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Fixed point theorems for λ-generalized contractions in D^*-metric spaces

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Abstract. The fixed point theorems for D^*-metric spaces were obtained by several authors. The notion of a D^*-metric space and λ-generalized contractions are presented in this paper and a fixed point theorem on a λ-generalized contraction of an f-orbitally complete D^*-metric space is obtained. Further, some consequences of this fixed point theorem are presented in this paper.

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1. Introduction

In 1992 B. C. Dhage [2] initiated a study of general metric spaces called D-metric spaces. Later several researchers made significant contributions to the study of fixed point theorems of D-metric spaces, in [1], [3] and [9]. Unfortunately, almost all fixed point theorems proved on D-metric spaces are not valid, in view of papers [7], [6] and [8]. R. Kannan introduced the concept of K-contraction in metric spaces and obtained fixed point results in metric spaces.

2. Definitions

As a modification of D-metric spaces, Shaban Sedghi, Nabi Shobe and Haiyun Zhou [10] have introduced D^*-metric spaces as follows:

Definition 2.1. [10] Let X be a non-empty set. A function $D^* : X^3 \to [0, \infty)$ is said to be a generalized metric or D^*-metric on X, if it satisfies the following properties:

(i) $D^*(x, y, z) \geq 0$ for all $x, y, z \in X$,

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(ii) $D^*(x, y, z) = 0$ if and only if x=y=z,

(iii) $D^*(x, y, z) = D^*(\sigma(x, y, z))$ for all $x, y, z \in X$, where $\sigma(x, y, z)$ is a permutation of the set $\{x, y, z\}$,

(iv) $D^*(x, y, z) \leq D^*(x, y, w) + D^*(w, z, z)$ for all $x, y, z, w \in X$.

The pair (X, D^*), where D^* is a generalized metric on X is called a D^*-metric space, or a generalized metric space.

Example 2.2. [10] Let (X, d) be a metric space. Define $D_1^* : X^3 \to [0, \infty)$ by $D_1^*(x, y, z) = \max\{d(x, y), d(y, z)d(z, x)\}$ for $x, y, z \in X$. Then (X, D_1^*) is a generalized metric space.

Note 2.3. *[11] Using (ii) and inequality (iv) of Definition 2.1, one can prove that, if (X, D^*) is a D^*-metric space, then*

$$D^*(x, x, y) = D^*(x, y, y) \tag{2.1}$$

for all $x, y \in X$. In fact, $D^*(x, x, y) \leq D^*(x, x, x) + D^*(x, y, y) = D^*(x, y, y)$ and $D^*(x, y, y) = D^*(y, y, x) \leq D^*(y, y, y) + D^*(y, x, x) = D^*(x, x, y)$ for all $x, y \in X$, proving (2.1).

Definition 2.4. Let (X, D^*) be a D^*-metric space. A sequence $\{x_n\}$ in X is said to

(i) Converge to some point x of X, if
$D^*(x_n, x_n, x) = D^*(x_n, x, x) \to D^*(x, x, x)$ as $n \to \infty$

(ii) be Cauchy if, for every $\epsilon > 0$, there is a natural number n_0 such that $D^*(x_n, x_n, x_m) < \epsilon$ for all $m, n \geq n_0$.

It is easy to see (A fact proved in [10]; Lemma 1.8 and Lemma 1.9) that, if $\{x_n\}$ converges to x in (X, D^*), then x is unique, and that $\{x_n\}$ is a Cauchy sequence in (X, D^*).

Definition 2.5. A D^*-metric space (X, D^*) is said to be complete, if every Cauchy sequence in it converges in it.

Note 2.6. *As noted in Example 2.2, given any metric space (X, d) it is possible to define a D^*-metric D_1^* by using the metric d. We shall call D_1^* as D^*-metric induced by the metric d. Thus, every metric space gives rise to at least one D^*-metric space (X, D_1^*). Also, if (X, D^*) is a D^*-metric space, then defining $d_0(x, y) = D^*(x, y, y)$ for $x, y \in X$, we can easily show that (X, d_0) is a metric space, and we shall call d_0 as metric induced by D^*.*

Theorem 2.7. *Let (X, d) be a metric space and D_1^* the D^*-metric induced by d (as given in Example 2.2). A sequence $\{x_n\}$ in (X, D_1^*) is a Cauchy sequence if and only if $\{x_n\}$ is a Cauchy sequence in (X, d).*

Proof. First note that we have $d(x,y) \leq D_1^*(x,y,y) \leq 2d(x,y)$ for all $x,y \in X$. The theorem follows immediately from the above inequality. In fact, if $\{x_n\}$ is a Cauchy sequence in (X,d), then, for any given $\epsilon > 0$, choose a natural number n_0 such that $m,n \geq n_0$ implies $d(x_m,x_n) < \epsilon/2$; and note that, for the same $n_0, m,n \geq n_0$ implies that $D^*(x_m,x_n,x_n) \leq 2d(x_m,x_n) < \epsilon$, proving that $\{x_n\}$ is a Cauchy sequence in (X, D_1^*). The other part of the theorem can be proved by using the left one among the two inequalities noted at the beginning of the proof. □

Corollary 2.8. *Suppose that (X,d) is a metric space, and let D_1^* be a D^*-metric induced by the metric d. Then the D^*-metric space (X, D_1^*) is complete if and only if (X,d) is complete.*

Proof. Follows from Theorem 2.7 . □

It has been proved in ([10]; Lemma 1.7) that, if (X, D^*) is a D^*-metric space, then D^* is a continuous function on X^3, in the sense that $\lim_{n\to\infty} D^*(x_n,y_n,z_n) = D^*(x,y,z)$, whenever $\{(x_n,y_n,z_n)\}$ is a sequence in X^3 converging to $(x,y,z) \in X^3$. Equivalently, $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$ and $\lim_{n\to\infty} z_n = z$ imply that $\lim_{n\to\infty} D^*(x_n,y_n,z_n) = D^*(x,y,z)$.

The purpose of this paper is to define certain types of contractions among self-maps of D^*-metric spaces, and to establish fixed point theorems for such self-maps.

Definition 2.9. Let f be a self map of a D^*-metric space (X, D^*). For any $x \in X$, the set $O_f(x:\infty) = \{f^n x : n \geq 0\} = \{x, fx, f^2x, ...\}$ is called the orbit of x under f.

Definition 2.10. Let f be a self-map of a D^*-metric space (X, D^*). If, for some $x \in X$, every Cauchy sequence in $O_f(x:\infty)$ converges a point in X, then (X, D^*) is said to be an f-orbitally complete D^*-metric space.

Remark 2.11. Trivially, a complete D^*-metric space is f-orbitally complete for any self-map f of X. However, the converse is not true. A self-map f of a D^*-metric space (X, D^*) is called a contraction, if there is a q with $0 \leq q < 1$ such that

$$D^*(fx,fy,fz) \leq q.D^*(x,y,z) \tag{2.2}$$

for all $x,y,z \in X$.

R. Kannan [4] defined a contraction for metric spaces in a different way, which we shall call a K-contraction. Analogously we define K-contraction for D^*-metric spaces as follows:

Definition 2.12. A self-map f of a D^*-metric space (X, D^*) is called a K-contraction, if there is a q with $0 \leq q < 1/3$ such that

$$D^*(fx,fy,fz) \leq q.\{D^*(x,fx,fx) + D^*(y,fy,fy) + D^*(z,fz,fz)\} \tag{2.3}$$

for all $x, y, z \in X$.

The concepts of contraction and K-contraction are independent. We now define a special type of contraction, called a λ-generalized contraction for D^*-metric spaces as follows:

Definition 2.13. A self-map f of a D^*-metric space (X, D^*) is called a λ-generalized contraction if, for every $x, y, z \in X$, there exist non-negative numbers q, r, s, t and v (depending on x, y and z) such that

$$\sup_{x,y,z \in X} \{q + r + s + t + 10v\} = \lambda < 1 \tag{2.4}$$

and

$$\begin{aligned} D^*(fx, fy, fz) \leq\ & q(x, y, z) D^*(x, y, z) + r(x, y, z) D^*(x, fx, fx) \\ & + s(x, y, z) D^*(y, fy, fy) + t(x, y, z) D^*(z, fz, fz) \\ & + v(x, y, z) \{D^*(x, fy, fy) + D^*(y, fz, fz) \\ & + D^*(z, fx, fx) + D^*(x, fz, fz) + D^*(y, fx, fx) \\ & + D^*(z, fy, fy) + D^*(x, fy, fz) \\ & + D^*(y, fz, fx) + D^*(z, fx, fy)\}, \end{aligned} \tag{2.5}$$

for all $x, y, z \in X$. From the definition it is clear that every contraction and K-contraction are λ-generalized contractions.

3. Main results

Fixed point theorem for λ-generalized contraction of D^*-metric spaces

Theorem 3.1. *Suppose f is a self-map of a D^*-metric space (X, D^*) and X is f-orbitally complete. If f is a λ-generalized contraction, then it has a unique fixed point $u \in X$. In fact,*

$$u = \lim_{n \to \infty} f^n x \tag{3.1}$$

for any $x \in X$ and

$$D^*(f^n x, u, u) \leq \frac{\lambda^n}{1 - \lambda} D^*(x, fx, fx) \tag{3.2}$$

for any $x \in X$ and $n \geq 1$.

Proof. Let $x \in X$ be an arbitrary element of X. Define the sequence $\{x_n\}$ by $x_0 = x$, $x_1 = fx_0$, $x_2 = fx_1 = f^2 x$, ..., $x_n = fx_{n-1} = f^n x$, ...and denote the orbit of x under f by $O_f(x : \infty) = \{x_n : n = 0, 1, 2, 3, ...\}$.

Consider

$$\begin{aligned}
D^*(x_n, x_{n+1}, x_{n+1}) &= D^*(fx_{n-1}, fx_n, fx_n) \\
&\le q(x_{n-1}, x_n, x_n)D^*(x_{n-1}, x_n, x_n) \\
&+ r(x_{n-1}, x_n, x_n)D^*(x_{n-1}, x_n, x_n) \\
&+ s(x_{n-1}, x_n, x_n)D^*(x_n, x_{n+1}, x_{n+1}) \\
&+ t(x_{n-1}, x_n, x_n)D^*(x_n, x_{n+1}, x_{n+1}) \\
&+ v(x_{n-1}, x_n, x_n)\{D^*(x_{n-1}, x_{n+1}, x_{n+1}) \\
&+ D^*(x_n, x_{n+1}, x_{n+1}) + D^*(x_n, x_n, x_n) \\
&+ D^*(x_{n-1}, x_{n+1}, x_{n+1}) + D^*(x_n, x_n, x_n) \\
&+ D^*(x_n, x_{n+1}, x_{n+1}) + D^*(x_{n-1}, x_{n+1}, x_{n+1}) \\
&+ D^*(x_n, x_{n+1}, x_n) + D^*(x_n, x_n, x_{n+1})\}
\end{aligned}$$

writing
$q_{n-1} = q(x_{n-1}, x_n, x_n)$, $r_{n-1} = r(x_{n-1}, x_n, x_n)$, $s_{n-1} = s(x_{n-1}, x_n, x_n)$, $t_{n-1} = t(x_{n-1}, x_n, x_n)$ and $v_{n-1} = v(x_{n-1}, x_n, x_n)$, we get

$$\begin{aligned}
D^*(x_n, x_{n+1}, x_{n+1}) &\le q_{n-1}D^*(x_{n-1}, x_n, x_n) + r_{n-1}D^*(x_{n-1}, x_n, x_n) \\
&+ s_{n-1}D^*(x_n, x_{n+1}, x_{n+1}) + t_{n-1}D^*(x_n, x_{n+1}, x_{n+1}) \\
&+ v_{n-1}\{3D^*(x_{n-1}, x_{n+1}, x_{n+1}) + 2D^*(x_n, x_{n+1}, x_{n+1}) \\
&+ 2D^*(x_n, x_n, x_{n+1})\}
\end{aligned}$$

since $D^*(x, x, y) = D^*(x, y, y)$ [Note 2.3], so we write
$D^*(x_{n-1}, x_{n+1}, x_{n+1}) = D^*(x_{n-1}, x_{n-1}, x_{n+1})$ and
$D^*(x_n, x_n, x_{n+1}) = D^*(x_n, x_{n+1}, x_{n+1})$. Therefore

$$\begin{aligned}
D^*(x_n, x_{n+1}, x_{n+1}) &\le q_{n-1}D^*(x_{n-1}, x_n, x_n) + r_{n-1}D^*(x_{n-1}, x_n, x_n) \\
&+ s_{n-1}D^*(x_n, x_{n+1}, x_{n+1}) + t_{n-1}D^*(x_n, x_{n+1}, x_{n+1}) \\
&+ v_{n-1}\{3D^*(x_{n-1}, x_{n-1}, x_{n+1}) + 4D^*(x_n, x_{n+1}, x_{n+1})\}
\end{aligned}$$

By using property (iv) of a D^*-metric space, we write

$$\begin{aligned}
D^*(x_{n-1}, x_{n-1}, x_{n+1}) &\le D^*(x_{n-1}, x_{n-1}, x_n) + D^*(x_n, x_{n+1}, x_{n+1}) \\
&\le D^*(x_{n-1}, x_n, x_n) + D^*(x_n, x_{n+1}, x_{n+1})\}
\end{aligned}$$

$$\begin{aligned}
D^*(x_n, x_{n+1}, x_{n+1}) &\le q_{n-1}D^*(x_{n-1}, x_n, x_n) + r_{n-1}D^*(x_{n-1}, x_n, x_n) \\
&+ s_{n-1}D^*(x_n, x_{n+1}, x_{n+1}) + t_{n-1}D^*(x_n, x_{n+1}, x_{n+1}) \\
&+ v_{n-1}\{3D^*(x_{n-1}, x_n, x_n) + 7D^*(x_n, x_{n+1}, x_{n+1})\} \\
&\le (q_{n-1} + r_{n-1} + 3v_{n-1})D^*(x_{n-1}, x_n, x_n) \\
&+ (s_{n-1} + t_{n-1} + 7v_{n-1})D^*(x_n, x_{n+1}, x_{n+1})
\end{aligned}$$

This implies that

$$\begin{aligned}
&(1 - s_{n-1} - t_{n-1} - 7v_{n-1})D^*(x_n, x_{n+1}, x_{n+1}) \\
&\quad \le (q_{n-1} + r_{n-1} + 3v_{n-1})D^*(x_{n-1}, x_n, x_n).
\end{aligned}$$

Hence $D^*(x_n, x_{n+1}, x_{n+1}) \leq (\frac{q_{n-1}+r_{n-1}+3v_{n-1}}{1-s_{n-1}-t_{n-1}-7v_{n-1}})D^*(x_{n-1}, x_n, x_n)$
$D^*(x_n, x_{n+1}, x_{n+1}) \leq \lambda D^*(x_{n-1}, x_n, x_n)$, where $\lambda = \frac{q_{n-1}+r_{n-1}+3v_{n-1}}{1-s_{n-1}-t_{n-1}-7v_{n-1}}$.
Assume that $\lambda < 1$, which implies $\lambda = \frac{q_{n-1}+r_{n-1}+3v_{n-1}}{1-s_{n-1}-t_{n-1}-7v_{n-1}} < 1$
$q_{n-1} + r_{n-1} + 3v_{n-1} < 1 - s_{n-1} - t_{n-1} - 7v_{n-1}$
$q_{n-1} + r_{n-1} + s_{n-1} + t_{n-1} + 10v_{n-1} < 1$ since
$\sup\limits_{x,y,z\in X} \{q + r + s + t + 10v\} = \lambda = q_{n-1} + r_{n-1} + s_{n-1} + t_{n-1} + 10v_{n-1} < 1$.
Thus by iteration, we get

$$D^*(x_n, x_{n+1}, x_{n+1}) \leq \lambda^n D^*(x_0, x_1, x_1) = \lambda^n D^*(x_0, fx_0, fx_0) \tag{3.3}$$

Therefore

$$\begin{aligned}
D^*(x_n, x_{n+p}, x_{n+p}) &\leq D^*(x_n, x_{n+1}, x_{n+1}) + D^*(x_{n+1}, x_{n+2}, x_{n+2}) \\
&+ D^*(x_{n+2}, x_{n+3}, x_{n+3}) + \ldots + D^*(x_{n+p-1}, x_{n+p}, x_{n+p}) \\
&\leq \lambda^n D^*(x_0, x_1, x_1) + \lambda^{n+1} D^*(x_0, x_1, x_1) \\
&+ \lambda^{n+2} D^*(x_0, x_1, x_1) + \ldots + \lambda^{n+p-1} D^*(x_0, x_1, x_1) \\
&\leq (\lambda^n + \lambda^{n+1} + \lambda^{n+2} + \ldots + \lambda^{n+p-1} + \ldots) D^*(x_0, x_1, x_1) \\
&\leq \frac{\lambda^n}{1-\lambda} D^*(x_0, x_1, x_1)
\end{aligned}$$

$$D^*(x_n, x_{n+p}, x_{n+p}) \leq \frac{\lambda^n}{1-\lambda} D^*(x_0, x_1, x_1) \tag{3.4}$$

Hence $D^*(x_n, x_{n+p}, x_{n+p}) \leq \lambda^n D^*(x_0, x_1, x_1)/(1-\lambda) \to 0$ as $n \to \infty$, since $0 \leq \lambda < 1$, and $\{x_n\}$ is a Cauchy sequence in $O_f(x : \infty)$. Since X is f-orbitally complete, there exists a $u \in X$ such that

$$u = \lim_{n\to\infty} x_n = \lim_{n\to\infty} f^n x_0 = \lim_{n\to\infty} f^n x.$$

To show that u is a fixed point of f,

$$\begin{aligned}
D^*(fu, fx_n, fx_n) &\leq qD^*(u, x_n, x_n) + rD^*(u, fu, fu) \\
&+ sD^*(x_n, fx_n, fx_n) + tD^*(x_n, fx_n, fx_n) \\
&+ v\{D^*(u, fx_n, fx_n) + D^*(x_n, fx_n, fx_n) \\
&+ D^*(x_n, fu, fu) + D^*(u, fx_n, fx_n) \\
&+ D^*(x_n, fu, fu) + D^*(x_n, fx_n, fx_n) \\
&+ D^*(u, fx_n, fx_n) + D^*(x_n, fx_n, fu) \\
&+ D^*(x_n, fu, fx_n)\}
\end{aligned}$$

$$\begin{aligned}
D^*(fu, fx_n, fx_n) &\le qD^*(u, x_n, x_n) + rD^*(u, x_{n+1}, x_{n+1}) \\
&+ rD^*(x_{n+1}, fu, fu) + sD^*(x_n, x_{n+1}, x_{n+1}) \\
&+ tD^*(x_n, x_{n+1}, x_{n+1}) + v\{D^*(u, x_{n+1}, x_{n+1}) \\
&+ D^*(x_n, x_{n+1}, x_{n+1}) + D^*(x_n, x_{n+1}, x_{n+1}) \\
&+ D^*(x_{n+1}, fu, fu) + D^*(u, x_{n+1}, x_{n+1}) \\
&+ D^*(x_n, x_{n+1}, x_{n+1}) + D^*(x_{n+1}, fu, fu) \\
&+ D^*(x_n, x_{n+1}, x_{n+1}) + D^*(u, x_{n+1}, x_{n+1}) \\
&+ D^*(x_n, x_{n+1}, x_{n+1}) + D^*(x_{n+1}, x_{n+1}, fu) \\
&+ D^*(x_n, x_{n+1}, x_{n+1}) + D^*(x_{n+1}, fu, x_{n+1})\} \\
&\le qD^*(u, x_n, x_n) + (r + 3v)D^*(u, x_{n+1}, x_{n+1}) \\
&+ (r + 4v)D^*(fx_n, fu, fu) \\
&+ (s + t + 6v)D^*(x_n, x_{n+1}, x_{n+1})
\end{aligned}$$

$$\begin{aligned}
D^*(fu, fx_n, fx_n) \le \lambda D^*(u, x_n, x_n) &+ \lambda D^*(u, x_{n+1}, x_{n+1}) \\
&+ \lambda D^*(fx_n, fu, fu) + \lambda D^*(x_n, x_{n+1}, x_{n+1}).
\end{aligned}$$

Therefore

$$\begin{aligned}
(1 - \lambda)D^*(fu, fx_n, fx_n) \le \lambda(&D^*(u, x_n, x_n) \\
&+ D^*(u, x_{n+1}, x_{n+1}) + D^*(x_n, x_{n+1}, x_{n+1}))
\end{aligned}$$

and

$$\begin{aligned}
D^*(fu, fx_n, fx_n) \le (\frac{\lambda}{1-\lambda})(&D^*(u, x_n, x_n) \\
&+ D^*(u, x_{n+1}, x_{n+1}) + D^*(x_n, x_{n+1}, x_{n+1}))
\end{aligned}$$

which implies that $\lim_{n\to\infty} D^*(fu, fx_n, fx_n) = 0$. Hence $fu = \lim_{n\to\infty} fx_n = \lim_{n\to\infty} x_{n+1} = u$, and u is a fixed point of f.

To prove that f has unique fixed point, suppose that $fu = u$ and $fw = w$ for some $u, w \in X$. Then, by the definition of λ-generalized contraction, it follows that

$$\begin{aligned}
D^*(u, w, w) = D^*(fu, fw, fw) &\le qD^*(u, w, w) + rD^*(u, fu, fu) \\
&+ sD^*(w, fw, fw) + tD^*(w, fw, fw) + v\{D^*(u, fw, fw) \\
&+ D^*(w, fw, fw) + D^*(w, fu, fu) + D^*(u, fu, fu) \\
&+ D^*(w, fu, fu) + D^*(w, fw, fw) + D^*(u, fw, fw) \\
&+ D^*(w, fw, fu) + D^*(w, fu, fw)\} \\
&\le (q + 6v)D^*(u, w, w) \le \lambda D^*(u, w, w)
\end{aligned}$$

Which implies that $(1-\lambda)D^*(u, w, w) = 0$, since $\lambda < 1$, $D^*(u, w, w) = 0$. That implies $u = w$. Thus f has unique fixed point. Since x is arbitrary in the above discussion, it follows that $u = \lim_{n\to\infty} f^n x$ for any $x \in X$ and hence equation 3.1 is proved. Finally, since $D^*(x_n, x_{n+p}, x_{n+p}) \leq \lambda^n D^*(x, fx, fx)/(1-\lambda)$ (by 3.4), on letting $p \to \infty$, we obtain $D^*(x_n, u, u) \leq \lambda^n D^*(x, fx, fx)/(1-\lambda)$, proving equation 3.2. □

Corollary 3.2. *Let f be a self-map of a D^*-metric space (X, D^*), and X be f-orbitally complete. If f is a contraction of (X, D^*), then it has a unique fixed point $u \in X$. In fact,*

$$u = \lim_{n\to\infty} f^n x \quad \textit{for} \quad \textit{any} \quad x \in X \quad \textit{and} \tag{3.5}$$

for any $x \in X$ and

$$D^*(f^n x, u, u) \leq \frac{\lambda^n}{1-\lambda} D^*(x, fx, fx) \tag{3.6}$$

for any $x \in X$ and $n \geq 1$.

Proof. Since every contraction is λ-generalized contraction, the Corollary follows from Theorem 3.1. □

Remark 3.3. The Banach contraction principle is a particular case of Corollary 3.2. For, if (X, d) is a complete metric space, then, by Corollary 2.8 , (X, D_1^*) is a complete D^*-metric space, and hence f-orbitally complete for any selfmap f of X. Also, if f is a contraction of (X, d), then the contractive condition can be written as

$$D_1^*(fx, fy, fy) \leq q.D_1^*(x, y, y)$$

for all $x, y \in X$, since $D_1^*(x, y, y) = d(x, y)$; so that f is a contraction on (X, D_1^*). Thus f is a contraction on the f-orbitally complete D^*-metric space (X, D_1^*), and the conclusions of Corollary 3.2 hold for f, and f satisfies the Banach contraction principle.

Corollary 3.4. *Suppose that f is a self-map of a D^*-metric space (X, D^*) and X is f-orbitally complete. If f is a K-contraction of (X, D^*), with constant q, then it has a unique fixed point $u \in X$. In fact,*

$$u = \lim_{n\to\infty} f^n x \tag{3.7}$$

for any $x \in X$ and

$$D^*(f^n x, u, u) \leq \frac{2q^n}{1-2q} D^*(x, fx, fx) \tag{3.8}$$

for all $x \in X$ and $n \geq 1$.

Proof. Since every contraction is a λ-generalized contraction, the Corollary follows from Theorem 3.1 by taking $\lambda = 2q$. □

Remark 3.5. Kannan's result ([5]; p. 406) is a particular case of the Corollary 3.4 . In fact, if (X, d) is a complete metric space, then, by Corollary 2.8, (X, D_1^*) is a complete D^*-metric space, and hence f-orbitally complete for any selfmap f of X. Also, if f is a K-contraction, with constant q, of (X, d), then the condition of K-contraction can be written as

$$D_1^*(fx, fy, fy) \leq q\{D_1^*(x, fx, fx) + D_1^*(y, fy, fy)\} \tag{3.9}$$

for all $x, y \in X$. Since $D_1^*(x, y, y) = d(x, y)$. Thus f is a K-contraction on (X, D_1^*), and f is a K-contraction on the f-orbitally complete D^*-metric space (X, D_1^*). Therefore the conclusions of Corollary 3.4 hold for f, which are the conclusions of Kannan's result.

4. Consequences of Theorem 3.1

Theorem 4.1. *Let f be a self-map of a D^*-metric space (X, D^*) and X be f-orbitally complete. If there is a positive integer k such that f^k is a λ-generalized contraction, then it has a unique fixed point $u \in X$. In fact,*

$$u = \lim_{n\to\infty} f^n x, \tag{4.1}$$

for any $x \in X$ and

$$D^*(f^n x, u, u) \leq \lambda^{n/k}.\rho(x, fx, fx) \tag{4.2}$$

for any $x \in X$ and $n \geq 1$,
where $\rho(x, fx, fx) = \max\{\lambda^{-1}D^*(f^r x, f^{r+k}x, f^{r+k}x) : r = 0, 1, 2, \ldots, k-1\}$.

Proof. Suppose that f^k is a λ-generalized contraction of an f-orbitally complete D^*-metric space (X, D^*). By Theorem 3.1, f^k has unique fixed point. Let u be the fixed point of f^k. Then we claim that fu is also a fixed point of f^k. In fact, $f^k(fu) = f^{k+1}u = f(f^k u) = fu$. By the uniqueness of fixed point of f^k, $fu = u$, showing that u is a fixed point of f. To prove the uniqueness of fixed point of f, let $u, v \in X$ be such that $fu = u$ and $fv = v$. Then $f^k u = u$ and $f^k v = v$ and hence u and v are fixed points of f^k, which has unique fixed point. Hence $u = v$. To prove equation 4.1, let n be any integer. Then by the division algorithm, $n = mk + j$, $0 \leq j < k$, $m \geq 0$ and, for any $x \in X$, $f^n x = (f^k)^m f^j x$. Since f^k is a λ-generalized contraction, by equation 3.2 we have

$$\begin{aligned} D^*(f^n x, u, u) &= D^*((f^k)^m f^j x, u, u) \\ &\leq \frac{\lambda^m}{1-\lambda} D^*(f^j x, f^k f^j x, f^k f^j x) \\ &= \frac{\lambda^m}{1-\lambda} D^*(f^j x, f^{k+j} x, f^{k+j} x) \end{aligned}$$

$D^*(f^n x, u, u) \leq \frac{\lambda^m}{1-\lambda} \max\{D^*(f^i x, f^{i+j}x, f^{i+j}x) : i = 0, 1, 2 \ldots, k-1\} \to 0$ as $m = m(n) \to \infty$. Thus $u = \lim\limits_{n\to\infty} f^n x$ for any $x \in X$. To prove equation

4.2, let n be any positive integer. Since f^k is a λ-generalized contraction and $n = mk + j$, $0 \leq j < k$, $m \geq 0$ with $m = [n/k]$, from equation 3.2 we have

$$\begin{aligned} D^*(f^n x, u, u) &= D^*(f^{mk} f^j x, u, u) \\ &\leq \frac{\lambda^m}{1-\lambda} D^*(f^j x, f^{k+j} x, f^{k+j} x) \\ &= \frac{(\lambda^{1/k})^{mk+j-j}}{1-\lambda} D^*(f^j x, f^{k+j} x, f^{k+j} x) \\ &\leq (\lambda^{1/k})^{mk+j-k} D^*(f^j x, f^{k+j} x, f^{k+j} x) \\ &\leq (\lambda^{1/k})^n \lambda^{-1} D^*(f^j x, f^{k+j} x, f^{k+j} x) \end{aligned}$$

Hence
$D^*(f^n x, u, u) \leq \lambda^{n/k} \max\{\lambda^{-1} D^*(f^i x, f^{i+k} x, f^{i+k} x) : i = 0, 1, 2 \ldots, k-1\}$. $\square$

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