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Cosine families of bounded linear operators on non-Archimedean Banach spaces

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Abstract. In this paper, we initiate the investigation of cosine families of bounded linear operators on non-Archimedean Banach spaces. We show some properties of non-Archimedean C_0-cosine operator functions.

Examples are given to support our work and we will discuss the solvability of some homogeneous p-adic second-order differential equations where the parameter of C_0-cosine family of bounded linear operators belongs to a clopen ball Ω_r of the ground field $\mathbb{K}$.

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1. Introduction and preliminaries

In the classical setting, $\mathbb{R}$ and $\mathbb{C}$ denote the set of real numbers and complex numbers, respectively. Cosine functions are closely related with differential equations on Archimedean Banach space X over $\mathbb{C}$:

$$u^{''}(t) - B^2u(t) = 0 \; or \; u^{''}(t) - Bu(t) = 0, \; t \in \mathbb{R},$$

with corresponding initial conditions and $B : D(B) \to X$ is a linear operator. Solutions are given by $u(t) = ch(Bt), \; t \in \mathbb{R}$ (ch: hyperbolic cosine). By direct calculation one verifies that thus defined functions satisfy the following functional equations:

$$\forall t, s \in \mathbb{R}, C(t+s) + C(t-s) = 2C(t)C(s), \; C(0) = I,$$

I is the unit operator on X. It has been observed by Augustin Louis, baron Cauchy $(1789 - 1857)$.

In [13], M. Sova attempted to make a systematic study of the basic properties of operator functions called cosine operator functions i.e., functions which satisfy the well-know functional equation of d'Alembert. In the real case such a functional equation is satisfied by the class of all $cos(at)$, ch(at) (ch: hyperbolic

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cosine, a arbitrary real number). Many studies were started for cosine families of linear operators on Banach or Hilbert spaces; for examples of the pioneering work of H.O. Fattorini, who treats in detail some of applications of the second Cauchy problem for linear differential equations and also who studied the uniformly bounded cosine functions in Hilbert spaces for more details, we refer to [1], and [7]. Moreover, I. Cioranescu and C. Lizama [2] characterized the spectrum of strongly continuous cosine functions defined in a Hilbert space in terms of properties of the infinitesimal generator. In [13], M. Sova proved that the infinitesimal generator of all uniformly continuous cosine family of linear operators on a Banach space is a bounded operator. Recently, R. Ameziane Hassani, A. Blali, A. El Amrani and K. Moussaouja showed by counterexample that the result mentioned above is not true in general on Fréchet spaces, and they proved that the infinitesimal generator of an uniformly continuous cosine family of operators in a class of Fréchet spaces (quojection) is necessarily continuous. For more details, we refer to [8], and also they proved if the Quojection-Fréchet spece X is a Grothendieck space with the Dunford-Pettis property, then every C_0-cosine family is necessarily uniformly continuous and therefore its infinitesimal generator is a continuous linear operator. M. Kostić studied the convoluted C-cosine functions, convoluted C-semigroups and composition property and automatic extension of convoluted C-cosine functions; for more details, we refer to [10].

In the non-Archimedean setting, [3] T. Diagana started with a brief conceptualization of C_0-groups on (ultrametric) free Banach spaces X. In contrast with the classical setting the parameter of a given C_0-group belongs to a clopen ball Ω_r of the ground field $\mathbb{K}$. In [6], A. El Amrani et al. introduced and studied new classes of linear operators so called C_0-groups, C-groups and cosine families of bounded linear operators on non-Archimedean Banach spaces over a non-Archimedean complete valued field $\mathbb{K}$ of characteristic zero (i.e., $char(\mathbb{K}) = 0$). Throughout this paper, X is a non-Archimedean (n.a) Banach space over a (n.a) non trivially complete valued field $\mathbb{K}$ with valuation $|\cdot|$, $B(X)$ denotes the set of all bounded linear operators from X into X, $\mathbb{Q}_p$ is the field of p-adic numbers ($p \geq 2$ being a prime) equipped with p-adic valuation $|.|_p$, $\mathbb{Z}_p$ denotes the ring of p-adic integers (the ring of p-adic integers $\mathbb{Z}_p$ is the unit ball of $\mathbb{Q}_p$). For more details, we refer to [9] and [12]. We denote the completion of algebraic closure of $\mathbb{Q}_p$ under the p-adic absolute value $|\cdot|_p$ by $\mathbb{C}_p$ [9]. Let $r > 0$, Ω_r is the clopen ball of $\mathbb{K}$ centred at 0 with radius r, e.g., $\Omega_r = \{t \in \mathbb{K} : |t| < r\}$. In contrast with the classical context, the p-adic hyperbolic cosine and cosine functions are given by

$$ch(q) = \sum_{n=0}^{+\infty} \frac{q^{2n}}{(2n)!} \; and \; cos(q) = \sum_{n=0}^{+\infty} (-1)^n \frac{q^{2n}}{(2n)!}$$

respectively, they are not always well defined and analytic for each $q \in \mathbb{Q}_p$. However, they do converges for all $q \in \mathbb{Q}_p$ such that $|q|_p < r = p^{\frac{-1}{p-1}}$. For more details, we refer to [9] and [12]. We begin with some preliminaries.

Definition 1.1 ([4], Definition 2.1)**.** Let X be a vector space over $\mathbb{K}$.

A non-negative real valued function $\|\cdot\| : X \to \mathbb{R}_+$ is called a non-Archimedean norm if:

(1) For all $x \in X$, $\|x\| = 0$ if and only if $x = 0$,

(2) For any $x \in X$ and $\lambda \in \mathbb{K}$, $\|\lambda x\| = |\lambda|\|x\|$,

(3) For any $x\,, y \in X$, $\|x + y\| \leq \max(\|x\|, \|y\|)$.

Property (3) of Definition 1.1 is referred to as the ultrametric or strong triangle inequality.

Definition 1.2 ([4], Definition 2.2)**.** A non-Archimedean normed space is a pair $(X; \|\cdot\|)$, where X is a vector space over $\mathbb{K}$ and $\|\cdot\|$ is a non-Archimedean norm on X.

Definition 1.3 ([4], Definition 2.2)**.** A non-Archimedean Banach space is a vector space endowed with a non-Archimedean norm, which is complete.

For more details on non-Archimedean Banach spaces and related issues, see for example [4], [5],[11] and [14].

Proposition 1.4 ([4], Proposition 2.16)**.** *(1) A closed subspace of a non-Archimedean Banach space is a non-Archimedean Banach space;*

(2) The direct sum of two non-Archimedean Banach spaces is a non-Archimedean Banach space.

In this section, we define and discuss properties of non-Archimedean Banach spaces which have bases.

Definition 1.5 ([4], Definition 2.5)**.** A non-Archimedean Banach space $(X, \|\cdot\|)$ over a non-Archimedean valued field (complete) $(\mathbb{K}, |\;|)$ is said to be a free non-Archimedean Banach space if there exists a family $(x_i)_{i\in I}$ of elements of X indexed by a set I such that each element $x \in X$ can be written uniquely like a pointwise convergent series defined by $x = \sum_{i\in I} \lambda_i x_i$, and $\|x\| = \sup_{i\in I} |\lambda_i|\|x_i\|$.

The family $(x_i)_{i\in I}$ is then called an orthogonal basis for X; which is equivalent to saying that $(x_i)_{i\in I}$ is a Schauder basis (or a basis), in this case (X is a $n.a$ Banach space), [11, theorem 2.3.11]. If, for all $i \in I$, $\|x_i\| = 1$, then $(x_i)_{i\in I}$ is called an orthonormal basis of X. For more details of orthogonality and the concepts of bases in the non-Archimedean case, we refer to [11] and [14].

However the treatment of those non-Archimedean Banach spaces in the general case can be found in [5] and [11]. Moreover, X is a free non-Archimedean Banach space over $\mathbb{K}$ if and only if X is isometrically isomorphic to $c_0(I, u)$ for certain index set I and an application $u : I \to \mathbb{R}_+^*$. By [11, Theorem 2.58] $c_0(I)$ is of countable type if and only if I is countable. For more details we refer to [11] and [14]. In this work the basis of free $n.a$ Banach spaces considered is countable $I = \mathbb{N}$.

Definition 1.6. [4] Let $(X, \|\cdot\|)$ be a non-Archimedean Banach space. The non-Archimedean Banach space $(B(X), \|\cdot\|)$ is the collection of all bounded linear operators from X into itself equipped with the operator-norm defined by

$$(\forall A \in B(X))\ \ \|A\| = \sup_{x \in X\setminus\{0\}} \frac{\|A\,(x)\,\|}{\|x\|}.$$

For more details on the theory of non-Archimedean linear operators, we refer to [3],[4],[5],[11] and [14].

Throughout this paper, X is a non-Archimedean $(n.a)$ Banach space over a $(n.a)$ non trivially complete valued field $\mathbb{K}$ of characteristic zero with valuation $|.|$, $B(X)$ is equipped with the norm of Definition 1.6 and for all $r > 0$, $\Omega_r = \{t \in \mathbb{K} : |t| < r\}$ and $\Omega_r^* = \Omega_r\setminus\{0\}$, denote the clopen ball with the center at 0 and the radius r, deprived of zero, and r chosen such that $t \in \Omega_r \mapsto C(t)$ is well defined.

Definition 1.7 ([6], Definition 2.24)**.** Let $r > 0$ be a real number. A function $C : \Omega_r \longrightarrow B(X)$ is called a C_0 or strongly continuous operator cosine function on X if

(i) C(0)=I,

(ii) For every $t,\ s \in \Omega_r,\ C(t+s) + C(t-s) = 2C(t)C(s)$,

(iii) For each $x \in X,\ \ t \longrightarrow C(t)x$ is continuous on Ω_r.

A cosine family of bounded linear operators $(C(t))_{t\in\Omega_r}$ is uniformly continuous if $\lim\limits_{t\to 0} \|C(t) - I\| = 0$.

The linear operator A defined by

$$D(A) = \{x \in X : \lim_{t\to 0} 2\frac{C(t)x - x}{t^2}\ \ exists\},$$

and

$$For\ each\ x \in D(A), Ax = \lim_{t\to 0} 2\frac{C(t)x - x}{t^2},$$

is called the infinitesimal generator of the cosine family $(C(t))_{t\in\Omega_r}$; where for all $x \in X$, $2x = x + x$.

We begin with the following lemmas.

Lemma 1.8 ([6], Lemma 2.26)**.** *Let $(C(t))_{t\in\Omega_r}$ be a strongly continuous cosine family on X, then for each $t \in \Omega_r$, $C(2t) = 2C(t)^2 - I$.*

Remark 1.9. Let $\mathbb{K} = \mathbb{Q}_p$. By Lemma 1.8, if $p \neq 2$, we have for all $t \in \Omega_r,\ C(\frac{t}{2})^2 = \frac{C(t)+I}{2}$.

Lemma 1.10 ([6], Lemma 2.27)**.** *Let $(C(t))_{t\in\Omega_r}$ be a strongly continuous cosine family on X , then:*

(i) For every $t \in \Omega_r$, $C(-t) = C(t)$,

(ii) For each t, $s \in \Omega_r$, $C(t)C(s) = C(s)C(t)$.

We have the following theorem.

Theorem 1.11 ([6], Theorem 2.32)**.** *Let $(C(t))_{t\in\Omega_r}$ be a strongly continuous cosine family satisfying : there is $M > 0$ such that for each $t \in \Omega_r$ $\|C(t)\| \leq M$, and let A be its infinitesimal generator. Then, for every $x \in D(A)$, $AC(s)x = C(s)Ax$ and $C(s)x \in D(A)$ for each $s \in \Omega_r$.*

Recall that $\mathbb{C}_p^+ = \{a \in \mathbb{C}_p : |1 - a| < 1\}$. For each $a \in \mathbb{C}_p^+$ where $p \neq 2$, the element

$$\sqrt{a} = a^{\frac{1}{2}} = \sum_{n\in\mathbb{N}} \binom{\frac{1}{2}}{n} (a-1)^n \tag{1.1}$$

is the unique positive square root of a. For more details see [12], section 49, page 143.

Example 1.12 ([6], Theorem 2.28)**.** Let $\mathbb{K} = \mathbb{C}_p$ with $p \neq 2$. Consider the ball Ω_r of $\mathbb{C}_p$ with $r = p^{\frac{-1}{p-1}}$. Let X be a free $n.a.$ Banach space over $\mathbb{C}_p$ and $(e_i)_{i\in\mathbb{N}}$ be an canonical base of X. Define for each $q \in \Omega_r$ and for $x = \sum_{i\in\mathbb{N}} x_i e_i$ the family of linear operators $C(q)x = \sum_{i\in\mathbb{N}} x_i \cosh(\sqrt{\mu_i} q) e_i$, where $(\mu_i)_{i\in\mathbb{N}} \subset \mathbb{C}_p^+$ is a sequence of positive elements of $\mathbb{C}_p$. It is routine to check that the family $(C(q))_{q\in\Omega_r}$ is well defined.

Proposition 1.13 ([6], Proposition 2.29)**.** *The family $(C(q))_{q\in\Omega_r}$ of linear operators given above is a cosine family of bounded linear operators, whose infinitesimal generator is the bounded diagonal operator A defined by $Ax = \sum_{i\in\mathbb{N}} \sqrt{\mu_i} x_i e_i$ for each $x = \sum_{i\in\mathbb{N}} x_i e_i \in X$.*

2. Main results

Recall that k is the residue class field of $\mathbb{K}$. Througout this paper, we assume that $\mathbb{K}$ is a complete non-Archimedean valued field of characteristic zero $\Big(char(\mathbb{K}) = 0\Big)$ with $char(k) = p$ $\Big(p$ is a prime number$\Big)$, we have the following example.

Example 2.1. Let X be a non-Archimedean Banach space over $\mathbb{K}$, let $A \in B(X)$ such that $\|A\| < r \left(= p^{\frac{-1}{p-1}}\right)$; it is easy to check that for all $t \in \Omega_r$, $C(t) = \sum_{n\in\mathbb{N}} \frac{t^{2n}}{(2n)!} A^n$ is a strongly continuous cosine family of bounded operators of infinitesimal generator A on X.

We have the following lemma.

Lemma 2.2. *Let X be a non-Archimedean Banach space over $\mathbb{K}$, let $(C(t))_{t\in\Omega_r}$ be a strongly continuous cosine family on X. Then for each $t\in\Omega_r$ and $n\in\mathbb{N}^*$ there exist $n+1$ constants $a_0,\cdots,a_n$ in $\mathbb{K}$ such that $C(nt)=a_0I+a_1C(t)+\cdots+a_nC(t)^n$.*

Proof. By induction. For $n=1$ the assertion is true and also for $n=2$ by Lemma 1.8. Now let $n\geq 2$, then by (ii) of Definition 1.7,

$$C((n+1)t)=-C((n-1)t)+2C(t)C(nt).$$

Hence if we assume that the assertion of the lemma is true for each $1\leq k\leq n$, then a simple computation shows that it is also true for $n+1$. This in fact proves our lemma. □

As in [3], we have the following remark.

Remark 2.3. (i). Let X be a non-Archimedean free Banach space over $\mathbb{K}$, let $(C(t))_{t\in\Omega_r}$ be a C_0-cosine family of linear operators on X and $(e_i)_{i\in\mathbb{N}}$ an orthogonal base of X, then for each $t\in\Omega_r$, $C(t)$ can be expressed as: for any $x=\sum_{i\in\mathbb{N}}x_ie_i\in X$, $C(t)(x)=\sum_{i\in\mathbb{N}}x_iC(t)(e_i)$, where

$$\left(\forall j\in\mathbb{N}\right)\ C(t)(e_j)=\sum_{i\in\mathbb{N}}a_{i,\,j}(t)\,e_i,\ \textit{with}\ \lim_{i\to\infty}|a_{i\ j}(t)|\|e_i\|=0.$$

(ii). Using (i), one can easily see that for each $t\in\Omega_r^*$,

$$\left(\forall j\in\mathbb{N}\right)\ \left(\frac{C(t)-I}{t^2}\right)e_j=\left(\frac{a_{jj}(t)-1}{t^2}\right)e_j+\sum_{i\neq j}\frac{a_{ij}(t)}{t^2}e_i,$$

with $\lim_{i\neq j,i\to\infty}|a_{ij}(t)|\|e_i\|=0$.
(iii). If $(C(t))_{t\in\Omega_r}$ is a C_0-cosine family of linear operators on X, then its infinitesimal generator A may or may not be a bounded linear operator on X.

Theorem 2.4. *Let X be a non-Archimedean Banach space over $\mathbb{K}$, let $A\in B(X)$ such that $\|A\|<r$ $\left(r=p^{\frac{-1}{p-1}}\right)$. Then A is the infinitesimal generator of a uniformly continuous cosine family of bounded operators $(C(t))_{t\in\Omega_r}$.*

Proof. Let X be a non-Archimedean Banach space over $\mathbb{K}$, let $A\in B(X)$ such that $\|A\|<r\left(=p^{\frac{-1}{p-1}}\right)$, then for all $t\in\Omega_r$, $C(t)=\sum_{n\in\mathbb{N}}\frac{t^{2n}}{(2n)!}A^n$ satisfies the conditions of Definition 1.7, we will show that, for each $t\in\Omega_r$,

$$C(t)=\sum_{n\in\mathbb{N}}\frac{t^{2n}}{(2n)!}A^n,\tag{2.1}$$

the series given by 2.1 converges in norm and defines a family of bounded linear operators on X by $|t|\|A\|<r$. It is easy to check that $C(0)=I$ and for all

$t,\ s \in \Omega_r,\ C(t+s)+C(t-s) = 2C(t)C(s)$. It remains to show that $(C(t))_{t\in\Omega_r}$ given above is a uniformly continuous cosine family. Indeed, for all $t \in \Omega_r$; one has $C(t) - I = t^2 A \left(\sum_{n\in\mathbb{N}} \frac{t^{2n}}{(2(n+1))!}A^n\right)$, then $\|C(t) - I\| \leq |t|_p^2\|A\|\|\zeta_t\|$, $\left(\textit{where } \zeta_t = \sum_{n\in\mathbb{N}} \frac{t^{2n}}{(2(n+1))!}A^n\right)$ converges and $\left(\forall t \in \Omega_r^*\right)$ $\|2\frac{C(t)-I}{t^2} - A\| \leq \|2A\|\|\sum_{n\geq 1} \frac{t^{2n}}{(2(n+1))!}A^n)\| < \|\sum_{n\geq 1} \frac{t^{2n}}{(2(n+1))!}A^n\|$, then

$$\lim_{t\to 0}\|C(t) - I\| = 0. \tag{2.2}$$

Consequently,

$$\lim_{t\to 0}\|2\frac{C(t) - I}{t^2} - A\| = 0. \tag{2.3}$$

Hence, A is the infinitesimal generator of a uniformly continuous cosine family of bounded operators $(C(t))_{t\in\Omega_r}$. □

Remark 2.5. (i). Note that the mapping $\Omega_r \mapsto B\,(X)\,,\ t \mapsto C(t)$ is analytic. Furthermore, $\frac{d^2C(t)}{dt^2} = AC\,(t) = C\,(t)\,A$.
(ii). An abstract version of Theorem 2.4, that is in general ultametric-valued field $\mathbb{K}$, remains an unsolved problem.

We have the following proposition.

Proposition 2.6. *Let X be a non-Archimedean Banach space over $\mathbb{K}$, let $(T(t))_{t\in\Omega_r}$ be a uniformly continuous group of bounded linear operators on X. Set for all $t \in \Omega_r$, $C(t) = \frac{T(t)+T(-t)}{2}$, $(C(t))_{t\in\Omega_r}$ is a uniformly continuous cosine family of bounded linear operators on X.*

Proof. Setting, for each $t \in \Omega_r$, $C(t) = \frac{T(t)+T(-t)}{2}$, $(C(t))_{t\in\Omega_r}$ is a uniformly continuous cosine family of bounded linear operators on X. In fact:

(i) $C(0) = \frac{T(0)+T(0)}{2} = I$,

(ii) For each $t,\ s \in \Omega_r$,

$$2C(t)C(s) = 2\left(\frac{T(t)+T(-t)}{2}\right)\cdot\left(\frac{T(s)+T(-s)}{2}\right) \tag{2.4}$$

$$= \frac{T(t+s)+T(-(t+s))}{2} + \frac{T(t-s)+T(-(t-s))}{2} \tag{2.5}$$

$$= C(t+s)+C(t-s). \tag{2.6}$$

(iii) $\|C(t) - I\| \leq \max\{\|\frac{e^{tA}-I}{2}\|; \|\frac{e^{-tA}-I}{2}\|\} \longrightarrow 0$ as $t \to 0$.

□

We have the following example.

Example 2.7. It is easy to check that $(e^{tA})_{t\in\Omega_r}$ is a uniformly continuous group on X. Set, for each $t \in \Omega_r$, $C(t) = \frac{e^{tA}+e^{-tA}}{2}$. Then $(C(t))_{t\in\Omega_r}$ is a uniformly cosine family on X. In fact

(i) $C(0) = \frac{e^{0A}+e^{-0A}}{2} = I$,

(ii) For each $t,\ s \in \Omega_r$,

$$2C(t)C(s) = 2\left(\frac{e^{tA}+e^{-tA}}{2}\right)\cdot\left(\frac{e^{sA}+e^{-sA}}{2}\right) \tag{2.7}$$

$$= \frac{e^{(t+s)A}+e^{-(t+s)A}}{2} + \frac{e^{(t-s)A}+e^{-(t-s)A}}{2} \tag{2.8}$$

$$= C(t+s)+C(t-s). \tag{2.9}$$

(iii) $\|C(t)-I\| \le \max\{\|\frac{e^{tA}-I}{2}\|; \|\frac{e^{-tA}-I}{2}\|\} \longrightarrow 0$ as $t \to 0$.

Proposition 2.8. *There exists a Banach space X over $\mathbb{Q}_p$ and strongly continuous cosine family $(C(t))_{t\in\mathbb{Q}_p}$ of bounded linear operators on X satisfying: there exists $M > 0$ such that for all $z \in X$, $t \in \mathbb{Q}_p$, $\|C(t)z\| \le (1+|t|_p^2 M)\|z\|$.*

Proof. Let $X = \mathbb{Q}_p \times \mathbb{Q}_p$ equipped with the non-Archimedean norm by:
For all $z = (x, y) \in \mathbb{Q}_p \times \mathbb{Q}_p$, $\|z\| = \max\{|x|_p,\ |y|_p\}$, where $|\cdot|_p$ is the p-adic absolute value. We consider $(C(t))_{t\in\mathbb{Q}_p}$ on $\mathbb{Q}_p \times \mathbb{Q}_p$:

$$\forall t \in \mathbb{Q}_p,\ C(t) = \begin{pmatrix} 1 & \frac{t^2}{2} \\ 0 & 1 \end{pmatrix}.$$

$(C(t))_{t\in\mathbb{Q}_p}$ is well-defined and defines a strongly continuous cosine family on X. In fact:

(i) C(0)=I,

(ii) For all $t,\ s \in \mathbb{Q}_p$,

$$C(t-s)+C(t+s) = \begin{pmatrix} 1 & \frac{(t-s)^2}{2} \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & \frac{(t+s)^2}{2} \\ 0 & 1 \end{pmatrix} \tag{2.10}$$

$$= \begin{pmatrix} 2 & \frac{(t-s)^2+(t+s)^2}{2} \\ 0 & 2 \end{pmatrix} \tag{2.11}$$

$$= \begin{pmatrix} 2 & t^2+s^2 \\ 0 & 2 \end{pmatrix}. \tag{2.12}$$

And

$$C(t)C(s) = \begin{pmatrix} 1 & \frac{t^2}{2} \\ 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} 1 & \frac{s^2}{2} \\ 0 & 1 \end{pmatrix} \tag{2.13}$$

$$= \begin{pmatrix} 1 & \frac{t^2+s^2}{2} \\ 0 & 1 \end{pmatrix}. \tag{2.14}$$

Hence $\forall\ t, s \in \mathbb{Q}_p,\ C(t-s)+C(t+s) = 2C(t)C(s)$.

(iii) For all $z \in X$ and $t \in \mathbb{Q}_p$, we have

$$\begin{aligned} \|C(t)z - z\| &= \| \begin{pmatrix} \frac{t^2 y}{2} \\ 0 \end{pmatrix} \| \\ &= |\frac{t^2 y}{2}|_p \\ &\leq M|t|^2\|z\| \text{ with } M = |\frac{1}{2}|_p. \end{aligned}$$

Consequently, for all $z \in X$, $\lim_{t\to 0} \|C(t)z - z\| = 0$. Hence $(C(t))_{t\in\Omega_r}$ is a C_0-cosine family on X.

(iv) On the other hand, for all $z = (x, y) \in X$ and $t \in \mathbb{Q}_p$, we have

$$\begin{aligned} \|C(t)z\| &= \| \begin{pmatrix} x + \frac{t^2 y}{2} \\ y \end{pmatrix} \| \\ &= \max\{|x + \frac{t^2 y}{2}|_p, |y|_p\} \\ &\leq \max\{|x|_p,\ |\frac{t^2 y}{2}|_p, |y|_p\} \\ &\leq \max\{|x|_p,\ (1 + M|t|_p^2)|y|\} \text{ with } M = |\frac{1}{2}|_p \\ &\leq (1 + M|t|_p^2)\max\{|x|_p,\ |y|_p\} \\ &\leq (1 + M|t|_p^2)\|z\|. \end{aligned}$$

□

We introduce the following definition.

Definition 2.9. A $(C(t))_{t\in\Omega_r}$ cosine family of linear operators with the infinitesimal generator A is said to be a cosine family of contraction, if for all $t \in \Omega_r$, $\|C(t)\| \leq 1$.

Example 2.10. Let X be a non-Archimedean Banach space over $\mathbb{C}_p$ with $p \neq 2$, let $A \in B(X)$ such that $\|A\| < r \left(r = p^{\frac{-1}{p-1}}\right)$, for all $t \in \Omega_r$, $C(t) = \dfrac{e^{-tA} + e^{tA}}{2}$, from Example 2.7. $(C(t))_{t\in\Omega_r}$ is a uniformly continuous cosine family on X and, for all $t \in \Omega_r$,

$$\|C(t)\| \leq \{\|\frac{e^{-tA} + e^{tA}}{2}\|\} \leq \max\{\|e^{-tA}\|,\ \|e^{tA}\|\} = 1.$$

Consequently, $(C(t))_{t\in\Omega_r}$ is a contraction.

We have the following definition.

Definition 2.11 ([6], Definition 2.16)**.** Let X and Y be two non-Archimedean Banach spaces over $\mathbb{K}$. For all $T \in B(X)$ and $S \in B(Y)$, the operator $T \oplus S$ is defined on the Banach space $X \oplus Y (= \{(x, y) : x \in X, y \in Y\} = \{x \oplus y \ : x \in X,\ y \in Y\}$ endowed with the $n.a.$ norm $\|x \oplus y\| = \max(\|x\|,\ \|y\|),)$ by

$$(\forall x \oplus y \in X \oplus Y)\ \ (T \oplus S)(x \oplus y) = Tx \oplus Sy = (Tx, Sy).$$

We have the following theorem.

Theorem 2.12. *Let $(C(t))_{t\in\Omega_r}$ is a C_0- cosine family of a generator A on X. Let $S(t) = C(t) \oplus I$ for all $t \in \Omega_r$. Then we have*

(i) $(S(t))_{t\in\Omega_r}$ be a C_0- cosine family on $X \oplus X$,

(ii) The generator of $(S(t))_{t\in\Omega_r}$ is the operator T defined on $D(T) = D(A) \oplus X$ such that for all $x \in D(A)$, $y \in X$, $T(x \oplus y) = Ax \oplus 0$.

Proof. (i) Since $(C(t))_{t\in\Omega_r}$ be a C_0- cosine family of a generator A on X, then

$$S(0) = C(0) \oplus I = I \oplus I = I_{X\oplus X}.$$

Let $x \oplus y \in X \oplus X$ and $t,\ s \in \Omega_r$, we have:

$$\begin{aligned} 2S(t)S(s)(x \oplus y) &= 2S(t)(C(s) \oplus I)(x \oplus y) \\ &= 2(C(t) \oplus I)(C(s)x \oplus y) \\ &= 2C(t)C(s)x \oplus 2y \\ &= C(t-s)(x) + C(t+s)(x) \oplus 2y \\ &= C(t-s)x \oplus y + C(t+s)x \oplus y \\ &= S(t-s)(x \oplus y) + S(t+s)(x \oplus y) \\ &= (S(t-s) + S(t+s))(x \oplus y). \end{aligned}$$

On the other hand,

$$\begin{aligned} \lim_{t\to 0} \|S(t)(x \oplus y) - x \oplus y\| &= \lim_{t\to 0} \|(C(t)x - x) \oplus 0\| \\ &= \lim_{t\to 0} \max\left(\|C(t)x - x\|, 0\right) \\ &= \lim_{t\to 0} \|C(t)x - x\| \\ &= 0. \end{aligned}$$

Therefore $(S(t))_{t\in\Omega_r}$ is a C_0- cosine family on $X \oplus X$.

(ii) Let $x \in D(A)$ and $y \in X$. We have:

$$\begin{aligned} \lim_{t\to 0} 2\frac{S\left(t\right)\left(x \oplus y\right) - x \oplus y}{t^2} &= \lim_{t\to 0} 2\frac{C\left(t\right)\left(x\right) \oplus y - x \oplus y}{t^2} \\ &= \lim_{t\to 0} \frac{2\left(C\left(t\right)\left(x\right) - x\right) \oplus 0}{t^2} \\ &= Ax \oplus 0. \end{aligned}$$

Then $D(T) = D(A) \oplus X$ and $T(x \oplus y) = A(x) \oplus 0$, for all $x \in D(A)$. $\square$

We have the following example.

Example 2.13. Assume that p is a prime number and $r = p^{\frac{-1}{p-1}}$. The 2×2 square matrix A over $\mathbb{C}_p \times \mathbb{C}_p$ given by:

$$A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \textit{ where } a,\ b \in \Omega_r,$$

generates a C_0 cosine operator $C(t)$ given by:

$$\forall t \in \Omega_r,\ C(t) = \begin{pmatrix} ch(ta) & 0 \\ 0 & ch(tb) \end{pmatrix}$$

where $ch(\cdot)$ denotes the p-adic hyperbolic cosine function.

Example 2.14. Let A be the multiplication operator on $X = C\left(\mathbb{Z}_p,\ \mathbb{Q}_p\right)$ defined by $(\forall u \in C\left(\mathbb{Z}_p,\ \mathbb{Q}_p\right)) \quad Au = Q(x)u,\ u(0) = u_0$, where $Q = \sum_{n=0}^{\infty} q_n f_n \in C\left(\mathbb{Z}_p,\ \mathbb{Q}_p\right)$, $q_n \in \mathbb{Q}_p$, $(f_n)_n$ is the base of X. Suppose that $\|Q\|_\infty = \sup_n |q_n| < r\left(= p^{\frac{-1}{p-1}}\right)$. The function defined by $(\forall t \in \Omega_r)$, $u(t) = \sum_{n \in \mathbb{N}} \left(\frac{(tA)^{2n}}{2n!}\right) u_0$, for some $u_0 \in X$, is the solution to the homogenous p-adic second-order differential equation $\frac{d^2}{dt^2}u(t) = A^2 u(t),\ t \in \Omega_r,\ u(0) = u_0$.

2.1. Question

- Can we caracterize the infinitesimal generator of C_0-cosine on infinite dimensional non-Archimedean Banach spaces ?

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