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On the Zariski topology over the primary-like spectrum

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Abstract. Let R be a commutative ring with identity and M be a unital R-module. The primary-like spectrum $\mathcal{PS}(M)$ has a topology which is a generalization of the Zariski topology on the prime spectrum $\mathrm{Spec}(R)$. We get several topological properties of $\mathcal{PS}(M)$, mostly for the case when the continuous mapping $\phi : \mathcal{PS}(M) \to \mathrm{Spec}(R/\mathrm{Ann}(M))$ defined by $\phi(Q) = \sqrt{(Q : M)}/\mathrm{Ann}(M)$ is surjective or injective. For example, if ϕ is surjective, then $\mathcal{PS}(M)$ is a connected space if and only if $\mathrm{Spec}(R/\mathrm{Ann}(M))$ is a connected space. It is shown that if ϕ is surjective, then a subset Y of $\mathcal{PS}(M)$ is irreducible if and only if Y is the closure of a singleton set. It is also proved that if the image of ϕ is a closed subset of $\mathrm{Spec}(R/\mathrm{Ann}(M))$, then $\mathcal{PS}(M)$ is a spectral space if and only if ϕ is injective.

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1. Introduction

Throughout this paper all rings are commutative with identity and all modules are unital. Let R be a ring and M be an R-module. For a submodule N of M, we let $(N : M) = \{r \in R \mid rM \subseteq N\}$. As usual, we denote the annihilator ideal $((0) : M)$ by $\mathrm{Ann}(M)$. A proper submodule P of M with $p = (P : M)$ is called a *prime submodule* (or a *p-prime submodule*) of M, if for $r \in R$ and $m \in M$, $rm \in P$ implies that either $r \in p$ or $m \in P$. The *prime spectrum* of M, denoted $\mathrm{Spec}(M)$, is the set of all prime submodules of M. Also for a prime ideal p of R, $\mathrm{Spec}_p(M)$ will denote the set of all p-prime submodules of M. The intersection of all prime submodules of M containing N, denoted $\mathrm{rad}\,N$, is called the *radical* of N. If there is no prime submodule containing N, $\mathrm{rad}\,N$ is defined to be M. In the ideal case, the radical of I is denoted by $\sqrt{I}$. As a generalization of a primary ideal one hand and a generalization of the prime submodule on the other hand, a proper submodule Q of M is called a *primary-like submodule*, if for $r \in R$ and $m \in M$, $rm \in Q$ implies either $r \in (Q : M)$ or $m \in \mathrm{rad}\,Q$ [9]. We say that a submodule N of a nonzero R-module M satisfies the *primeful property* if for each prime ideal p of R with $(N : M) \subseteq p$, there exists a prime submodule P containing N such

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that $(P : M) = p$. If the zero submodule of M satisfies the primeful property, then M is called a *primeful module* [12]. For example, finitely generated modules and projective modules over domains are two classes of primeful modules [12, Propositin 3.8, Corollary 4.3]. The *primary-like spectrum* of M, denoted $\mathcal{PS}(M)$, is defined to be the set of all primary-like submodules of M satisfying the primeful property. If N is a submodule of M satisfying the primeful property, then $(\operatorname{rad} N : M) = \sqrt{(N : M)}$ [12, Proposition 5.3]. It is easily seen that if Q is a primary-like submodule satisfying the primeful property, then $p = \sqrt{(Q : M)}$ is a prime ideal of R. Therefore by a *p-primary-like submodule* Q of M, we mean that Q is a primary-like submodule satisfying the primeful property with $p = \sqrt{(Q : M)}$. The set of such submodules is denoted by $\mathcal{PS}_p(M)$. It should be noted that if $Q \in \mathcal{PS}_p(M)$ and m is a maximal ideal of R containing p, then there is a prime submodule P containing Q such that $(P : M) = m$. It follows that $\operatorname{rad} Q \neq M$ for all $Q \in \mathcal{PS}(M)$.

In recent years, several generalizations of the Zariski topology from rings to modules have been introduced and studied from various points of views (see, for example, [2, 5, 7, 11, 13, 9]).

One of them is the *Zariski topology* on $\operatorname{Spec}(M)$ which is described by taking the set $\{V(N) \mid N \text{ is a submodule of } M\}$ as the set of closed sets of $\operatorname{Spec}(M)$, where $V(N) = \{P \in \operatorname{Spec}(M) \mid (P : M) \supseteq (N : M)\}$ [11, 7].

Now, we set $\nu(N) = \{Q \in \mathcal{PS}(M) \mid \sqrt{(Q : M)} \supseteq \sqrt{(N : M)}\}$ for every submodule N of M. As in the case of the Zariski topology on $\operatorname{Spec}(M)$, the class of varieties $\Omega(M) = \{\nu(N) \mid N \text{ is a submodule of } M\}$ satisfies all axioms of closed sets in a topological space [9, Lemma 1]. Throughout this paper, it is assumed that $\mathcal{PS}(M)$ is equipped with this topology which enjoys analogs of many of the properties of the Zariski topology on $\operatorname{Spec}(M)$. We have already obtained some of the topological properties of this space in [9]. For instance, in [9, Lemma 5], it has been shown that the set $\mathcal{B} = \{\mathcal{PS}(M) \setminus \nu(rM) \mid r \in R\}$ forms a basis for this topology on $\mathcal{PS}(M)$. Furthermore, every finite intersection of the elements of $\mathcal{B}$ is a quasi-compact subspace of $\mathcal{PS}(M)$ [9, Theorem 3].

In this paper, we examine the properties of certain mappings between the primary-like spectrum $\mathcal{PS}(M)$ of M and the spectrums $\operatorname{Spec}(R/Ann(M))$ and $\operatorname{Spec}(M)$, in particular considering when these mappings are continuous or homeomorphisms (Proposition 2.8, Theorem 2.9 and Corollary 2.10). It is shown that $\mathcal{PS}(M)$ is connected if and only if $\operatorname{Spec}(R/Ann(M)$ is a connected space (Proposition 2.12). Hochster's characterization of a spectral space involves an irreducibility discussion in $\mathcal{PS}(M)$. It is shown that for any finitely generated module M, every irreducible subspace of $\mathcal{PS}(M)$ is the closure of a singleton set (Theorem 3.8). In particular, if M is a finitely generated R-module, then $\mathcal{PS}(M)$ is a spectral space, i.e., $\mathcal{PS}(M)$ is homeomorphic with $\operatorname{Spec}(S)$ for some commutative ring S (Theorem 4.4).

2. Continuous mappings between spectrums

As shown in [11, Proposition 3.1], $\psi : \mathrm{Spec}(M) \to \mathrm{Spec}(R/\mathrm{Ann}(M))$ defined by $\psi(P) = (P : M)/Ann(M)$ is a continuous mapping. In [9, Proposition1], we have introduced the mappings $\phi : \mathcal{PS}(M) \to \mathrm{Spec}(R/\mathrm{Ann}(M))$ by $\phi(Q) = \sqrt{(Q : M)}/Ann(M)$ which is continuous, and plays a role analogous to that of ψ. Here, we introduce $\rho : \mathcal{PS}(M) \to \mathrm{Spec}(M)$ defined by $\rho(Q) = S_p(Q + pM)$, in which $p = \sqrt{(Q : M)}$ and

$$S_p(Q + pM) = \{m \in M \mid \exists c \in R \setminus p,\ cm \in Q + pM\}.$$

By [12, Proposition 4.4], ρ is well defined. Note that $\phi = \psi \circ \rho$. It is shown that ρ is a continuous mapping (Proposition 2.8), and the conditions under which ρ is injective, surjective, closed and open are examined.

An R-module M is called a multiplication module, if every submodule of M has the form IM. In this case, we can take $I = (N : M)$ (see, for example, [8]). It is easy to see that if M is a multiplication R-module, then ψ is injective.

Proposition 2.1. *Let M be an R-module. Consider the following statements.*

(1) If $\nu(Q) = \nu(Q')$ for $Q, Q' \in \mathcal{PS}(M)$, then $Q = Q'$.

(2) $|\mathcal{PS}_p(M)| \leq 1$ for every $p \in Spec(R)$.

(3) ϕ is injective.

(4) ρ is injective.

Then (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Rightarrow$ (4). *Moreover, if M is a multiplication R-module, then* (4) $\Rightarrow$ (3).

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) Follows from [9, Proposition 2].
(3) $\Rightarrow$ (4) Clear.
(4) $\Rightarrow$ (3) Since M is a multiplication module, ψ is injective. Now since $\phi = \psi\circ\rho$ and ρ is injective, we conclude that ϕ is injective. □

The following example shows that (4) $\Rightarrow$ (3) in Proposition 2.1 is not true in general.

Example 2.2. Let V be a vector space over a field F with $\dim_F V > 1$. It is evident that $\mathcal{PS}(V)$ and $\mathrm{Spec}(V)$ are the set of all proper vector subspaces of V. Now, since $\phi(Q) = \phi(Q') = 0$ for all distinct subspaces $Q, Q' \in \mathcal{PS}(V)$, ϕ is not injective. On the other hand ρ is injective, because if $\rho(Q) = \rho(Q')$ for $Q, Q' \in \mathcal{PS}(V)$, then $S_{(0)}(Q) = S_{(0)}(Q')$ which follows that $Q = Q'$.

Proposition 2.3. *Let M be an R-module. Consider the following statements:*

(1) $\mathcal{PS}_p(M) \neq \emptyset$ for every $p \in V(Ann(M))$.

(2) ϕ is surjective.

(3) ψ is surjective.

(4) $pMp \neq Mp$ *for every* $p \in V(Ann(M))$.

(5) $S_p(pM)$ *is a* p*-prime submodule of* M *for every* $p \in V(Ann(M))$.

(6) $\mathrm{Spec}_p(M) \neq \emptyset$, *for every* $p \in V(Ann(M))$.

Then $(1) \Leftrightarrow (2) \Rightarrow (3) \Leftrightarrow (4) \Leftrightarrow (5) \Leftrightarrow (6)$. *Moreover, all of these conditions are equivalent in the following cases:*

(a) M *is a multiplication* R*-module.*

(b) M *is a projective* R*-module.*

(c) M *is a faithfully flat* R*-module.*

Proof. $(1) \Leftrightarrow (2) \Rightarrow (3)$ Clear.
$(3) \Leftrightarrow (4) \Leftrightarrow (5) \Leftrightarrow (6)$ Follows from [12, Theorem 2.1].
$(6) \Rightarrow (1)$ (a) Let M be a multiplication R-module. Let $p \in V(Ann(M))$ and $P \in \mathrm{Spec}_p(M)$. Since M is a multiplication module, we have $P = pM$. Suppose $p' \in \mathrm{Spec}(R)$ and $p \subseteq p'$. By [12, Theorem 2.1], there exists a prime submodule P' of M such that $(P' : M) = p'$. It follows that $P = pM \subseteq p'M = P'$. Hence P satisfies the primeful property and so $\mathcal{PS}_p(M) \neq \emptyset$.
(b) Let M be a projective R-module. Let $p \in V(Ann(M))$ and $P \in \mathrm{Spec}_p(M)$. By [1, Corollary 2.3], pM is a prime submodule of M. Also, pM satisfies the primeful property and $(pM : M) = p$ by [12, Corollary 4.3 and Proposition 4.5]. Therefore $pM \in \mathcal{PS}_p(M)$.
(c) Let M be a faithfully flat R-module. Let $p \in V(Ann(M))$ and $P \in \mathrm{Spec}_p(M)$. By [4, Corollary 2.6 (ii)], pM is a prime submodule of M. Also, pM satisfies the primeful property and $(pM : M) = p$ by [12, Corollary 4.3 and Proposition 4.5]. Therefore $pM \in \mathcal{PS}_p(M)$. □

The following example shows that $(3) \Rightarrow (2)$ in Proposition 2.3 is not true in general.

Example 2.4. Let Ω be the set of all prime integers p and $M = \prod_{p \in \Omega} \frac{\mathbb{Z}}{p\mathbb{Z}}$. By [12, Example 1] the submodule (0) of M satisfies the primeful property (i.e., ψ is surjective) and $\mathrm{rad}(0) = (0)$. It follows that (0) is not a primary-like submodule of M, since it is not a prime submodule of M. Thus $\underline{(0)} \notin \mathcal{PS}(M)$. Now, we show that ϕ is not surjective. If, on the contrary, $\phi(Q) = \overline{(0)}$ for some $Q \in \mathcal{PS}(M)$, then $(Q : M) = 0$. Let $(Q : M) \subseteq p$ for $(0) \neq p \in \mathrm{Spec}(\mathbb{Z})$. Since pM is the only prime submodule with $(pM : M) = p$ and Q satisfies the primeful property, we have $Q \subseteq pM$. Thus $Q \subseteq \cap_{p \in \Omega} pM = (0)$ and so $Q = (0)$. This is a contradiction, since $(0) \notin \mathcal{PS}(M)$.

Corollary 2.5. *If* $\mathcal{PS}_p(M)$ *is a singleton set for every* $p \in Spec(R)$*, then* ϕ *is a bijective map and therefore* $\mathrm{Spec}(R/\mathrm{Ann}(M))$ *is a singleton set.*

Proof. By Proposition 2.1 and Proposition 2.3. □

Corollary 2.6. *Let M be a multiplication R-module. If ρ is injective and ψ is surjective, then ϕ is bijective. In this case $\mathcal{PS}(M)$ and* $\mathrm{Spec}(R/\mathrm{Ann}(M))$ *are homeomorphic.*

Proof. By Proposition 2.1, Proposition 2.3 and [9, Theorem 1]. □

Proposition 2.7. *Let M be a finitely generated multiplication R-module. Then ϕ and ρ are surjective, and ψ is bijective.*

Proof. Since M is a multiplication R-module, it is evident that ψ is injective. Let $\bar{p} \in \mathrm{Spec}(R/\mathrm{Ann}(M))$. Since M is finitely generated, by [8, Theorem 2.5], $pM \neq M$. Thus by [8, Corollary 2.11], $pM \in \mathrm{Spec}(M)$ and $\psi(pM) = \bar{p}$. Hence ψ is surjective and so by Proposition 2.3, ϕ is surjective. Now, let $P \in \mathrm{Spec}(M)$. Then there exists a $Q \in \mathcal{PS}(M)$ such that $\phi(Q) = \psi(P)$, i.e., $\sqrt{(Q:M)} = (P:M)$. By [12, Proposition 4.4], $(S_p(Q+pM):M) = (P:M)$ where $p = (P:M)$. Since M is a multiplication R-module, we have $\rho(Q) = S_p(Q+pM) = P$. Hence ρ is surjective. □

Let M be an R-module. From now on, we will denote $R/\mathrm{Ann}(M)$ by $\bar{R}$ and any ideal $I/\mathrm{Ann}(M)$ of $\bar{R}$ by $\bar{I}$. By [11, Proposition 3.1], $\psi^{-1}(V(\bar{I})) = V(IM)$, for every ideal $I \in V(Ann(M))$. Also, by [9, Proposition 1], we have $\phi^{-1}(V(\bar{I})) = \nu(IM)$, for every ideal $I \in V(Ann(M))$. Therefore both ψ and ϕ are continuous. Now we give a similar result for ρ.

Proposition 2.8. *Let M be a R-module. Then $\rho^{-1}(V(N)) = \nu(N)$, for every submodule N of M. Therefore ρ is a continuous mapping.*

Proof. Let $Q \in \rho^{-1}(V(N))$. Then $\rho(Q) \in V(N)$, and so $(S_p(Q+pM):M) \supseteq (N:M)$ in which $p = \sqrt{(Q:M)}$. Hence we have

$$\sqrt{(Q:M)} \supseteq \sqrt{(S_p(Q+pM):M)} \supseteq \sqrt{(N:M)}.$$

Thus $Q \in \nu(N)$, so that $\rho^{-1}(V(N)) \subseteq \nu(N)$. For the reverse inclusion, let $Q \in \nu(N)$. It follows that, $p = \sqrt{(Q:M)} \supseteq \sqrt{(N:M)} \supseteq (N:M)$. Thus

$$(S_p(Q+pM):M) \supseteq (pM:M) \supseteq ((N:M)M:M) = (N:M),$$

which shows that $S_p(Q+pM) \in V(N)$, i.e., $\rho(Q) \in V(N)$. Thus $Q \in \rho^{-1}(V(N))$ so that $\nu(N) \subseteq \rho^{-1}(V(N))$. □

In [11, Theorem 3.6], it has been shown that that if ψ is a surjective map, then $\psi(V(N)) = V(\overline{\sqrt{(N:M)}})$ and $\psi(\mathrm{Spec}(M) - V(N)) = \mathrm{Spec}(\bar{R}) - V(\overline{\sqrt{(N:M)}})$, for every submodule N of M. Also, by [9, Theorem 1], $\phi(\nu(N)) = V(\overline{\sqrt{(N:M)}})$ and $\phi(\mathcal{PS}(M) - \nu(N)) = \mathrm{Spec}(\bar{R}) - V(\overline{\sqrt{(N:M)}})$, for every submodule N of M. Now we give a similar result for ρ.

Theorem 2.9. *Let M be an R-module. Then if ρ is surjective, then for every submodule N of M, $\rho(\nu(N)) = V(N)$ and $\rho(\mathcal{PS}(M) - \nu(N)) = \mathrm{Spec}(M) - V(N)$. Therefore ρ is closed and open.*

Proof. Let N be a submodule of M. Using Proposition 2.8, $\rho^{-1}(V(N)) = \nu(N)$. Then $\rho(\nu(N)) = \rho(\rho^{-1}(V(N))) = V(N)$. Also, we have

$$\begin{aligned}\rho(\mathcal{PS}(M) - \nu(N)) &= \rho(\rho^{-1}(\mathrm{Spec}(M)) - \rho^{-1}(V(N))) \\ &= \rho(\rho^{-1}(\mathrm{Spec}(M) - (V(N)))) \\ &= \mathrm{Spec}(M) - V(N).\end{aligned}$$

□

Corollary 2.10. *Let ρ be as before. Then ρ is a bijection if and only if ρ is a homeomorphism.*

Proof. By Theorem 2.9. □

Corollary 2.11. *Let M be a finitely generated multiplication R-module. Then ρ is a homeomorphism if and only if ϕ is a homeomorphism.*

Proof. $\Rightarrow$) By Proposition 2.1, Proposition 2.7 and [9, Proposition 1].
$\Leftarrow$) By Proposition 2.1, Proposition 2.7 and Theorem 2.9. □

Proposition 2.12. *Let ϕ be a surjective map. Then the following statements are equivalent.*

(1) $\mathcal{PS}(M)$ *is connected;*

(2) $\mathrm{Spec}(\bar{R})$ *is connected;*

(3) The ring $\bar{R}$ contains no idempotent other than $\bar{0}$ and $\bar{1}$;

(4) $\mathrm{Spec}(M)$ *is connected.*

Proof. (1)$\Rightarrow$(2) Since ϕ is a continuous map, ϕ preserves connectedness. Hence $\mathrm{Spec}(\bar{R})$ is connected.
(2)$\Rightarrow$(1) If $\mathcal{PS}(M)$ is disconnected, then $\mathcal{PS}(M)$ must contain a non-empty proper subset Y that is both open and closed. Accordingly, $\phi(Y)$ is a non-empty subset of $\mathrm{Spec}(\bar{R})$ that is both open and closed by Theorem 2.9. Since Y is open, $Y = \mathcal{PS}(M) - \nu(N)$ for some submodule N of M whence by Theorem 2.9, $\phi(Y) = \mathrm{Spec}(\bar{R}) - \phi^{-1}(V(\overline{\sqrt{(N : M)}}))$. Therefore, if $\phi(Y) = \mathrm{Spec}(\bar{R})$, then $V(\overline{\sqrt{(N : M)}})) = \emptyset$. Thus $\overline{\sqrt{(N : M)}} = \bar{R}$, and so $N = M$. It follows that $Y = \mathcal{PS}(M) - \nu(N) = \mathcal{PS}(M) - \nu(M) = \mathcal{PS}(M)$, which is impossible, since Y is a proper subset of $\mathcal{PS}(M)$. Thus $\phi(Y)$ is a proper subset of $\mathrm{Spec}(\bar{R})$ so that $\mathrm{Spec}(\bar{R})$ is disconnected, a contradiction.
(2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) follows from [11, Corollary 3.8]. □

Lemma 2.13. *Let M and M' be R-modules and N' a submodule of M'. Let $f : M \to M'$ be an epimorphism. Then the following hold.*

(1) If N' is a primary-like submodule of M', then $f^{-1}(N')$ is a primary-like submodule of M.

(2) If N' satisfies the primeful property, then so does $f^{-1}(N')$.

Proof. (1) Suppose $rm' \in f^{-1}(N')$ and $r \notin (f^{-1}(N') : M)$. Thus $rf(m') \in N'$ and $r \notin (N' : M')$. Now, since $f^{-1}(\operatorname{rad} N') \subseteq \operatorname{rad}(f^{-1}(N'))$, we have $m' \in \operatorname{rad}(f^{-1}(N'))$, as required.
(2) Let p be a prime ideal such that $(f^{-1}(N') : M) \subseteq p$. Suppose $r \in (N' : M')$. Hence $f(rM) = rf(M) = rM' \subseteq N'$. Thus $rM \subseteq f^{-1}(N')$ and so $r \in p$. Therefore $(N' : M') \subseteq p$. Then there exists a prime submodule P' of M' containing N' such that $(P' : M') = p$. It is easily seen that $(f^{-1}(P') : M) = p$. Thus $f^{-1}(N')$ satisfies the primeful property. □

Theorem 2.14. *Let M and M' be R-modules and $f : M \to M'$ be a epimorphism. Then the following hold:*

(1) The mapping $\sigma_f : \operatorname{Spec}(M') \to \operatorname{Spec}(M)$ defined by $P' \mapsto f^{-1}(P')$ is an injective continuous map.

(2) The mapping $\mu_f : \mathcal{PS}(M') \to \mathcal{PS}(M)$ defined by $Q' \mapsto f^{-1}(Q')$ is an injective continuous map.

(3) If $g : M' \to M''$ is an epimorphism, then $\mu_{g\circ f} = \mu_f \circ \mu_g$ and $\sigma_{g\circ f} = \sigma_f \circ \sigma_g$.

(4) $\mu(1_{\operatorname{Spec}(M)}) = 1_{\mathcal{PS}(M)}$ and $\sigma(1_M) = 1_{\operatorname{Spec}(M)}$ in which $1_{\mathcal{PS}(M)}$ and $1_{\operatorname{Spec}(M)}$ are identity maps over $\operatorname{Spec}(M)$ and $\mathcal{PS}(M)$ respectively.

(5) $\rho_M \circ \mu_f = \sigma_f \circ \rho_{M'}$ in which ρ_M and $\rho_{M'}$ are the same ρ related to M and M' respectively.

Proof. (1) Follows from [11, Proposition 3.9].
(2) It is clear that μ_f is well-defined by Lemma 2.13. It is also injective, since f is surjective. Let $Q' \in \mathcal{PS}(M')$ and $\nu(N)$ be a closed subset of $\mathcal{PS}(M)$. Now, by using the fact that $\nu(N) = \nu(\sqrt{(N : M)}M)$, we have

$$\begin{aligned} Q' \in {\mu_f}^{-1}(\nu(N)) &\Leftrightarrow Q' \in {\mu_f}^{-1}(\nu(\sqrt{(N : M)}M)) \\ &\Leftrightarrow f^{-1}(Q') \supseteq \sqrt{(N : M)}M \\ &\Leftrightarrow Q' \supseteq f(\sqrt{(N : M)}M) \\ &\Leftrightarrow Q' \supseteq \sqrt{(N : M)}M' \\ &\Leftrightarrow Q' \in \nu(\sqrt{(N : M)}M'). \end{aligned}$$

Therefore ${\mu_f}^{-1}(\nu(N)) = \nu(\sqrt{(N : M)}M')$ and so μ_f is continuous.
(3), (4) Clear.
(5) Let $Q' \in \mathcal{PS}(M')$. Then we have $(\rho_M \circ \mu_f)(Q') = S_p(f^{-1}(Q') + pM)$

and $(\sigma_f \circ \rho_{M'})(Q') = f^{-1}(S_{p'}(Q' + p'M'))$, where $p = \sqrt{(f^{-1}(Q') : M)}$ and $p' = \sqrt{(Q' : M')}$. It is easily seen that that $p = p'$, and then we have

$$\begin{aligned} x \in S_p(f^{-1}(Q') + pM) &\Leftrightarrow cx \in f^{-1}(Q') + pM \text{ for some } c \in R - p \\ &\Leftrightarrow cf(x) \in Q' + pM \text{ for some } c \in R - p \\ &\Leftrightarrow f(x) \in S_p(Q' + pM') \\ &\Leftrightarrow x \in f^{-1}(S_p(Q' + pM')) \\ &\Leftrightarrow x \in f^{-1}(S_{p'}(Q' + p'M')). \end{aligned}$$

Thus $\rho_M \circ \mu_f = \sigma_f \circ \rho_{M'}$. □

Lemma 2.15. *Let M and M' be R-modules. Let $f : M \to M'$ be a epimorphism and N a submodule of M containing $Ker f$. Then the following hold.*

(1) If N is a primary-like submodule of M, then $f(N)$ is a primary-like submodule of M'.

(2) If N satisfies the primeful property, then $f(N)$ satisfies the primeful property.

Proof. (1) First note that $f(N)$ is a proper submodule of M', since N is a proper submodule containing $Ker f$. Assume that $rf(m) \in f(N)$ for $r \in R$ and $m \in M$. Thus there exists $n \in N$ such that $rm - n \in Ker f$. Hence $rm \in N$, and thus $r \in (N : M)$ or $m \in \operatorname{rad} N$. Since $(N : M) = (f(N) : M')$ and $f(\operatorname{rad} N) = \operatorname{rad}(f(N))$, then $f(N)$ is a primary-like submodule of M'.
(2) Let p be a prime ideal containing $(f(N) : M')$. Then p is a prime ideal containing $(N : M)$ and so there is a prime submodule P containing N such that $(P : M) = p$. Since P contains $Ker f$, the submodule $f(P)$ of M' is a prime submodule containing $f(N)$ such that $(f(P) : M') = p$. □

Theorem 2.16. *Let M and M' be R-modules. Let $f : M \to M'$ be an epimorphism. Then $\mathcal{PS}(M')$ is homeomorphic to the topological subspace $\mathcal{W}$ of $\mathcal{PS}(M)$ consists of all primary-like submodules of M containing $Ker f$.*

Proof. Consider $\delta_f : \mathcal{W} \to \mathcal{PS}(M')$ defined by $\delta_f(Q) = f(Q)$. By Lemma 2.15, δ_f is well-defined. Also δ_f is continuous. Indeed, since f is surjective and Q is a primary-like submodule containing $Ker f$, we have

$$\begin{aligned} Q \in {\delta_f}^{-1}(\nu(f(N))) &\Leftrightarrow f(Q) \in \nu(f(N)) \\ &\Leftrightarrow \sqrt{(f(Q) : M')} \supseteq \sqrt{(f(N) : M')} \\ &\Leftrightarrow \sqrt{(Q : M)} \supseteq \sqrt{(N : M)} \\ &\Leftrightarrow Q \in \nu(N). \end{aligned}$$

It shows that ${\delta_f}^{-1}(\nu(f(N))) = \nu(N) \cap \mathcal{W}$. Moreover, by letting μ_f as in Theorem 2.14, $(\delta_f \circ \mu_f)(Q') = f(f^{-1}(Q')) = Q'$ for all $Q' \in \mathcal{PS}(M')$. Thus $\delta_f \circ \mu_f = 1_{\mathcal{PS}(M')}$. Also if $Q \in \mathcal{W}$, then $(\mu_f \circ \delta_f)(Q) = f^{-1}(f(Q)) \supseteq Q$. For the reverse inclusion, let $x \in f^{-1}(f(Q))$. Then $f(x) = f(q)$, for some $q \in Q$ so that $x - q \in Kerf$. It follows that $x \in Q$, since $Kerf \subseteq Q$. Hence $(\mu_f \circ \delta_f)(Q) = Q$ for every $Q \in \mathcal{W}$, and so $\mu_f \circ \delta_f = 1_{\mathcal{W}}$. Thus $\mathcal{PS}(M')$ is homeomorphic to $\mathcal{W}$. □

3. Irreduciblity in $\mathcal{PS}(M)$

A topological space X is called *irreducible* if $X \neq \emptyset$ and every finite intersection of non-empty open sets of X is non-empty. A subset Y of a topological space X is called *irreducible* if the subspace Y of X is irreducible. Equivalently, a subspace Y of X is irreducible if for every pair of closed subsets C_1, C_2 of X with $Y \subseteq C_1 \cup C_2$, we have $Y \subseteq C_1$ or $Y \subseteq C_2$ (see, for example, [6, P. 94]).

Let M be an R-module and Y be a subset of $\mathcal{PS}(M)$. We will denote the closure of Y in $\mathcal{PS}(M)$ by $cl(Y)$, and also the intersection of all elements of Y by $\gamma(Y)$ (note that if $Y = \emptyset$, then $\gamma(Y) = M$).

Proposition 3.1. *Let M be an R module. Then for every $Q \in \mathcal{PS}(M)$, $\nu(Q)$ is irreducible.*

Proof. Since $\{Q\}$ is an irreducible subset of $\mathcal{PS}(M)$, $\overline{\{Q\}}$ is an irreducible subset of $\mathcal{PS}(M)$ by [3, page 13, Exercise 20]. Now, by [9, Corollary 2], $cl(\{Q\}) = \nu(Q)$. Therefore $\nu(Q)$ is an irreducible subset of $\mathcal{PS}(M)$. □

Let C be a closed subset of a topological space X. An element $x \in C$ is called a *generic point* of X if $C = cl(\{x\})$.

Corollary 3.2. *Let M be an R-module and $Q, Q' \in \mathcal{PS}(M)$. If $\sqrt{(Q : M)} = \sqrt{(Q' : M)}$. Then Q is a generic point for the irreducible closed subset $\nu(Q')$.*

Proof. First note that, by Proposition 3.1, $\nu(Q')$ is an irreducible closed subset of $\mathcal{PS}(M)$. Also, by [9, Corollary 2], $\overline{\{Q\}} = \nu(Q) = \nu(Q')$. Thus Q is a generic point of $\nu(Q')$. □

Proposition 3.3. *Let M be an R-module and $Y \subseteq \mathcal{PS}(M)$. If $\gamma(Y)$ is a primary-like submodule of M, then Y is irreducible in $\mathcal{PS}(M)$.*

Proof. Suppose that $\gamma(Y)$ is a primary-like submodule of M. Then by [9, Proposition 3], $\nu(\gamma(Y)) = cl(Y)$. Hence $cl(Y)$ is irreducible by Proposition 3.1. Thus Y is irreducible by [3, page 13, Exersise 20]. □

Corollary 3.4. *Let M be an R-module and $Y \subseteq \mathcal{PS}(M)$. If Y is linearly ordered by inclusion, then Y and $\mathcal{PS}_p(M)$ are irreducible in $\mathcal{PS}(M)$ for every prime ideal p of R.*

Proof. Since $\gamma(Y)$ and $\gamma(\mathcal{PS}_p(M))$ are primary-like submodules of M, we conclude that Y and $\mathcal{PS}_p(M)$ are irreducible by Proposition 3.3. □

Corollary 3.5. *Let m be a maximal ideal of R and M an R-module. Then $\mathcal{PS}_m(M)$ is an irreducible closed subset of $\mathcal{PS}(M)$.*

Proof. Let m is a maximal ideal of R. Then $\mathcal{PS}_m(M)$ is irreducible by Corollary 3.4. Also, since

$$\begin{aligned}\nu(mM) &= \{Q \in \mathcal{PS}(M) \mid \sqrt{(Q:M)} \supseteq \sqrt{(mM:M)}\}\\ &= \{Q \in \mathcal{PS}(M) \mid \sqrt{(Q:M)} \supseteq m\}\\ &= \{Q \in \mathcal{PS}(M) \mid \sqrt{(Q:M)} = m\}\\ &= \mathcal{PS}_m(M),\end{aligned}$$

$\mathcal{PS}_m(M)$ is a closed subset of $\mathcal{PS}(M)$. □

Proposition 3.6. *Let M be an R-module, $Y \subseteq \mathcal{PS}(M)$ and $p = \sqrt{(\gamma(Y):M)}$. If p is a prime ideal of R and $\mathcal{PS}_p(M) \neq \emptyset$, then Y is irreducible in $\mathcal{PS}(M)$.*

Proof. Let $Q \in \mathcal{PS}_p(M)$. Since $p = \sqrt{(Q:M)} = \sqrt{(\gamma(Y):M)}$, we have $\nu(Q) = \nu(\gamma(Y)) = cl(Y)$ by [9, Proposition3]. Hence, $cl(Y)$ is irreducible and so Y is irreducible by [3, page 13, Exersise 20]. □

Proposition 3.7. *Let M be an R-module, Y be an irreducible subset of $\mathcal{PS}(M)$ and $A = \{\sqrt{(Q:M)} : Q \in Y\}$. Then A is an irreducible subset of $Spec(R)$, and thus $\gamma(A) = \sqrt{(\gamma(Y):M)}$ is a prime ideal of R.*

Proof. Suppose that Y is irreducible. Since ϕ is continuous by Proposition 2.8, $\phi(Y) = Y'$ is an irreducible subset of $\mathrm{Spec}(\bar{R})$. Therefore, we have $\gamma(Y') = \overline{\sqrt{(\gamma(Y):M)}}$, and so $\gamma(Y')$ is a prime ideal of $\bar{R}$ by [6, page 129, Proposition 14]. Thus $\gamma(A) = \sqrt{(\gamma(Y):M)}$ for some subset A of $\mathrm{Spec}(R)$, and hence $\gamma(A)$ is a prime ideal of R. Thus by [6, page 129, Proposition 14], A is an irreducible subset of $\mathrm{Spec}(R)$. □

Theorem 3.8. *Let M be an R-module and ϕ be surjective. If $Y \subseteq \mathcal{PS}(M)$, then Y is an irreducible closed subset of $\mathcal{PS}(M)$ if and only if $Y = \nu(Q)$ for some $Q \in \mathcal{PS}(M)$. In particular, every irreducible closed subset of $\mathcal{PS}(M)$ has a generic point.*

Proof. By Proposition 3.1, for any $Q \in \mathcal{PS}(M)$, $\nu(Q)$ is an irreducible closed subset of $\mathcal{PS}(M)$. Conversely, if Y is an irreducible closed subset of $\mathcal{PS}(M)$, then $Y = \nu(N)$ for some submodule N of M in which $\sqrt{(\gamma(\nu(N)):M)} = \sqrt{(\gamma(Y):M)} = p$, and this ideal is a prime ideal of R by Proposition 3.7. Since ϕ is surjective, there exists a p-primary-like submodule $Q \in \mathcal{PS}(M)$ such that $\sqrt{(Q:M)} = p$. It follows that $p = \sqrt{(\gamma(\nu(N)):M)} = \sqrt{(Q:M)}$. Hence $\nu(\gamma(\nu(N))) = \nu(Q)$. Thus $Y = \nu(Q)$, by [9, Proposition3]. □

Proposition 3.9. *Let M be an R module and ϕ be surjective. Then*

(1) The assignment $Q \mapsto \nu(Q)$ is a surjection from $\mathcal{PS}(M)$ to the set of all irreducible closed subsets of $\mathcal{PS}(M)$.

(2) The assignment $\nu(Q) \mapsto \overline{\sqrt{(Q:M)}}$ is a bijection from the set of all irreducible closed subsets of $\mathcal{PS}(M)$ to $\mathrm{Spec}(\bar{R})$.

(3) *The assignment* $\nu(Q) \mapsto V(\psi^{-1}(\overline{\sqrt{(Q:M)}}))$ *is a bijection from the set of all irreducible closed subsets of* $\mathcal{PS}(M)$ *to the set of all irreducible closed subsets of* $\operatorname{Spec}(M)$.

Proof. (1) By Theorem 3.8.
(2) It is easy to see that the given assignment is well-defined and an injection. Suppose $\bar{p} \in \operatorname{Spec}(\bar{R})$. Since ϕ is surjective, $\bar{p} = \overline{\sqrt{(Q:M)}}$ for some $Q \in \mathcal{PS}(M)$. Thus the assighnment $\nu(Q) \mapsto \bar{p}$, and so the given assignment is a surjection.
(3) First note that, for any $Q \in \mathcal{PS}(M)$, the closed subset $V(\psi^{-1}(\overline{\sqrt{(Q:M)}}))$ of $\operatorname{Spec}(M)$ is irreducible by [11, Theorem 5.7]. Thus the given assignment is well defined, since for $Q, Q' \in \mathcal{PS}(M)$, $\nu(Q) = \nu(Q')$ implies that $\sqrt{(Q:M)} = \sqrt{(Q':M)}$. Now, let $\psi^{-1}(\overline{\sqrt{(Q:M)}}) = P$ and $\psi^{-1}(\overline{\sqrt{(Q':M)}}) = P'$ for $Q, Q' \in \mathcal{PS}(M)$. If $V(P) = V(P')$, then $(P:M) = (P':M)$. Thus, since ψ is surjective, $\sqrt{(Q:M)} = \sqrt{(Q':M)}$. It follows that $\nu(Q) = \nu(Q')$, and then the given assignment is injective. Next, for the surjectivity, let $Q \in \mathcal{PS}(M)$ and $V(\psi^{-1}(\overline{\sqrt{(Q:M)}})) = V(P)$. Thus $\nu(P)$ is mapped to $V(\psi^{-1}(\overline{\overline{\sqrt{(P:M)}}}))$ which is $V(P)$. Thus the given assignment is a bijection. □

Theorem 3.10. *Let M be a finitely generated R-module. Then the following statements are equivalent.*

(1) $\operatorname{Spec}(M)$ *is an irreducible space;*

(2) $\mathcal{PS}(M)$ *is an irreducible space;*

(3) $\operatorname{Supp}(M)$ *is an irreducible space;*

(4) $\sqrt{Ann(M)}$ *is a prime ideal of R;*

(5) $\mathcal{PS}(M) = \nu(pM)$ *for some* $p \in \operatorname{Supp}(M)$*;*

(6) $\operatorname{Spec}(M) = V(pM)$ *for some* $p \in \operatorname{Supp}(M)$.

Proof. (1) ⇒ (2) Since $\operatorname{Spec}(M)$ is an irreducible space, by [11, Theorem 5.7], $\operatorname{Spec}(M) = V(P)$ for some $P \in \operatorname{Spec}(M)$. Let $\bar{p} = \psi(P)$. Since ϕ is surjective, there is an element $Q \in \mathcal{PS}(M)$ such that $\phi(Q) = \overline{\sqrt{(Q:M)}} = \bar{p}$. We show that $\mathcal{PS}(M) = \nu(Q)$. Suppose that $Q' \in \mathcal{PS}(M)$ and $p' = \sqrt{(Q':M)}$. Now, since $\rho(Q') \in V(P)$, we have

$$\begin{aligned}\overline{\sqrt{(Q':M)}} = \overline{p'} &= \overline{(S_{p'}(Q'+p'M):M)} = \overline{(\rho(Q'):M)}\\ &\supseteq \overline{(P:M)} = \bar{p} = \overline{\sqrt{(Q:M)}},\end{aligned}$$

which follows that $\sqrt{(Q':M)} \supseteq \sqrt{(Q:M)}$. Thus $Q' \in \nu(Q)$, and therefore $\mathcal{PS}(M) = \nu(Q')$, i.e., $\mathcal{PS}(M)$ is an irreducible space by Theorem 3.8.
(2) ⇒ (3) Since ϕ is a surjective continuous map, $\operatorname{Spec}(\bar{R})$ is irreducible by the assumption. Now since by [3, page 13, Ex. 21], $\operatorname{Spec}(\bar{R})$ and $V(Ann(M))$

are homeomorphic and by [12, Proposition 3.4] $\mathrm{Supp}(M) = V(Ann(M))$, we conclude that $\mathrm{Supp}(M)$ is an irreducible space.
(3) $\Rightarrow$ (4) By [6, page 102, Proposition 14], $\gamma(\mathrm{Supp}(M))$ is a prime ideal of R. Now since $\gamma(\mathrm{Supp}(M)) = \gamma(V(Ann(M))) = \sqrt{Ann(M)}$, we are done.
(4) $\Rightarrow$ (5) First note that $\sqrt{Ann(M)} \subseteq \sqrt{(Q:M)}$, for each $Q \in \mathcal{PS}(M)$. Since M is finitely generated, by [12, Proposition 3.8], there exists $P \in \mathrm{Spec}(M)$ such that $(P:M) = \sqrt{(P:M)} = \sqrt{Ann(M)}$. Therefore

$$\begin{aligned}\mathcal{PS}(M) &= \{Q \in \mathcal{PS}(M) \mid \sqrt{(Q:M)} \supseteq \sqrt{(P:M)}\}\\ &= \nu(P) = \nu(\sqrt{(P:M)}M) = \nu(\sqrt{Ann(M)}M).\end{aligned}$$

Thus $\mathcal{PS}(M) = \nu(pM)$ in which $p = \sqrt{Ann(M)}$ and $p \in \mathrm{Supp}(M)$.
(5) $\Rightarrow$ (6) Let $P \in \mathrm{Spec}(M)$. Since ϕ is surjective, there is an element $Q \in \mathcal{PS}(M)$ such that $\phi(Q) = \overline{(P:M)}$, and so $\overline{\sqrt{(Q:M)}} = \overline{(P:M)}$. Now, since $\mathcal{PS}(M) = \nu(pM)$ for some $p \in \mathrm{Supp}(M)$, we have

$$(pM:M) \subseteq \sqrt{(pM:M)} \subseteq \sqrt{(Q:M)} = (P:M).$$

Thus $P \in V(pM)$ so that $\mathrm{Spec}(M) = V(pM)$ for some $p \in \mathrm{Supp}(M)$.
(6) $\Rightarrow$ (1) Let $\mathrm{Spec}(M) = V(pM)$ for some $p \in \mathrm{Supp}(M)$. Since ψ is surjective, there exists $P \in X$ such that $(P:M) = p$. Hence by [11, Result 3], we have $\mathrm{Spec}(M) = V(pM) = V((P:M)M) = V(P)$. Thus, by [11, Theorem 5.7], $\mathrm{Spec}(M)$ is irreducible. □

4. $\mathcal{PS}(M)$ as a spectral space

A topological space X is a T_0-*space* if and only if for any two distinct points in X there exists an open subset of X which contains one of the points but not the other. It is well-known that, for any ring R, *Spec*(R) is a T_0-space for the Zariski topology. In [11, page 429], it has been shown that if M is a vector space, then $\mathrm{Spec}(M)$ is not a T_0-space. This example can be used again to show that $\mathcal{PS}(M)$ is not also a T_0 space. In fact, if M is a vector space, then $\nu(N) = \mathcal{PS}(M)$ for every proper subspace N of M so that $\mathcal{PS}(M)$ has the trivial topology.

Proposition 4.1. *Let M be a multiplication R-module. If for every $Q \in \mathcal{PS}(M)$ the ideal $(Q:M)$ is a radical ideal, then $\mathcal{PS}(M)$ is a T_0-space.*

Proof. Let $Q \in \mathcal{PS}(M)$ and $(Q:M)$ is a radical ideal of R. Then Q is a prime submodule of M, and so $\mathcal{PS}(M)$ is a topological subspace of $\mathrm{Spec}(M)$. Thus $\mathcal{PS}(M)$ is a T_0-space by [11, Corollary 6.2]. □

Theorem 4.2. *Let M be an R-module. Then the following statements are equivalent.*

(1) $\mathcal{PS}(M)$ is a T_0-space.

(2) If $\nu(Q) = \nu(Q')$ for $Q, Q' \in \mathcal{PS}(M)$, then $Q = Q'$.

Proof. (1) Let $Q \neq Q'$ for some $Q, Q' \in \mathcal{PS}(M)$. Since $\mathcal{PS}(M)$ is a T_0-space, $cl(Q) \neq clQ'$. Thus by [9, Corollary 2], we have $\nu(Q) \neq \nu(Q')$.
(2) Let $Q \neq Q'$ for some $Q, Q' \in \mathcal{PS}(M)$. Then by the assumption $\nu(Q) \neq \nu(Q')$. Therefore, by [9, Corollary 2], $cl(Q) \neq cl(Q')$. Hence $\mathcal{PS}(M)$ is a T_0-space. □

Recall that a *spectral space* is a topological space homeomorphic to the prime spectrum of a ring equipped with the Zariski topology. By Hochster's characterization [10], the topological space X is spectral if and only if the following statements hold:

(1) X is a T_0-space.

(2) X is quasi-compact.

(3) the quasi-compact open subsets of X are closed under finite intersection and form an open base.

(4) each irreducible closed subset of X has a generic point.

For any ring R, $Spec(R)$ is well-known to satisfy these condition (see [6, P. 401-403]).

Theorem 4.3. *Let M be a finitely generated multiplication R-module. Then $\mathcal{PS}(M)$ is a spectral space.*

Proof. By [9, Theorem3], Theorem 3.8, [9, Corollary3] and [11, Corollary 6.2]. □

Theorem 4.4. *Let M be an R-module. Consider the following statements:*

(1) $\mathcal{PS}(M)$ is a spectral space;

(2) $\mathcal{PS}(M)$ is a T_0-space;

(3) If $\nu(Q) = \nu(Q')$ for $Q, Q' \in \mathcal{PS}(M)$, then $Q = Q'$;

(4) $|\mathcal{PS}_p(M)| \leq 1$ for every $p \in Spec(R)$;

(5) ϕ is injective.

Then (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5).
Moreover if ϕ is surjective , then (5) $\Rightarrow$ (1).

Proof. (1)$\Rightarrow$(2) Clear.
(2)$\Rightarrow$(3) By Theorem 4.2.
(3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) By Proposition 2.1.
(5) $\Rightarrow$ (1) By[9, Theorem 1], $\mathcal{PS}(M)$ is homeomorphic to the spectral space Spec($\bar{R}$). Thus $\mathcal{PS}(M)$ is a spectral space. □

Proposition 4.5. *Let M be an R-module and $|\mathcal{PS}_p(M)| = 1$ for every $p \in Spec(R)$. Then $\mathcal{PS}(M)$ is a spectral space.*

Proof. Since Spec($\bar{R}$) is a spectral space, $\mathcal{PS}(M)$ is also a spectral space by Corollary 2.5 and Corollary 2.10. □

Theorem 4.6. *Let M be a multiplication R-module such that $\mathcal{PS}(M)$ is a non-empty finite set. Then $\mathcal{PS}(M)$ is a spectral space if and only if $|\mathcal{PS}_p(M)| \leq 1$ for every $p \in Spec(R)$.*

Proof. Since $\mathcal{PS}(M)$ is a non-empty finite set, then the conditions (2) and (3) in Hochster's characterization are clearly true. Suppose $Y = \{y_1, y_2, \ldots, y_n\}$ is an irreducible closed subset of $\mathcal{PS}(M)$. Thus $Y = \overline{\{y_i\}}$ for some i where $1 \leq i \leq n$, i.e., Y has a generic point. Hence, by [9, Theorem 4.3], $\mathcal{PS}(M)$ is a spectral space if and only if $\mathcal{PS}(M)$ is a T_0-space if and only if $|\mathcal{PS}_p(M)| \leq 1$ for every $p \in Spec(R)$. □

Theorem 4.7. *Let M be an R-module and $Im(\phi)$ be a closed subset of* Spec($\bar{R}$). *Then $\mathcal{PS}(M)$ is a spectral space if and only if ϕ is injective.*

Proof. Let $Y = Im(\phi)$ be a closed subset of Spec($\bar{R}$). Then Y is a spectral subspace of Spec($\bar{R}$). Assume that ϕ is injective. Then the bijection $\phi : \mathcal{PS}(M) \to Y$ is continuous by Proposition 2.8. We show that ϕ is a closed map. Let N be a submodule of M, and $Y' = Y \cap V(\overline{\sqrt{(N:M)}})$. Then Y' is a closed subset of Y, and so by Proposition 2.8 we have

$$\phi^{-1}(Y') = \phi^{-1}(Y) \cap \phi^{-1}(V\overline{\sqrt{(N:M)}}) = \nu(\sqrt{(N:M)}M) = \nu(N).$$

Hence $\phi(\nu(N)) = \phi(\phi^{-1}(Y')) = Y'$ is a closed subset of Y. Thus $\phi : \mathcal{PS}(M) \to Y$ is a homeomorphism and so $\mathcal{PS}(M)$ is a spectral space. Conversely, assume $\mathcal{PS}(M)$ is a spectral space. Hence by Theorem 4.4, ϕ is injective. □

Example 4.8. (1) Every proper submodule of $\mathbb{Z}$-module $\mathbb{Z}(p^\infty)$ is primary-like. However $Spec_L(\mathbb{Z}(p^\infty)) = Spec(\mathbb{Z}(p^\infty)) = \emptyset$.

(2) For $\mathbb{Z}$-module $\mathbb{Q}$, $Spec(\mathbb{Q}) = \{0\}$ and $Spec_L(\mathbb{Q}) = \emptyset$, because $\mathbb{Q}$ has no submodules satisfying the primeful property.

(3) For a vector space V over a field F, $Spec_L(V) = Spec(V)$= the set of all proper vector subspaces of V.

(4) Let $M = \mathbb{Q} \oplus \mathbb{Z}_p$, where $\mathbb{Z}_p$ is the cyclic group of order p. Then Spec$(M) = \{\mathbb{Q} \oplus 0, 0 \oplus \mathbb{Z}_p\}$ [13, Example 2.6]. Although $\{0 \oplus 0, \mathbb{Q} \oplus 0, 0 \oplus \mathbb{Z}_p\} \cup \{N : N \nsubseteq \mathbb{Q} \oplus 0,\ N \nsubseteq 0 \oplus \mathbb{Z}_p\}$ is the set of all primary like submodules of M. However $\mathcal{PS}(M) = \emptyset$.

(5) Let $M = \mathbb{Z}(p^\infty) \oplus \mathbb{Z}_p$. Then M is not a multiplication $\mathbb{Z}$-module [13, Example 3.7]. Spec$(M) = pM = \mathbb{Z}(p^\infty) \oplus 0$. By an easy verification $\{< 1/p^i + \mathbb{Z} > \oplus 0 :\ i \in \mathbb{Z}\} \cup \{\mathbb{Z}(p^\infty) \oplus 0, 0 \oplus \mathbb{Z}_p\} \cup \{N : N \nsubseteq \mathbb{Z}(p^\infty) \oplus 0\}$ is the set of all primary-like submodules of M. But $N = 0 \oplus \mathbb{Z}_p \notin \mathcal{PS}(M)$. Also $\operatorname{rad} N = M$ and $(N : M) = 0$. Thus for the primary-like submodule N, $\sqrt{(N:M)} \subsetneq (\operatorname{rad} N : M)$.

(6) Let $M = \prod_{p\in\Omega} \frac{\mathbb{Z}}{p\mathbb{Z}}$ and $M' = \bigoplus_{p\in\Omega} \frac{\mathbb{Z}}{p\mathbb{Z}}$ be $\mathbb{Z}$-modules where Ω is the set of prime integers. Then M' is a 0-prime submodule of M which does not satisfy the primeful property. In fact $\mathrm{Spec}(M) = \{M'\} \cup \{pM : p \in \Omega\}$. However, $(\mathrm{rad}\, M' : M) = \sqrt{(M' : M)} = 0$ [9, Example 2.12].
It is easy to see that the zero submodule 0 satisfies the primeful property and $\mathrm{rad}\, 0 = 0$. But 0 is not a primary-like submodule of M, because 0 is not a prime submodule of M.

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