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On $n-$absorbing primary submodules

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Abstract. Let R be a commutative ring with $1 \neq 0$, N a proper submodule of an $R-$module M, and n a positive integer. In this paper, we define N to be an $n-$absorbing primary submodule of M if whenever $a_1 \dots a_n x \in N$ for $a_1, \dots, a_n \in R$ and $x \in M$, then either $a_1 \dots a_n \in (N :_R M)$ or there are $(n-1)$ of the a_i 's whose product with x is in $M - rad(N)$. A number of results concerning $n-$absorbing primary submodules are given.

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1. Introduction

Throughout this paper all rings are commutative with $1 \neq 0$ and all modules are considered to be unitary.

Recently, extensive researches have been done on prime and primary ideals and submodules. Let R be a commutative ring with identity. Of course a proper ideal I of R is said to be *a prime ideal* if $ab \in I$ implies that $a \in I$ or $b \in I$ where $a, b \in R$. There are several ways to generalize the notion of prime ideals and submodules, see for example [3, 5, 14, 15, 11, 22, 30, 32]. Badawi in [7] generalized the concept of prime ideals in a different way. He defined a proper ideal I of R to be *a 2−absorbing ideal* of R if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. In [9], Badawi and Darani generalized the concept of 2−absorbing ideals to the concept of weakly 2−absorbing ideals. They defined a proper ideal I of R to be *a weakly 2−absorbing ideal* of R if whenever $a, b, c \in R$ and $0 \neq abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. Later in [4], Anderson and Badawi introduced the concept of $n-$absorbing ideals of R. According to their definition, a proper ideal I of the ring R is said to be *an $n-$absorbing* (resp., *strongly $n-$absorbing*) *ideal* if whenever $x_1 \dots x_{n+1} \in I$ for $x_1, \dots, x_{n+1} \in R$ (resp., $I_1 \dots I_{n+1} \subseteq I$ for ideals $I_1, \dots, I_{n+1}$ of R), then there are n of the x_i 's (resp., n of the I_i 's) whose product is in I. In [33], the concepts of 2−absorbing and weakly 2−absorbing ideals of the ring R generalized to that of submodules of an $R-$module M as follows: A proper submodule N of an $R-$module M is called *a 2−absorbing* (resp., *weakly 2−absorbing*) *submodule* of M if whenever $a, b \in R$, $m \in M$

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and $abm \in N$ (resp.,$0 \neq abm \in N$), then $ab \in (N :_R M)$ or $am \in N$ or $bm \in N$. In [34], Darani and Soheilnia generalized the concept of $n-$absorbing ideals of the ring R to that of submodules of an $R-$module M. They defined a proper submodule N of an $R-$module M to be *an $n-$absorbing* (resp., *strongly $n-$absorbing*) *submodule* if whenever $a_1 \ldots a_n m \in N$ for $a_1, \ldots, a_n \in R$ and $m \in M$ (resp., $I_1 \ldots I_n L \subseteq N$ for ideals $I_1, \ldots, I_n$ of R and submodule L of M), then either $a_1 \ldots a_n \in (N :_R M)$ (resp., $I_1 \ldots I_n \subseteq (N :_R M)$) or there are $(n-1)$ of the a_i *'s* (resp., I_i *'s*) whose product with m (resp., with L) is in N. In [8], the concept of primary ideals of the ring R generalized to the concept of $2-$absorbing primary ideals. A proper ideal I of R is called *a $2-$absorbing primary ideal* of R if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$, where $\sqrt{I} = \{x \in R : x^k \in I \ for\ some\ positive\ integer\ k\}$ is the radical ideal of I in R. In [24], Mostafanasab et.al. generalized the concept of $2-$absorbing primary ideal of the ring R to that of submodules of an $R-$module M. They defined a proper submodule N of an $R-$module M to be *a $2-$absorbing primary submodule* of M if whenever $a, b \in R$, $m \in M$ and $abm \in N$, then $am \in M-rad(N)$ or $bm \in M-rad(N)$ or $ab \in (N :_R M)$. In [35], Darani et.al. generalized the concept of $2-$absorbing primary submodule to weakly $2-$absorbing primary submodule. They defined a proper submodule N of an $R-$module M to be *a weakly $2-$absorbing primary submodule* of M if whenever $a, b \in R$, $m \in M$ and $0 \neq abm \in N$, then $am \in M-rad(N)$ or $bm \in M-rad(N)$ or $ab \in (N :_R M)$. Most of the concepts concerning prime and primary ideals and submodules have been studied and generalized to graded ring theory, see for example [13, 17, 19, 26, 28]. The motivation of this paper is to continue the study of the family of $n-$absorbing ideals and submodules, also to identify new properties in that subject. The remainder of this paper is organized as follows:
In Section 2, we give some basic definitions and results that are used in the sequel of this paper. Section 3 includes the results and theorems concerning $n-$absorbing primary submodules. We give a useful characterization of an $n-$absorbing primary submodule, (see Theorem 3.6). The first main result of this section is (Theorem 3.11). We show that if N is a submodule of a finitely generated multiplication $R-$module M with $M-rad(N)$ a primary submodule of M, then N is an $n-$absorbing primary submodule of M. One important part of this section is in the case when R is a Noetherian domain and M a torsion-free multiplication $R-$module (see Theorem 3.20). Section 4 includes the conclusion.

2. Preliminary notes

As usual, if N is a proper submodule of an $R-$module M, then the *residual of N by M* is the set $(N :_R M) = \{r \in R : rM \subseteq N\}$ which is an ideal of R. In particular, if $m \in M$, then $(0 :_R m) = \{r \in R : rm = 0\}$ is called the *annihilator of* m. Also, the set $(0 :_R M)$ is an ideal of R called the *annihilator of* M.
The radical $M-rad(N)$ is defined to be the intersection of all prime submodules

of M containing N. If M has no prime submodule containing N, then we say $M - rad(N) = M$. The radical of the module M is defined to be $M - rad(0)$. Recall that a proper submodule N of an $R-$module M is *prime* (resp., *primary*) *submodule* of M if for $r \in R$ and $m \in M$ with $rm \in N$, then either $m \in N$ or $r \in (N :_R M)$ (resp., $r^k \in (N :_R M)$ for some positive integer k). In this case, one can easily verify that $p = (N :_R M)$ (resp., $p = \sqrt{(N :_R M)}$) is a prime ideal of R and we say N is *a $p-$prime* (resp., *$p-$primary*) *submodule.* An $R-$module M is called *faithful* if its annihilator is 0. M is called *a multiplication module* if for each submodule N of M, we have $N = PM$ for some ideal P of R. In this case we can take $P = (N :_R M)$, see [16]. For more details on multiplication modules, one can consult [10] and [2].
An $R-$module M is called *a cancellation module* if $PM = IM$ (for ideals P and I of R) implies $P = I$. Finitely generated faithful multiplication modules are cancellation modules ([29],Corollary to Theorem 9). If M is a finitely generated faithful multiplication $R-$module hence (a cancellation module), then it is easy to see that $(PN :_R M) = P(N :_R M)$ for each submodule N of M and each ideal P of R.
Let M be an $R-$module, and I a prime ideal of R. We say that I is an associated prime of M (or that I is associated to M) if I is the annihilator of some $x \in M$. The set of associated primes of M is denoted by $Ass_R(M)$.

3. Results and discussion

Definition 3.1. Let n be a positive integer. A proper ideal I of a commutative ring R is said to be an $n-$absorbing primary ideal of R if whenever $a_1 \dots a_{n+1} \in I$ for $a_1, \dots, a_{n+1} \in R$, then either $a_1 \dots a_n \in I$ or a product of n of the a_i *'s* (other than $a_1 \dots a_n$) is in $\sqrt{I}$.

Equivalently, one can define $n-$absorbing primary ideals in the following way:
A proper ideal I of a commutative ring R is said to be an $n-$absorbing primary ideal of R if whenever $a_1 \dots a_{n+1} \in I$ for $a_1, \dots, a_{n+1} \in R$, then either $a_1 \dots a_n \in I$ or there exists $1 \leq i \leq n$ such that $a_1 \dots a_{i-1}a_{i+1} \dots a_{n+1} \in \sqrt{I}$. From the definition, one can see that any $n-$absorbing ideal of R is an $n-$absorbing primary ideal of R. However, the converse is not true in general.

Example 3.2. Let $R = \mathbb{Z}$ and $I = (50)$ be an ideal of R. We have $5.5.2 \in I$, $5.5 \notin I$ and $5.2 \notin I$. Thus, I is not a $2-$absorbing ideal of R. However, we have that $5.2 \in \sqrt{I}$ and hence I is a $2-$absorbing primary ideal of R.

Definition 3.3. Let n be a positive integer. A proper submodule N of an $R-$module M is said to be an $n-$absorbing primary submodule of M if whenever $a_1 \dots a_n x \in N$ for $a_1, \dots, a_n \in R$ and $x \in M$, then either $a_1 \dots a_n \in (N :_R M)$ or there are $(n - 1)$ of the a_i *'s* whose product with x is in $M - rad(N)$.

Equivalently, a proper submodule N of an $R-$ module M is called an $n-$absorbing primary submodule of M if whenever $a_1 \dots a_n x \in N$ for

$a_1, \ldots, a_n \in R$ and $x \in M$, then either $a_1 \ldots a_n \in (N :_R M)$ or there exists $1 \leq i \leq n$ such that $a_1 \ldots a_{i-1}a_{i+1} \ldots a_n x \in M - rad(N)$.

Definition 3.4. Let n be a positive integer. A proper submodule N of an $R-$module M is said to be a weakly $n-$absorbing primary submodule of M if whenever $0 \neq a_1 \ldots a_n x \in N$ for $a_1, \ldots, a_n \in R$ and $x \in M$, then either $a_1 \ldots a_n \in (N :_R M)$ or there are $(n-1)$ of the a_i *'s* whose product with x is in $M - rad(N)$.

Theorem 3.5. *If N is an $n-$absorbing primary submodule of an $R-$module M, then it's an $m-$absorbing primary submodule of M for every positive integer $m > n$.*

Proof. Let N be an $n-$absorbing primary submodule of M. We need to show that N is an $(n+1)-$absorbing primary submodule of M. Let $a_1 a_2 \ldots a_{n+1} x \in N$ for $a_1, a_2, \ldots, a_{n+1} \in R$ and $x \in M$. Now set $a_1 a_2 = \overline{a}$. Then $\overline{a} \ldots a_{n+1} x \in N$ implies $\overline{a} \ldots a_{n+1} \in (N :_R M)$ or $\overline{a} \ldots a_{i-1} a_{i+1} \ldots a_{n+1} x \in M - rad(N)$ or $a_3 a_4 \ldots a_{n+1} x \in M - rad(N)$ for some $3 \leq i \leq n+1$. Hence, N is an $m-$absorbing primary submodule of M for $m > n$. □

Now, we give a characterization of an $n-$absorbing primary submodule:

Theorem 3.6. *Let N be a proper submodule of an $R-$module M. Then the following statements are equivalent:*

(i) *N is an $n-$absorbing primary submodule of M;*

(ii) *If $a_1 \ldots a_n \notin (N :_R M)$, where $a_1, \ldots, a_n \in R$, then $(N :_M a_1 \ldots a_n) \subseteq \bigcup_{i=1}^{n} (M - rad(N) :_M a_1 a_2 \ldots a_{i-1} a_{i+1} \ldots a_n)$.*

Proof. $(i) \implies (ii)$ Assume that $a_1, \ldots, a_n \in R$ are such that $a_1 \ldots a_n \notin (N :_R M)$. Let $x \in (N :_M a_1 \ldots a_n)$. Then $a_1 \ldots a_n x \in N$, and so there are $(n-1)$ of the a_i *'s* whose product with x is in $M - rad(N)$. Then there exists $k \in \{1, 2, \ldots, n\}$ such that $a_1 a_2 \ldots a_{k-1} a_{k+1} \ldots a_n x \in M - rad(N)$, which implies that $x \in (M - rad(N) :_M a_1 a_2 \ldots a_{k-1} a_{k+1} \ldots a_n) \subseteq \bigcup_{i=1}^{n} (M - rad(N) :_M a_1 a_2 \ldots a_{i-1} a_{i+1} \ldots a_n)$.
$(ii) \implies (i)$ Let $a_1 a_2 \ldots a_n x \in N$ for some $a_1, a_2, \ldots, a_n \in R$ and $x \in M$. Assume that $a_1 a_2 \ldots a_n \notin (N : M)$. This implies that $x \in (N :_M a_1 a_2 \ldots a_n) \subseteq \bigcup_{i=1}^{n} (M - rad(N) :_M a_1 a_2 \ldots a_{i-1} a_{i+1} \ldots a_n)$. Thus, we have $a_1 a_2 \ldots a_{i-1} a_{i+1} \ldots a_n x \in M - rad(N)$ for some $i \in \{1, 2, \ldots, n\}$. Therefore, N is an $n-$absorbing primary submodule of M. □

Recall from [27] that a ring R is said to be a $u-$ring if $I \subseteq \bigcup_{i=1}^{n} I_i$ for some ideals $I, I_1, I_2, \ldots, I_n$ of R implies that $I \subseteq I_k$ for some $k \in \{1, 2, \ldots, n\}$.

Similar to the concept of a $u-$ring, define the concept of a $u-$module as follows:

Definition 3.7. An $R-$module M is said to be a $u-$module if $N \subseteq \bigcup_{i=1}^{n} N_i$ for some submodules $N, N_1, N_2, \ldots, N_n$ of M implies that $N \subseteq N_k$ for some $k \in \{1, 2, \ldots, n\}$.

Theorem 3.8. *Let M be a finitely generated multiplication $u-$module over the ring R. If N is an $n-$absorbing primary submodule of M, then $(N :_R M)$ is an $n-$absorbing primary ideal of R.*

Proof. Let $a_1, \ldots, a_{n+1} \in R$ be such that $a_1 \ldots a_{n+1} \in (N :_R M)$. This implies that $a_{n+1}M \subseteq (N :_M a_1 a_2 \ldots a_n)$. Now assume that $a_1 a_2 \ldots a_n \notin (N :_R M)$. Then by Theorem 3.6, we have $a_{n+1}M \subseteq \bigcup_{i=1}^{n}(M - rad(N) :_N a_1 a_2 \ldots a_{i-1} a_{i+1} \ldots a_n)$. Since M is a $u-$module, we conclude that $a_{n+1}M \subseteq (M - rad(N) :_M a_1 a_2 \ldots a_{k-1} a_{k+1} \ldots a_n)$ for some $k \in \{1, 2, \ldots, n\}$. Thus, we have $a_1 a_2 \ldots a_{k-1} a_{k+1} \ldots a_{n+1} \in (M - rad(N) :_R M) = \sqrt{(N :_R M)}$. Therefore, $(N :_R M)$ is an $n-$absorbing primary ideal of R. □

Theorem 3.9. *Let N be a submodule of an $R-$module M. If $M - rad(N)$ is prime submodule of M, then N is an $n-$absorbing primary submodule of M.*

Proof. Let $a_1 a_2 \ldots a_n x = a_1(a_2 \ldots a_n x) \in N \subseteq M - rad(N)$ for some $a_1, a_2, \ldots, a_n \in R$ and $x \in M$. Assume that $a_2 \ldots a_n x \notin M - rad(N)$. Since $M - rad(N)$ is prime, we conclude that $a_1 \in (M - rad(N) :_R M)$, which implies that $a_1 a_2 \ldots a_{i-1} a_{i+1} \ldots a_n x \in M - rad(N)$. Therefore, N is an $n-$absorbing primary submodule of M. □

Proposition 3.10. *([24], Proposition* 2.5.*) Let M be a finitely generated multiplication $R-$module and N be a submodule of M. Then the following statements are equivalent:*

(i) $M - rad(N)$ is a primary submodule of M.

(ii) $M - rad(N)$ is a prime submodule of M.

Theorem 3.11. *Let M be a finitely generated multiplication $R-$module and N be a submodule of M. If $M - rad(N)$ is primary submodule of M, then N is an $n-$absorbing primary submodule of M.*

Proof. Assume that M is a finitely generated multiplication $R-$module and $M - rad(N)$ is a primary submodule of M, then by Proposition 3.10, $M - rad(N)$ is a prime submodule of M and therefore N is an $n-$absorbing primary submodule of M by Theorem 3.9. □

Theorem 3.12. *Let M be a faithful (resp., finitely generated faithful) multiplication $R-$module. If $M - rad(N)$ is a prime (resp., primary) submodule of M, then N^n is an $n-$absorbing primary submodule of M for every positive integer n.*

Proof. Assume that M is a faithful (resp., finitely generated faithful) multiplication $R-$module and $M - rad(N)$ is a prime (resp., primary) submodule of M. Then there exists an ideal P of R such that $N = PM$. Since for any faithful multiplication module M, we have $M - rad(IM) = \sqrt{I}M$ for any ideal I of R by ([1], Theorem 1(3)). Then $M - rad(N^n) = \sqrt{P^n}M = M - rad(N)$ which is a prime (resp., primary) submodule of M. Therefore, N^n is an $n-$absorbing primary submodule of M for every positive integer n by Theorem 3.9 and Theorem 3.11. □

Proposition 3.13. *([24], Proposition 2.14.) Let M be a multiplication $R-$module and K, N be submodules of M. Then we have the following:*

(i) $\sqrt{(KN :_R M)} = \sqrt{(K :_R M)} \cap \sqrt{(N :_R M)}$.

(ii) $M - rad(KN) = M - rad(K) \cap\ M - rad(N)$.

(iii) $M - rad(K \cap\ N) = M - rad(K) \cap\ M - rad(N)$.

Theorem 3.14. *Let M be a multiplication $R-$module and $N_1, \ldots, N_m$ are $n-$absorbing primary submodules of M with the same $M-$radical. Then $N_1 \bigcap \ldots \bigcap\ N_m$ is an $n-$absorbing primary submodule of M.*

Proof. $M - rad(N_1 \bigcap \ldots \bigcap\ N_m) = \bigcap_{i=1}^{m} M - rad(N_i)$ (by Proposition 3.13). Assume that $a_1 \ldots a_n x \in\ N$ for $a_1, \ldots, a_n \in\ R$ and $x \in\ M$ and $a_1 \ldots a_n \notin (N_1 \bigcap \ldots \bigcap\ N_m :_R M)$. Then $a_1 \ldots a_n \notin (N_i :_R M)$ for some $i \in \{1, 2, \ldots, m\}$. Hence there are $(n-1)$ of the a_i *'s* whose product with x is in $M - rad(N_i)$. But N_i *'s* have the same $M-$radical, so $N_1 \bigcap \ldots \bigcap\ N_m$ is an $n-$absorbing primary submodule of M. □

Theorem 3.15. *Let M be a multiplication $R-$module. If N_j is an n_j-absorbing primary submodule with the same radical of M for all $j \in \{1, \ldots, m\}$, then $N_1 \cap \ldots \cap\ N_m$ is an $n-$absorbing primary submodule of M with $n = n_1 + \ldots + n_m$.*

Proof. Since N_j is n_j-absorbing primary submodule with $n_j \leq\ n$, then by Theorem 3.5, N_j is an $n-$absorbing primary submodule of M. By Theorem 3.14, $N_1 \cap \ldots \cap\ N_m$ is an $n-$absorbing primary submodule of M. □

Recall that a commutative ring R with nonzero identity is said to be *a divided ring* if for every prime ideal I of R, we have $I \subseteq\ aR$ for all $a \in\ R \setminus I$, see [12]. Also the reader can consult [6] and [21] for more information on divided rings. Also, in [31], Tekir et.al. extended the concept of divided rings to modules as follows:

Definition 3.16. An $R-$module M is said to be *a divided module* if every prime submodule P of M is comparable with Rm for each $m \in\ M$, or equivalently, $P \subseteq\ Rm$ for each $m \in\ M - P$.

Theorem 3.17. *Every proper submodule N of a divided $R-$module M is an $n-$absorbing primary submodule of M.*

Proof. Suppose that N is a proper submodule of the $R-$module M. By ([31], Proposition 1), prime submodules of a divided module are linearly ordered. So $M - rad(N)$ is a prime submodule of M. Hence, we are done by definition. □

Remark 3.18. Assume that $I = (0 :_R M)$ and $A = R/I$. It is easy to see that:

(i) N is an $n-$absorbing primary $R-$submodule of M if and only if N is an $n-$absorbing primary $A-$submodule of M.

(ii) $(N :_R M)$ is an $n-$absorbing primary ideal of R if and only if $(N :_A M)$ is an $n-$absorbing primary ideal of A.

Theorem 3.19. *Let M be an $R-$module and S be a multiplicatively closed subset of R. If N is an $n-$absorbing primary submodule of M and $S^{-1}N \neq S^{-1}M$, then $S^{-1}N$ is an $n-$absorbing primary submodule of $S^{-1}M$.*

Proof. Let $a_1, \ldots, a_n \in R$, $s_1, \ldots, s_n \in S$ and $\frac{x}{s} \in S^{-1}M$ be such that $\frac{a_1}{s_1}\frac{a_2}{s_2}\ldots\frac{a_n}{s_n}\frac{x}{s} \in S^{-1}N$. Then there exists $m \in S$ such that $ma_1a_2\ldots a_nx \in N$. As N is an $n-$absorbing primary submodule of M, we get either $a_1a_2\ldots a_n \in (N :_R M)$ or $ma_1\ldots a_{i-1}a_{i+1}\ldots a_nx \in M-rad(N)$ for some $1 \leq i \leq n$. The first case implies that
$\frac{a_1}{s_1}\frac{a_2}{s_2}\ldots\frac{a_n}{s_n} = \frac{a_1a_2\ldots a_n}{s_1s_2\ldots s_n} \in S^{-1}(N :_R M) \subseteq (S^{-1}N :_{S^{-1}R} S^{-1}M)$.
The second case implies that
$\frac{a_1}{s_1}\frac{a_2}{s_2}\ldots\frac{a_{i-1}a_{i+1}}{s_{i-1}s_{i+1}}\ldots\frac{a_nx}{s_ns} \in S^{-1}(M-rad(N)) \subseteq S^{-1}M-rad(S^{-1}N)$.
Hence $S^{-1}N$ is an $n-$absorbing primary submodule of $S^{-1}M$. □

Let $T(R)$ be the total quotient ring of the commutative ring R. A non zero ideal I of R is called *an invertible ideal of* R if $II^{-1} = R$, where $I^{-1} = \{x \in T(R) : xI \subseteq R\}$. In [25], Naoum and Al-Alwan generalized the concept of an invertible ideal to the concept of an invertible submodule:
Let M be an $R-$module and let $S = R\backslash\{0\}$. Then $G = \{g \in S : gx = 0 \text{ for some } x \in M \text{ implies } x = 0\}$ is a multiplicatively closed subset of R. Let N_1 be a submodule of M and let $N_2 = \{m \in R_G : mN_1 \subseteq M\}$. A submodule N_1 is said to be *invertible in* M if $N_2N_1 = M$. A nonzero $R-$module M is said to be *a Dedekind module* if each nonzero submodule of M is invertible. For more information on Dedekind and generalized Dedekind modules, the reader can consult [1].

Theorem 3.20. *Let R be a Noetherian domain, M a torsion-free multiplication $u-$module over R. Then the following statements are equivalent:*

(i) M is a Dedekind module;

(ii) If N is a nonzero $n-$absorbing primary submodule of M, then either $N = A^n$ for some maximal submodule A of M and some positive integer n or $N = A_1^nA_2^m$ for some maximal submodules A_1 and A_2 of M and some positive integers n, m;

(iii) If N is a nonzero $n-$absorbing primary submodule of M, then either $N = P^n$ for some prime submodule P of M and some positive integer n or $N = N_1^nN_2^m$ for some prime submodules N_1 and N_2 of M and some positive integers n, m.

Proof. $(i) \Longrightarrow (ii)$ Since every multiplication module over a Noetheian ring is a Noetherian module, so M is Noetherian and hence finitely generated. As N is an $n-$absorbing primary submodule of M, so by Theorem 3.8, $(N :_R M)$ is an $n-$absorbing primary ideal of R. Now, $N = IM = (N :_R M)M$ for some proper ideal I of R. Since a finitely generated torsion free multiplication

module M over a domain R is a Dedekind module iff R is a Dedekind domain by ([20], Theorem 2.13). Then, we have either $I = L^n$ for some maximal ideal L of R and some positive integer n or $I = L_1^n L_2^m$ for some maximal ideals L_1 and L_2 of R and some positive integers $n,\ m$ by ([7], Theorem 2.11.). Hence, either $N = L^n M = (LM)^n = A^n$ where $A = LM$ or $N = (L_1 M)^n (L_2 M)^m = A_1^n A_2^m$ where $A_1 = L_1 M$ and $A_2 = L_2 M$.
$(ii) \Longrightarrow (iii)$ It is clear.
$(iii) \Longrightarrow (i)$ We need to show that R is a Dedekind domain. Let I be an ideal of R and L be a maximal ideal of R be such that $L^2 \subset I \subset L$. Then $\sqrt{I} = L$ and so that $M - rad(IM) = LM$, since M is a faithful multiplication $R-$module. Then by Theorem 3.11, IM is an $n-$absorbing primary submodule of M. Now by (iii), either $IM = P^n$ for some prime submodule P of M and some positive integer n or $IM = N_1^n N_2^m$ for some prime submodules N_1 and N_2 of M and some positive integers $n,\ m$. Since M is a cancellation module, then $I = J^n$ for some prime ideal J of R and some positive integer n or $I = J_1^n J_2^m$ for some prime ideals J_1 and J_2 of R and some positive integers $n,\ m$ in which any of the two cases make a contradiction. Thus there are no ideals properly between L^2 and L. Therefore, R is a Dedekind domain by ([18], Theorem 39.2). □

Lemma 3.21. *([23], Corollary 1.3) Let M and $\overline{M}$ be $R-$modules with $f : M \longrightarrow \overline{M}$ an $R-$module epimorphism. If N is a submodule of M containing $Ker(f)$, then $f(M - rad(N)) = \overline{M} - rad(f(N))$.*

Theorem 3.22. *Let M and $\overline{M}$ be $R-$modules and let $f : M \longrightarrow \overline{M}$ be an $R-$module homomorphism. Then we have the following:*

(i) If $\overline{N}$ is an $n-$absorbing primary submodule of $\overline{M}$, then $f^{-1}(\overline{N})$ is an $n-$absorbing primary submodule of M.

(ii) If f is epimorphism and N is an $n-$absorbing primary submodule of M containing $Ker(f)$, then $f(N)$ is an $n-$absorbing primary submodule of $\overline{M}$.

Proof. (i) Let $a_1, \ldots, a_n \in R$ and $x \in M$ such that $a_1 \ldots a_n x \in f^{-1}(\overline{N})$. Then $a_1 \ldots a_n f(x) \in \overline{N}$. Thus, either $a_1 \ldots a_n \in (\overline{N} :_R \overline{M})$ or there are $(n-1)$ of the $a_i\ 's$ whose product with $f(x)$ is in $\overline{M} - rad(\overline{N})$ and hence, either $a_1 \ldots a_n \in (f^{-1}(\overline{N}) :_R M)$ or there are $(n-1)$ of the $a_i\ 's$ whose product with x is in $f^{-1}(\overline{M} - rad(\overline{N}))$. Now, by using the inclusion $f^{-1}(\overline{M} - rad(\overline{N})) \subseteq M - rad(f^{-1}(\overline{N}))$, we have $f^{-1}(\overline{N})$ is an $n-$absorbing primary submodule of M.
(ii) Let $a_1, \ldots, a_n \in R$ and $\overline{y} \in \overline{M}$ be such that $a_1 \ldots a_n \overline{y} \in f(N)$. By assumption there exists $x \in M$ such that $\overline{y} = f(x)$ and so $f(a_1 \ldots a_n x) \in f(N)$. Since, $Ker(f) \subseteq N$, we have $a_1 \ldots a_n x \in N$. Then either $a_1 \ldots a_n \in (N :_R M)$ or there are $(n-1)$ of the $a_i\ 's$ whose product with x is in $M - rad(N)$. Thus, either $a_1 \ldots a_n \in (f(N) :_R \overline{M})$ or there are $(n-1)$ of the $a_i\ 's$ whose product with $\overline{y}$ is in $f(M - rad(N)) = \overline{M} - rad(f(N))$. Therefore, $f(N)$ is an $n-$absorbing primary submodule of $\overline{M}$. □

Corollary 3.23. *Let L and N be submodules of an $R-$module M such that $L \subseteq N$. If N is an $n-$absorbing primary submodule of M, then N/L is an $n-$absorbing primary submodule of M/L.*

Proof. Follows directly from Theorem 3.22 (ii). □

Theorem 3.24. *Let L and N be submodules of an $R-$module M such that $L \subset N \subset M$. If L is an $n-$absorbing primary submodule of M and N/L is a weakly $n-$absorbing primary submodule of M/L, then N is an $n-$absorbing primary submodule of M.*

Proof. Let $a_1, \dots, a_n \in R$ and $x \in M$ such that $a_1 \dots a_n x \in N$. If $a_1 \dots a_n x \in L$, then either $a_1 \dots a_n \in (L :_R M) \subseteq (N :_R M)$ or there are $(n-1)$ of the a_i $'s$ whose product with x is in $M - rad(L) \subseteq M - rad(N)$. So assume that $a_1 \dots a_n x \notin L$. Then $0 \neq a_1 \dots a_n(x + L) \in N/L$ implies that either $a_1 \dots a_n \in (N/L :_R M/L)$ or there are $(n-1)$ of the a_i $'s$ whose product with $(x+L)$ is in $M/L - rad(N/L) = \frac{M-rad(N)}{L}$. It means that either $a_1 \dots a_n \in (N :_R M)$ or there are $(n-1)$ of the a_i $'s$ whose product with x is in $M - rad(N)$. Therefore, N is an $n-$absorbing primary submodule of M. □

According to [24]:
Let R_i be a commutative ring with identity and M_i be an R_i-module, for $i = 1, 2$. Let $R = R_1 \times R_2$. Then $M = M_1 \times M_2$ is an $R-$module and each submodule of M is of the form $N = N_1 \times N_2$ for some submodules N_1 of M_1 and N_2 of M_2. In addition, if M_i is a multiplication R_i-module, for $i = 1, 2$, then M is a multiplication $R-$module. In this case, for each submodule $N = N_1 \times N_2$ of M we have $M - rad(N) = M_1 - rad(N_1) \times M_2 - rad(N_2)$.

Theorem 3.25. *Let M_1 be a multiplication R_1-module and M_2 be a multiplication R_2-module and let $R = R_1 \times R_2$ and $M = M_1 \times M_2$. Then the following hold:*

(i) A proper submodule L_1 of M_1 is an $n-$absorbing primary submodule if and only if $N = L_1 \times M_2$ is an $n-$absorbing primary submodule of M.

(ii) A proper submodule L_2 of M_2 is an $n-$absorbing primary submodule if and only if $N = M_1 \times L_2$ is an $n-$absorbing primary submodule of M.

Proof. (i) Assume that $N = L_1 \times M_2$ is an $n-$absorbing primary submodule of M. Since N is a proper submodule of M, so $L_1 \neq M_1$. Let $\overline{M} = \frac{M}{\{0\} \times M_2}$. Then $\overline{N} = \frac{N}{\{0\} \times M_2}$ is an $n-$absorbing primary submodule of $\overline{M}$ by Corollary 3.23. Since $\overline{M}$ is module-isomorphic to M_1 and $\overline{N}$ is module-isomorphic to L_1, so L_1 is an $n-$absorbing primary submodule of M_1.
Conversely, assume that L_1 is an $n-$absorbing primary submodule of M_1, then it is easy to see that $N = L_1 \times M_2$ is an $n-$absorbing primary submodule of M.
(ii) Proceed similarly to (i). □

Lemma 3.26. *If I is an $n-$absorbing primary ideal of R, then $\sqrt{I}$ is an $n-$absorbing ideal of R*

Proof. Let $a_1, \ldots, a_{n+1} \in R$ be such that $a_1 \ldots a_{n+1} \in \sqrt{I}$ and the product of a_{n+1} with $(n-1)$ of $a_1, \ldots, a_n \notin \sqrt{I}$. Since $a_1 \ldots a_{n+1} \in \sqrt{I}$, then $(a_1 \ldots a_{n+1})^k = a_1^k \ldots a_{n+1}^k \in I$ for some positive integer k. Since I is an $n-$absorbing primary ideal of R and the product of a_{n+1} with $(n-1)$ of $a_1, \ldots, a_n$ is not in $\sqrt{I}$, we conclude that $a_1^k \ldots a_n^k = (a_1 \ldots a_n)^k \in I$, and thus $a_1 \ldots a_n \in \sqrt{I}$. Therefore, $\sqrt{I}$ is an $n-$absorbing ideal of R. □

Theorem 3.27. *Let I be an $n-$absorbing primary ideal of the ring R and let M be a faithful multiplication $R-$module with $Ass_R(M/\sqrt{I}M)$ a totally ordered set. Then $a_1 \ldots a_n x \in IM$ implies that $a_1 \ldots a_{n-1} x \in \sqrt{I}M$ or $a_n x \in \sqrt{I}M$ or $a_1 \ldots a_n \in I$, whenever $a_1, \ldots, a_n \in R$ and $x \in M$.*

Proof. Assume that $a_1, \ldots, a_n \in R$, $x \in M$ and $a_1 \ldots a_n x \in IM$. If $(\sqrt{I}M :_R a_j x) = R$ for some $1 \leq j \leq n$, then we are done. Now, suppose that $(\sqrt{I}M :_R a_j x)$ are proper ideals of R for all $1 \leq j \leq n$. Since $Ass_R(M/\sqrt{I}M)$ is a totally ordered set, then $\bigcup_{j=1}^{n}(\sqrt{I}M :_R a_j x)$ is an ideal of R and so there exists a maximal ideal P such that $\bigcup_{j=1}^{n}(\sqrt{I}M :_R a_j x) \subseteq P$. We claim that $a_1 x \notin T_P(M) = \{\overline{x} \in M : (1-y)\overline{x} = 0 \text{ } for \text{ } some \text{ } y \in P\}$. To prove the claim, assume on the contrary that $a_1 x \in T_P(M)$. This implies that $(1-y)a_1 x = 0$ for some $y \in P$, thus $(1-y)a_1 x \in \sqrt{I}M$ and so $1-y \in (\sqrt{I}M :_R a_1 x) \subseteq P$, a contradiction.
Now by ([16], Theorem 1.2), there are $y \in P$ and $\overline{x} \in M$ such that $(1-y)M \subseteq R\overline{x}$. Thus, $(1-y)x = s\overline{x}$ for some $s \in R$. As $a_1 \ldots a_n x \in IM$, so $(1-y)(a_1 \ldots a_n x) = b\overline{x}$ for some $b \in I$. Thus $(a_1 \ldots a_n s - b)\overline{x} = 0$ and so $(1-y)(a_1 \ldots a_n s - b)M \subseteq (a_1 \ldots a_n s - b)R\overline{x} = 0$. But M is faithful, so $(1-y)(a_1 \ldots a_n s - b) = 0$. Therefore, $(1-y)(a_1 \ldots a_n s) = (1-y)b \in I$. Then $(1-y)(a_1 \ldots a_{n-1})s \in \sqrt{I}$ or $(1-y)a_n \in \sqrt{I}$ or $a_1 \ldots a_n s \in I$, because I is an $n-$absorbing primary ideal of R. If $(1-y)(a_1 \ldots a_{n-1})s \in \sqrt{I}$, then $(1-y)(a_1 \ldots a_{n-1}) \in \sqrt{I}$ or $(1-y)s \in \sqrt{I}$ or $(a_1 \ldots a_{n-1})s \in \sqrt{I}$, because $\sqrt{I}$ is an $n-$absorbing ideal of R by Lemma 3.26. If $(1-y)(a_1 \ldots a_{n-1}) \in \sqrt{I}$, then $(1-y)(a_1 \ldots a_{n-1} x) \in \sqrt{I}M$ and so $1-y \in (\sqrt{I}M :_R a_1 \ldots a_{n-1} x) \subseteq P$, a contradiction. If $(1-y)s \in \sqrt{I}$, then $(1-y)^2 x = (1-y)s\overline{x} \in \sqrt{I}M$ which implies that $(1-y)^2 \in (\sqrt{I}M :_R x) \subseteq (\sqrt{I}M :_R a_1 \ldots a_{n-1} x) \subseteq P$, a contradiction. Similarly, we can get that $(1-y)a_n \notin \sqrt{I}$. Now $a_1 \ldots a_{n-1} s \in \sqrt{I}$ implies that $(1-y)a_1 \ldots a_{n-1} x = a_1 \ldots a_{n-1} s\overline{x} \in \sqrt{I}M$ and so $1-y \in (\sqrt{I}M :_R a_1 \ldots a_{n-1} x) \subseteq P$, a contradiction. If $a_1 \ldots a_n s \in I$, then $a_1 \ldots a_{n-1} s \in \sqrt{I}$ or $a_n s \in \sqrt{I}$ or $a_1 \ldots a_n \in I$ of which the first two cases are impossible, thus $a_1 \ldots a_n \in P$. □

4. Conclusion

In this paper, we considered $n-$absorbing primary submodules. Weakly $n-$absorbing primary submodules have been defined and have not been studied in depth. Future research on weakly $n-$absorbing primary submodules over commutative rings can therefore be constructed.

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