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Symmetric properties of elementary operators

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Abstract. We consider the elementary operator $M_{A,B}$, acting on the Hilbert-Schmidt class $C_2(\mathcal{H})$, given by $M_{A,B}(T) = ATB$ with A and B bounded operators on $\mathcal{H}$. In this work, we establish necessary and sufficient conditions on A and B for $M_{A,B}$ to be a 2-symmetric and 3-symmetric. We also characterize binormality of elementary operators.

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1. Introduction

In this work, $\mathcal{H}$ denotes a complex Hilbert space with inner product $\langle .,. \rangle$. $\mathcal{B}(\mathcal{H})$ denotes the algebra of all bounded linear operators on $\mathcal{H}$.

For $A, B \in \mathcal{B}(\mathcal{H})$, we have the left multiplication operator L_A defined by

$$L_A(X) = AX, \forall X \in \mathcal{B}(\mathcal{H});$$

the right multiplication operator R_B defined by

$$R_B(X) = XB, \forall X \in \mathcal{B}(\mathcal{H});$$

the basic elementary operator (two-side multiplication)

$$M_{A,B} = L_A R_B;$$

the Jordan elementary operator $U_{A,B}$ on $\mathcal{B}(\mathcal{H})$ by

$$U_{A,B} = M_{A,B} + M_{B,A}.$$

An elementary operator on $\mathcal{B}(\mathcal{H})$ is a finite sum $R = \sum_{i=1}^{n} M_{A_i,B_i}$ of basic ones. For more facts about the elementary operators, we refer the reader to [7, 8] and the references therein.

Let J be a non-zero linear subspace of the space $\mathcal{B}(\mathcal{H})$. We say that J is a symmetric norm ideal if it is equipped with a norm $\|.\|_J$ satisfying the following conditions:

i) if $A, B \in \mathcal{B}(\mathcal{H})$ and $X \in J$ then $AX \in J$ and $XB \in J$.

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ii) J is Banach space with respect to the norm $\|.\|_J$.
iii) $\|X\|_J = \|X\|$ for all rank 1 operators $X \in J$.
iv) $\|AXB\| \leq \|A\| \|X\|_J \|B\|$ for all A, $B \in \mathcal{B}(\mathcal{H})$ and $X \in J$.

Familiar examples of symmetric norm ideals are the Schatten p-ideals $\left(C_p(\mathcal{H}), \|.\|_p\right)$ such that $1 \leq p \leq \infty$ on a Hilbert space $\mathcal{H}$. (see [5, 15]).

The space $C_p(\mathcal{H})$ consists of compact operators K such that $\sum_j s_j^p(K) < \infty$, where $\{s_j(K)\}_j$ denotes the sequence of the singular values of K.

For $K \in C_p(\mathcal{H})$ $(1 \leq p \leq \infty)$ we set

$$\|K\|_p = \left(\sum_j s_j^p(K)\right)^{\frac{1}{p}},$$

where by convention $\|K\|_\infty = s_1(K)$ is the usual operator norm of K.

For $p = 2$, the espace $(C_2(\mathcal{H}), \|.\|_2)$ is a Hilbert space (it is called the Hilbert-Schmidt class) with inner product, defined by

$$\langle X, Y\rangle = tr(XY^*) \quad (X, Y \in C_2(\mathcal{H})),$$

where $tr(.)$ denotes the usual trace of operators. Furthermore, $C_2(\mathcal{H})$ is an ideal of the algebra of all bounded operators on $\mathcal{H}$. We direct the reader to [5, 7, 9, 12, 13, 15] and the references therein.

Let A and B be bounded operators on $\mathcal{H}$, and $M_{A,B}$ a bounded operator on $C_2(\mathcal{H})$ defined by $M_{A,B}(T) = ATB$. The adjoint $M_{A,B}^*$ is given by $M_{A,B}^*(T) = A^*TB^*$ (see [1, 6, 8]).

We recall the definition of an m-symmetric operator, as given in [3, 4, 11, 10]. If $T \in \mathcal{B}(\mathcal{H})$, then T is said to be an m-symmetric if and only if

$$\sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} T^{*j} T^{m-j} = 0.$$

In particular, if T is a 2-symmetric or 3-symmetric operator, then it must satisfy the operator equation

$$T^2 - 2T^*T + T^{*2} = 0,$$

or

$$T^3 - 3T^*T^2 + 3T^{*2}T - T^{*3} = 0, \text{ respectively.}$$

In this work, we give necessary and sufficient conditions on A and B under which the elementary operator $M_{A,B}$ is 2-symmetric, 3-symmetric and binormal on $C_2(\mathcal{H})$. Our characterization follows from a theorem of Fong and Sourour (see [8]). This theorem was used by Magajana [14] to characterize subnormal elementary operators on $C_2(\mathcal{H})$, also Botelho and Jamison used this theorem to characterize m-isometry elementary operators on $C_2(\mathcal{H})$ (see [2]).

We consider $\{A_i\}_{i=1,....m}$ and $\{B\}_{i=1,....m}$ bounded operators on the Hilbert space $\mathcal{H}$ and ϕ an operator acting on $C_2(\mathcal{H})$ as follows:

$$\phi(T) = A_1TB_1 + A_2TB_2 + ... + A_mTB_m\ ,$$

with not all the A_i equal to 0.

Theorem 1.1 (Fong and Sourour [8]). *If* $\phi(T) = 0$ *for all* $T \in C_2(\mathcal{H})$, *then* $\{B_1, B_2, ...B_m\}$ *is linearly dependent. Furthemore, if* $\{B_1, B_2, ...B_n\}$ $(n \leq m)$ *is a maximal linearly independent subset of* $\{B_1, B_2, ...B_m\}$, *and* (c_{kj}) *denote constants for which,*

$$B_j = \sum_{k=1}^{n} c_{kj} B_k, \quad 1 \leq j \leq m,$$

then $\phi(T) = 0$ for all $T \in C_2(H)$ if and only if

$$A_k = -\sum_{j=n+1}^{m} c_{kj} A_j, \;\; 1 \leq k \leq n.$$

2. Main results

In this section we present our main results. First, we give necessary and sufficient conditions for the elementary operator $M_{A,B}(T) = ATB$ on $C_2(\mathcal{H})$ to be 2-symmetric.

Theorem 2.1. *Let* $M_{A,B}$ *be an elementary operator of two-sided multiplication, acting on* $\mathbf{C}_2(\mathcal{H})$, *such that* $M^2_{A^*,B} = -M^2_{A,B^*}$. *Then, the operator* $M_{A,B}$ *is* 2*-symmetric if and only if there exists a scalar* λ *so that* $A^2 + A^{*2} = 2\lambda A^* A$ *and* $BB^* = \lambda\left(B^2 + B^{*2}\right)$ *such that* $\frac{1}{2} \leq |\lambda| \leq 1$.

Proof. If $M_{A,B}$ is 2-symmetric, then for any $T \in C_2(\mathcal{H})$ we have that

$$M^2_{A,B}(T) - 2M^*_{A,B}M_{A,B}(T) + M^{*2}_{A,B}(T) = 0. \tag{2.1}$$

We know that $M^*_{A,B}(T) = A^*TB^*$. Moreover, it easy to check that $M^2_{A,B}(T) = A^2TB^2$, $M^*_{A,B}M_{A,B}(T) = A^*A\ TBB^*$ and $M^{*2}_{A,B}(T) = A^{*2}TB^{*2}$. Since $M^2_{A^*,B} = -M^2_{A,B^*}$, this means that $A^{*2}TB^2 = -A^2TB^{*2}$. Thus, the above equation (2.1) implies that

$$\left(A^2 + A^{*2}\right) T \left(B^2 + B^{*2}\right) = 2A^*ATBB^*.$$

Fong-Sourour's theorem implies that there exists a nonzero scalar λ so that

$$A^2 + A^{*2} = 2\lambda A^*A \text{ and } BB^* = \lambda\left(B^2 + B^{*2}\right).$$

We have

$$A^2 + A^{*2} = 2\lambda A^*A,$$

therefore

$$\left\|A^2 + A^{*2}\right\| = 2|\lambda| \left\|A^*A\right\|.$$

Then

$$2|\lambda| \left\|A^*A\right\| \leq \left\|A^2\right\| + \left\|A^{*2}\right\|,$$

since $\left\|A^2\right\| \leq \|A\|^2$ and $\|A^*A\| = \|A\|^2$, so $|\lambda| \leq 1$.

By applying similar technique on the equation $BB^* = \lambda\left(B^2 + B^{*2}\right)$ we can show that $|\lambda| \geq \frac{1}{2}$.

Conversely, it is straightforward to show that the conditions on A and B stated in the theorem imply that $M_{A,B}$ is a 2-symmetric. □

The next theorem gives a characterization of 3-symmetricity for elementary operators of two-sided multiplication.

Theorem 2.2. *Let $M_{A,B}$ be an elementary operator of two-sided multiplication, acting on $\mathbf{C}_2(\mathcal{H})$ with $A, B \in \mathcal{B}(\mathcal{H})$ such that $\left\{A^3, A^{*2}A\right\}$ and $\left\{B^3, BB^{*2}\right\}$ are two linearly independent subsets of $\mathcal{B}(\mathcal{H})$. The operator $M_{A,B}$ is 3-symmetric if and only if there exist scalars λ and μ so that one of the following statements holds:*

(i) $3A^{*2}A = \lambda A^{*3} - \mu A^3$, $B^3 = \overline{\lambda}B^2B^* + \mu BB^{*2}$.

(ii) $A^3 = 3\lambda A^*A^2 + \mu A^{*2}A$, $B^2B^* = \lambda B^3 + \mu B^{*3}$ and $|\lambda| = |\mu|$.

(iii) $A^{*3} = \lambda A^3$, $A^*A^2 = \mu A^{*2}A$, $B^3 = \lambda B^{*3}$, $BB^{*2} = \mu B^2B^*$ and $|\lambda| = |\mu| = 1$.

Proof. If $M_{A,B}$ is 3-symmetric, then for any $T \in C_2(\mathcal{H})$ we have that

$$A^{*3}TB^{*3} - 3A^{*2}ATBB^{*2} + 3A^*A^2TB^2B^* - 3A^3TB^3 = 0.$$

We first assume that $\left\{A^3, A^{*2}A\right\}$ is a maximal linearly independent subset of $\left\{A^3, A^{*2}A, A^{*3}, A^*A^2\right\}$. Fong-Sourour's theorem the implies that there exist scalars $\alpha_1, \alpha_2, \beta_1, \beta_2$ so that

$$\begin{cases} A^{*3} = \alpha_1 A^3 + 3\alpha_2 A^{*2}A & \text{(a)} \\ 3A^*A^2 = \beta_1 A^3 + 3\beta_2 A^{*2}A & \text{(b)} \end{cases} \tag{2.2}$$

and

$$\begin{cases} B^3 = \alpha_1 B^{*3} + \beta_1 B^2B^* & \text{(c)} \\ BB^{*2} = \alpha_2 B^{*3} + \beta_2 B^2B^* & \text{(d)} \end{cases} \tag{2.3}$$

By (2.2) we have $A^3 = \overline{\alpha_1}\left(\alpha_1 A^3 + 3\alpha_2 A^{*2}A\right) + \overline{\alpha_2}\left(\beta_1 A^3 + 3\beta_2 A^{*2}A\right)$. Therefore,

$$\left(|\alpha_1|^2 + \overline{\alpha_2}\beta_1 - 1\right)A^3 + 3\left(\overline{\alpha_1}\alpha_2 + \overline{\alpha_2}\beta_2\right)A^{*2}A = 0,$$

and $3A^{*2}A = \overline{\beta_1}\left(\alpha_1 A^3 + 3\alpha_2 A^{*2}A\right) + \overline{\beta_2}\left(\beta_1 A^3 + 3\beta_2 A^{*2}A\right)$. Therefore,

$$\left(\overline{\beta_1}\alpha_1 + \overline{\beta_2}\beta_1\right)A^3 + 3\left(|\beta_2|^2 + \alpha_2\overline{\beta_1} - 1\right)A^{*2}A = 0.$$

So, $|\alpha_1|^2+\overline{\alpha_2}\beta_1-1=0$, $\overline{\alpha_1}\alpha_2+\overline{\alpha_2}\beta_2=0$, $\overline{\beta_1}\alpha_1+\overline{\beta_2}\beta_1=0$ and $|\beta_2|^2+\alpha_2\overline{\beta_1}-1=0$. This implies that $|\alpha_1|=|\beta_2|$.

If we assume that $\alpha_2\neq 0$, then (a) and (d) become

$$3A^*A^2=\frac{1}{\alpha_2}A^{*3}-\frac{\alpha_1}{\alpha_2}A^3 \text{ and } B^{*3}=\frac{1}{\alpha_2}BB^{*2}-\frac{\beta_2}{\alpha_2}B^2B^*,$$

and if we set $\lambda=\frac{1}{\alpha_2}$ and $\mu=-\frac{\alpha_1}{\alpha_2}$ we get $3A^{*2}A=\lambda A^{*3}-\mu A^3$ and $B^3=\overline{\lambda}B^2B^*+\mu BB^{*2}$ as listed in (i).

If $\alpha_2=0$, then $|\alpha_1|=|\beta_2|=1$ so (a) and (d) reduce to $A^{*3}=\alpha_1 A^3$ and $BB^{*2}=\beta_2 B^2B^*$, respectively. If in addition we assume that $\beta_1\neq 0$, then (b) and (c) become

$$A^3=\frac{1}{\beta_2}\left(3A^*A^2\right)-\frac{\beta_2}{\beta_1}\left(A^{*2}A\right) \text{ and } B^2B^*=\frac{1}{\beta_1}B^3-\frac{\alpha_1}{\beta_1}B^{*3},$$

respectively. We now set $\lambda=\frac{1}{\beta_1}$ and $\mu=-\frac{\beta_2}{\beta_1}$. Hence we find (ii).

If $\alpha_2=\beta_1=0$, then $|\alpha_1|=|\beta_2|=1$ and the system (2.2) reduces to $A^{*3}=\alpha_1A^3$, $A^*A^2=\beta_2A^{*2}A$, $B^3=\alpha_1B^{*3}$ and $BB^{*2}=\beta_2B^2B^*$. When we set $\lambda=\alpha_1$ and $\mu=\beta_2$, we get (iii).

Now, we assume that $\left\{A^3, A^{*2}A, A^{*3}\right\}$ is a maximal linearly independent subset of $\left\{A^3, A^{*2}A, A^{*3}, A^*A^2\right\}$. Then Fong-Sourour's theorem implies the existence of scalars α_1,α_2 and α_3, so that $A^*A^2=\alpha_1A^3+\alpha_2A^{*2}A+\alpha_3A^{*3}$, therefore $B^3=\alpha_1BB^{*2}$, $BB^{*2}=\alpha_2B^2B^*$ and $B^{*3}=-\alpha_3BB^{*2}$. Then $\left\{B^3, BB^{*2}\right\}$ is linearly dependent subset of $\left\{A^3, A^{*2}A, A^{*3}, A^*A^2\right\}$. This contradicts our initial assumption. Similar reasoning applies if $\left\{A^3, A^{*2}A, A^*A^2\right\}$ is a maximal linearly independent subset of $\left\{A^3, A^{*2}A, A^{*3}, A^*A^2\right\}$.

Conversely, it is straightforward to verify that those relations listed in any of the items (i)-(iii) imply that $M_{A,B}$ is a 3-symmetric operator. This completes the proof. □

We recall that an operator is said to be binormal, if T^*T and TT^* commute. For more details about this class of operators we refer to [16].

Finally, we give necessary and sufficient conditions for the elementary operator $M_{A,B}(T)=ATB$ on $C_2(\mathcal{H})$ to be binormal.

Proposition 2.3. *Let $M_{A,B}$ be an elementary operator of two-sided multiplication acting on $\mathbf{C}_2(\mathcal{H})$ with $A,B\in\mathcal{B}(\mathcal{H})$. The operator $M_{A,B}$ is a binormal if and only if there exists a scalar λ so that $AA^{*2}A=\lambda A^*A^2A^*$ and $B^*B^2B^*=\lambda BB^{*2}B$ with $|\lambda|=1$.*

Proof. If $M_{A,B}$ is a binormal, then for any $T\in C_2(\mathcal{H})$ we have that

$$AA^{*2}ATBB^{*2}B=A^*A^2A^*TB^*B^2B^*.$$

We apply Fong-Sourour's theorem and we find

$$AA^{*2}A = \lambda A^{*}A^{2}A^{*} \text{ and } B^{*}B^{2}B^{*} = \lambda BB^{*2}B.$$

So $\left\|AA^{*2}A\right\| = |\lambda| \left\|A^{*}A^{2}A^{*}\right\|$, since $\left\|AA^{*2}A\right\| = \left\|A^{*}A^{2}A^{*}\right\|$. Hence $|\lambda| = 1$. The converse implication is straightforward. □

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