0$ and $P_2 - P_1$, ϕ is the positive angle between $P_3 - P_2$ and $P_2 - P_1$. At the beginning point, the curvature is

$$\kappa(0) = \frac{6b\sin(\theta)}{(3+\lambda)^2 a^2}. \tag{3.6}$$

If $\kappa(0) = 0$, then θ must be zero. For $\theta = 0$ and $\kappa(1) = c$, we have

$$b = \frac{cd^2(3+\mu)^2}{6\sin\phi}. \tag{3.7}$$

The derivative of the curvature at the beginning point is

$$\kappa'(0) = \frac{6de^{-\mu}\sin\phi}{a^2(3+\lambda)^2}. \tag{3.8}$$

From the condition $\kappa'(0) = 0$ of a Bloss curve, ϕ must be zero. If θ and ϕ are both equal to zero, $\mathbf{f}(t)$ is not a curve, it is a line. Therefore, there is no planar $\lambda\mu-$ Bezier-like curve with two shape parameters satisfying Bloss curve conditions, other than lines. □

Moreover, we have the following result.

Theorem 3.3. *If the control points of a planar cubic trigonometric Bezier curve with two shape parameters given in* (2.14) *are*

$$\begin{aligned} \mathbf{P}_1 &= \mathbf{P}_0 + a\mathbf{t}_0, \\ \mathbf{P}_2 &= \mathbf{P}_1 + b\cos\theta\mathbf{t}_0 + b\sin\theta\mathbf{n}_0, \\ \mathbf{P}_3 &= \mathbf{P}_2 + d\cos(\theta+\phi)\mathbf{t}_0 + d\sin(\theta+\phi)\mathbf{n}_0 \end{aligned} \tag{3.9}$$

where

$$a = b = \frac{2(1+2\mu)^2\sin\phi(\cos(\phi)+\sin\phi)^2}{c(2+\mu)^2}, \tag{3.10}$$

$$d = \frac{2(1+2\mu)\sin\phi(\cos(\phi)+\sin\phi)}{c(2+\mu)^2}, \tag{3.11}$$

$\lambda = \mu = 1$, *the positive angle between* $P_1 - P_0$ *and* $P_2 - P_1$ *is* $\theta = 0$, $0 < \phi < \frac{\pi}{2}$ *is the positive angle between* $P_3 - P_2$ *and* $P_2 - P_1$, *then this curve is a Bloss curve.*

Proof. Let $\mathbf{f}(t)$ be a planar cubic trigonometric Bezier curve. From (2.1), the curvature of a Bezier curve $\mathbf{f}(t)$ is obtained as

(3.12)

$$\begin{aligned} \kappa(t) = & \Bigg\{ \bigg(a\cos(\tfrac{\pi t}{2})(4+5\lambda-3\lambda\cos(\pi t)) + (-4\cos(\tfrac{\pi t}{2})(a(1+2\lambda)-b\cos\theta) \\ & +4d(2+\mu-3\mu\cos(\tfrac{\pi t}{2}))\cos(\alpha)\sin^2(\tfrac{\pi t}{2}))\sin(\tfrac{\pi t}{2})\bigg)\bigg(8b\cos(\pi t)\sin\theta \\ & -4d\left((\mu-4)(1+2\cos(\tfrac{\pi t}{2}))+9\mu\cos(\pi t)\right)\sin^2(\tfrac{\pi t}{4})\sin(\alpha)\bigg) \\ & -8\bigg(-4cos(\pi t)(a(1+2\lambda)-b\cos\theta)-2d\big((\mu-4)(1+2\cos(\tfrac{\pi t}{2})) \\ & +9\mu\cos(\pi t)\big)\cos(\alpha)\sin^2(\tfrac{\pi t}{4})+a(\lambda-4+9\lambda\cos(\pi t))\sin(\tfrac{\pi t}{2})\bigg) \\ & \bigg(\tfrac{1}{2}b\sin(\pi t)\sin\theta+2d\cos(\tfrac{\pi t}{4})(2+\mu-3\mu\cos(\tfrac{\pi t}{2}))\sin^3(\tfrac{\pi t}{4})\sin(\alpha)\bigg)\Bigg\} \Bigg/ \\ & \Bigg\{ \big(a\cos(\tfrac{\pi t}{2})(4+5\lambda-3\lambda\cos(\pi t))+(-4\cos(\tfrac{\pi t}{2})(a(1+2\lambda)-b\cos\theta)+ \\ & 4d(2+\mu-3\mu\cos(\tfrac{\pi t}{2}))Cos(\theta+\phi)\sin^2(\tfrac{\pi t}{2}))\sin(\tfrac{\pi t}{2})\big)^2 \\ & +16\big(\tfrac{1}{2}b\sin(\pi t)\sin\theta+2d\cos(\tfrac{\pi t}{4})(2+\mu-3\mu\cos(\tfrac{\pi t}{2}))\sin^3(\tfrac{\pi t}{4})\sin(\alpha)\big)^2\Bigg\}^{3/2} \end{aligned}$$

where $\alpha = \theta + \phi$, θ is the positive angle between $P_1 - P_0$ and $P_2 - P_1$, ϕ is the positive angle between $P_3 - P_2$ and $P_2 - P_1$.

At the beginning point, the curvature is

$$\kappa(0) = \frac{2b\sin(\theta)}{(2+\lambda)^2 a^2}. \tag{3.13}$$

If $\kappa(0) = 0$, then θ must be zero. For $\theta = 0$ and $\kappa(1) = c$, we have

$$b = \frac{cd^2(2+\mu)^2}{2\sin\phi}. \tag{3.14}$$

The derivative of the curvature at the beginning point is

$$\kappa'(0) = 0. \tag{3.15}$$

Since $\kappa(1) = c$, the derivative of the curvature at the endpoint must be

$$\kappa'(1) = \frac{3cd\pi(-2-4\mu+cd(2+\mu)^2\cot\phi)}{2(2+\mu)} = 0. \tag{3.16}$$

From this equation, we obtain

$$d = \frac{2(1+2\mu)\tan\phi}{c(2+\mu)^2}. \tag{3.17}$$

If

$$a = b = \frac{2(1+2\mu)^2\sin\phi\tan\phi}{c(2+\mu)^2}, \tag{3.18}$$

$$d = \frac{2(1+2\mu)\tan\phi}{c(2+\mu)^2}, \tag{3.19}$$

$\lambda = \mu = 1$, the positive angle between $P_1 - P_0$ and $P_2 - P_1$ is $\theta = 0$, $0 < \phi < \frac{\pi}{2}$ is the positive angle between $P_3 - P_2$ and $P_2 - P_1$ then $\kappa(0) = \kappa'(0) = \kappa'(1) = 0$ and $\kappa'(t)$ does not change sign in $t \in [0,1]$. Thus this curve is a spiral and it is also a Bloss curve.

□

We now check the conditions for a planar cubic trigonometric Bezier curve to be a Grabowski curve.

Theorem 3.4. *A planar cubic trigonometric Bezier curve with two shape parameters given in* (2.14) *is not a Grabowski curve.*

Proof. Let $\mathbf{f}(t)$ be a planar cubic trigonometric Bezier curve. From Theorem 3.3, if $\mathbf{f}(t)$ is a Bloss curve then the positive angle between $P_1 - P_0$ and $P_2 - P_1$ is $\theta = 0$. Also, since

$$a = b = \frac{2(1+2\mu)^2\sin\phi\tan\phi}{c(2+\mu)^2} \tag{3.20}$$

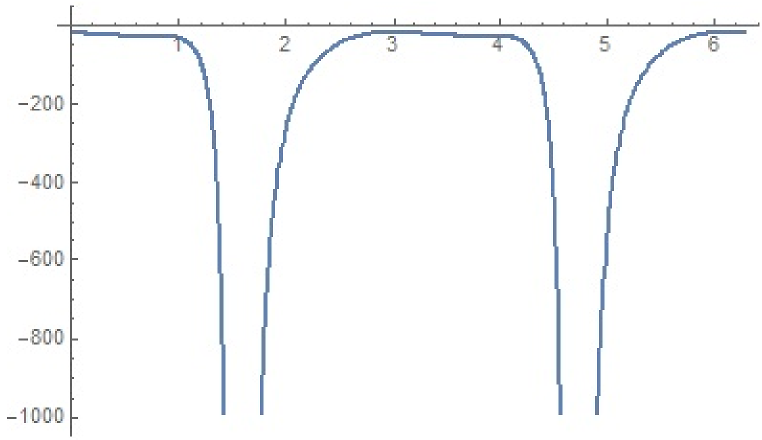

Figure 2: Graph of $\kappa''(1)$ for $\phi \in (0, 2\pi)$

and

$$d = \frac{2(1+2\mu)\tan\phi}{c(2+\mu)^2}, \tag{3.21}$$

at the beginning point,

$$\kappa(0) = 0, \tag{3.22}$$

$$\kappa'(0) = 0 \tag{3.23}$$

and at the ending point,

$$\kappa'(1) = 0. \tag{3.24}$$

From the conditions of Grabowski curve in definition 2.5, at the beginning and ending point, the second derivative of curvature of $\mathbf{f}(t)$ must be zero, i.e,

$$\kappa''(0) = 0 \tag{3.25}$$

and

$$\kappa''(1) = 0. \tag{3.26}$$

Firstly, from Theorem 3.3, $\mathbf{f}(t)$ is a Bloss curve $\lambda = \mu = 1$. Therefore we obtain

$$\kappa''(0) = 0. \tag{3.27}$$

Also,

$$\kappa''(1) = \frac{1}{2}c\pi^2(\cos(2\phi) - 5)\sec^2(\phi) \tag{3.28}$$

is not equal to zero for $\phi \in (0, \pi/2)$, see Figure 2. So, this curve is not a Grabowski curve.

□

Concluding remarks

Bloss curves are used in highway design and it is an important subject of the research area of civil engineering. In computer-aided engineering design literature, there are many new curves defined in papers. It will be convenient to find appropriate curves satisfying Bloss curves conditions. In this way, there will be many new tools for highway designers to use in practical applications our next research problem will be in this direction.

References

[1] Farin, G. *Curves and Surfaces for CAGD: A Practical Guide*, 5th ed. Morgan Kaufmann Publishers Inc., San Francisco, CA, USA, 2001.

[2] Habib, Z., and Sakai, M. Spiral transition curves and their applications. *Sci. Math. Jpn. 61*, 2 (2005), 195–206.

[3] Habib, Z., and Sakai, M. Cubic spiral transition matching G^2 Hermite end conditions. *Numer. Math. Theory Methods Appl. 4*, 4 (2011), 525–536.

[4] Habib, Z., and Sakai, M. Fairing arc spline and designing by using cubic Bezier spiral segments. *Math. Model. Anal. 17*, 2 (2012), 141–160.

[5] Han, X.-A., Ma, Y., and Huang, X. The cubic trigonometric bezier curve with two shape parameters. *Appl. Math. Lett. 22*, 2 (2009), 226–231.

[6] Kobryń, A. *Transition curves for highway geometric design*, vol. 14 of *Springer Tracts on Transportation and Traffic*. Springer, Cham, 2017.

[7] Millman, R. S., and Parker, G. D. *Elements of differential geometry.* Prentice-Hall Inc., Englewood Cliffs, N. J., 1977.

[8] Walton, D. J., and Meek, D. S. A planar cubic bezier spiral. *Journal of Computational and Applied Mathematics 72*, 1 (1996), 85 – 100.

[9] Walton, D. J., and Meek, D. S. A further generalisation of the planar cubic Bézier spiral. *J. Comput. Appl. Math. 236*, 11 (2012), 2869–2882.

[10] Zhu, Y., and Han, X. Curves and surfaces construction based on new basis with exponential functions. *Acta Appl. Math. 129* (2014), 183–203.

Received by the editors June 2, 2019
First published online January 7, 2021