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Extragradient method for approximating a common solution for a fixed point and variational inequality problems in Hilbert space

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Abstract. In this paper, we introduce an extragradient method to approximate a common solution of variational inequality problem and a fixed point problem for an asymptotically nonexpansive mapping in a real Hilbert space. We prove that the sequence generated by the iterative algorithm converges strongly to a common solution of a variational inequality problem and the fixed point problem for an asymptotically nonexpansive mapping. The results presented in this paper extend and generalize many previously known results in the literature. Some applications of main results are also provided.

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1. Introduction

Variational inequality theory describes a broad spectrum of very interesting developments involving a link among various fields of mathematics, physics, economics and engineering sciences. Some of these developments have made mutually enriching contacts with other fields. During the last five decades which have elapsed since its discovery, variational inequality theory has stimulated efforts and an ever increasing number of research workers are using variational inequality techniques. The important developments were the formulations that variational inequality can be used to study the problems of fluid flow through porous media, contact problems in elasticity, transportation problems and economics equilibrium. Ideas explaining these formulations led to the developments of new and powerful techniques to solve a wide class of linear and nonlinear problems. See, for example [2, 3, 6, 8, 10, 12].

Throughout this paper unless otherwise stated, H denotes a real Hilbert space with inner product $\langle .,. \rangle$ with the induced norm $\|.\|$ and C denotes a

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nonempty closed convex subset of H. Let $\{x_n\}$ be a sequence in H, then $x_n \to x$ (respectively, $x_n \rightharpoonup x$) denotes strong (respectively, weak) convergence of the sequence $\{x_n\}$ to a point $x \in H$. We denote by $\mathbb{N}$ and $\mathbb{R}$ the sets of all positive integers and all real numbers, respectively.

Recall that for every point $x \in H$, there exists a unique point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \forall y \in C.$$

The function $P_C : H \to C$ is called the metric projection of H onto C. It is well known that P_C is nonexpansive mapping, that is,

$$\|P_C x - P_C y\| \leq \|x - y\|, \quad \forall x, y \in H,$$

and satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H. \tag{1.1}$$

Further, P_C is characterized by the following properties:

(i) $\langle P_C x - P_C y, x - y\rangle \geq 0, \ \forall x, y \in H.$

(ii) $\|x - P_C x\|^2 + \|y - P_C x\|^2 \leq \|x - y\|^2, \ \forall x, y \in H.$

A mapping $T : C \to C$ is said to be asymptotically nonexpansive [7] if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that for every $x, y \in C$ and for each $n \in \mathbb{N}$

$$\|T^n x - T^n y\| \leq k_n\|x - y\|.$$

If $k_n = 1$ for all $n \in \mathbb{N}$, then T is said to be a nonexpansive mapping.

Note that if a mapping $T : C \to C$ is asymptotically nonexpansive with the asymptotic sequence $\{k_n\} \subset [1, \infty)$, then T is uniformly $k-$Lipschitzian, that is,

$$\|T^n x - T^n y\| \leq k\|x - y\|, \forall x, y \in C,$$

where $k = \sup_{n\in\mathbb{N}} k_n$, for each $n \in \mathbb{N}$.

The fixed point problem (in short, FPP) for the mapping $T : C \to C$ is to find $x \in C$ such that

$$Tx = x. \tag{1.2}$$

The solution set of FPP (1.2) is denoted by $F(T)$, that is,

$$F(T) = \{x \in C : Tx = x\}.$$

A mapping $A : C \to H$ is called an α-inverse strongly monotone mapping if there exists a real number $\alpha > 0$ such that for every $x, y \in C$

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2.$$

A mapping A is called a monotone mapping if for every $x, y \in C$

$$\langle Ax - Ay, x - y\rangle \geq 0.$$

It is known that every α-inverse strong monotone mapping is monotone and is also $\frac{1}{\alpha}$-Lipschitz continuous mapping. The variational inequality problem (in short, VIP) is to find $z \in C$ such that

$$\langle Az, y - z\rangle \geq 0, \ \forall y \in C. \tag{1.3}$$

The solution set of the variational inequality problem (1.3) is denoted by $VI(C, A)$, that is,

$$VI(C, A) = \{z \in C : \langle Az, y - z\rangle \geq 0, \forall y \in C\}.$$

With the connection to the variational inequality problem, it is easy to see that

$$x \in VI(C, A) \Longleftrightarrow x = P_C(x - \lambda Ax) \ \ \forall \lambda > 0.$$

The so-called extragradient method was introduced in 1976 by Korpelevich [9] as follows:

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C(x_n - \lambda_n Ax_n), \\ x_{n+1} = P_C(x_n - \lambda_n Ay_n) \end{cases} \tag{1.4}$$

for all $n \geq 0$, where $\lambda_n \in (0, \frac{1}{k})$, C is a closed convex subset of $\mathbb{R}^n$ and A is a monotone and $k-$Lipschitz continuous mapping of C into $\mathbb{R}^n$. He proved that if $VI(C, A)$ is nonempty, then the sequences $\{x_n\}$ and $\{y_n\}$, generated by (1.4), converge to the same point $z \in VI(C, A)$.

In 2003, Takahashi and Toyoda [14] introduced the following iteration process for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of a variational inequality problem for an inverse strongly-monotone mapping

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)TP_C(x_n - \lambda_n Ax_n), \tag{1.5}$$

for every $n = 0, 1, 2, \ldots,$ where $x = x_0 \in C$ and $\{\lambda_n\} \subseteq (0, 2\alpha)$. They showed that, if $F(T) \cap VI(C, A) \neq \emptyset$, then the sequence $\{x_n\}$ generated by (1.5), converges weakly to some $z \in F(T) \cap VI(C, A)$.

In 2006, motivated by the Korpelevich extragradient method, Zeng and Yao [19] introduced a new extragradient method and proved the following theorem.

Theorem 1.1. *([19]) Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be a monotone and $k-$Lipschitz continuous mapping. Let $S : C \to C$ be a nonexpansive mapping such that $F(S) \cap VI(C, A) \neq \emptyset$. Let $\{x_n\}$ and $\{y_n\}$ be sequences in C defined as follows:*

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C(x_n - \lambda_n Ax_n), \\ x_{n+1} = \alpha_n x_0 + (1 - \alpha_n)SP_C(x_n - \lambda_n Ay_n), \forall n \geq 0, \end{cases} \tag{1.6}$$

where $\{\lambda_n\}$ and $\{\alpha_n\}$ satisfy the conditions *(i)* $\lambda_n k \in (0, 1-\delta)$ *for some* $\delta \in (0,1)$.

(ii) $\{\alpha_n\} \subseteq (0,1)$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$.

Then the sequences $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to the same point* $P_{F(S)\cap VI(C,A)}x_0$*, provided that* $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

In 2007, Yao et al. [17] introduced the iterative scheme:

$$\begin{cases} x_1 = u \in C, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S P_C(x_n - \lambda_n A y_n), \forall n \geq 0, \end{cases} \tag{1.7}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in $[0,1]$. They proved the convergence of this sequence to common elements of $VI(C,A)$ and $F(S)$ for a monotone and $k-$Lipschitz continuous mapping $A : C \to H$ and a nonexpansive mapping $S : C \to C$ for which $F(S) \cap VI(C,A) \neq \emptyset$.

Recently, Nadezhkina and Takahashi [11] and Yao and Yao [18] proposed some new iterative schemes for finding elements in $F(T) \cap VI(C,A)$.

In this paper, motivated by iterative schemes considered in [10, 11, 19, 18], we constructed an iterative algorithm to approximate a common element of the set of fixed points of asymptotically nonexpansive mapping and the set of solution of variational inequality problem and proved a strong convergence theorem in Hilbert space settings.

2. Preliminaries

We now introduce preliminaries which will be used in this paper.

Recall that a mapping f from C into it self is called a ρ-contraction mapping if there exists $\rho \in (0,1)$ such that

$$\|f(x) - f(y)\| \leq \rho \|x - y\|, \forall x, y \in C.$$

Definition 2.1 ([4])**.** Let C be a closed convex subset of a Hilbert space H. A mapping $T : C \to C$ is called asymptotically regular on C if and only if,

$$\lim_{n\to\infty} \|T^n x - T^{n+1} x\| = 0, \forall x \in C.$$

Lemma 2.2 (*[5]*)**.** *Let* T *be an asymptotically nonexpansive mapping on a closed and convex subset* C *of a real Hilbert space* H*. Let* $\{x_n\}$ *be a sequence in* C*. Then* $I - T$ *is demiclosed at* 0*. That is, if* $x_n \rightharpoonup x$ *and* $x_n - Tx_n \to 0$*, then* $x \in F(T)$.

Lemma 2.3 (*[16]*)**.** *Let* $\{\delta_n\}$ *be a sequence of non negative real numbers, satisfying*

$$\delta_{n+1} \leq (1 - \varepsilon_n)\delta_n + \varepsilon_n \beta_n + \gamma_n, \forall n \geq 0,$$

where $\{\varepsilon_n\}$*,* $\{\beta_n\}$ *and* $\{\gamma_n\}$ *satisfies the conditions:*

(i) $\{\varepsilon_n\} \subset [0,1]$, $\sum_{n=1}^{\infty} \varepsilon_n = \infty$ *or equivalently,* $\prod_{n=1}^{\infty}(1-\varepsilon_n) = 0$,

(ii) $\limsup_{n\to\infty} \beta_n \leq 0$,

(iii) $\gamma_n \geq 0$, $\sum_{n=1}^{\infty} \gamma_n \leq \infty$.

Then,

$$\lim_{n\to\infty} \delta_n = 0.$$

Lemma 2.4 *([1]). Let H be a real Hilbert space. Then, for any given $x, y \in H$, we have the following inequality:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle.$$

Lemma 2.5 *([15]). Let $\{t_n\}$ be a sequence of nonnegative real numbers such that $t_{n+1} \leq (1-a_n)t_n + a_n\beta_n$, $n \geq 0$ where $\{a_n\}$ is a sequence in $(0,1)$ and $\{\beta_n\}$ is a sequence in $\mathbb{R}$ such that*

(C1) $\sum_{n=0}^{\infty} a_n = \infty$ *or equivalently* $\prod_{n=0}^{\infty}(1-a_n) = 0$,

(C2) $\limsup_{n\to\infty} \beta_n \leq 0$. *Then* $\lim_{n\to\infty} t_n = 0$.

Definition 2.6. A set valued mapping $T : H \to 2^H$ is called monotone if, for all $x, y \in H$, we have

$$\langle x-y, f-g\rangle \geq 0, \ \forall f \in Tx, g \in Ty.$$

Such an operator is maximal monotone if its graph $G(T)$ is not properly contained in the graph of any other monotone operator. It is known that a monotone mapping T is a maximal if and only if, for $(x, f) \in H \times H$, $\langle x-y, f-g\rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$.

Let A be a monotone, k-Lipschitz-continuous mapping of C onto H and let $N_C v$ be the normal cone to C at $v \in C$ such that
$N_C v = \{w \in H : \langle v-u, w\rangle \geq 0, \forall u \in C\}$.
Define

$$Tv = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then, T is maximal monotone and $0 \in Tv$ if and only $v \in VI(C, A)$, see (Rockafellar, [13]).

3. Main results

Let $A : C \to H$ be an α-inverse strongly monotone mapping , $f : C \to C$ be a ρ-contraction mapping and let $T : C \to C$ be an asymptotically non expansive mapping. Let $\{\alpha_n\} \subset [0, 1]$ and $\lambda_n \in (0, 2\alpha)$. For any $x_1 \in C$, we find $y_1 \in C$ such that

$$y_1 = P_C(x_1 - \lambda_1 Ax_1).$$

Now we can compute $x_2 \in C$ by

$$x_2 = \alpha_1 f(x_1) + (1 - \alpha_1)TP_C(y_1 - \lambda_1 Ay_1).$$

Also, we can find $y_2 \in C$ such that

$$y_2 = P_C(x_2 - \lambda_2 Ax_2).$$

After that, we can compute $x_3 \in C$ by

$$x_3 = \alpha_2 f(x_2) + (1 - \alpha_2)T^2 P_C(y_2 - \lambda_2 Ay_2).$$

Inductively, we can generate the sequence $\{x_n\} \subset C$ as follows:

$$\begin{cases} x_1 \in C, \\ y_n = P_C(x_n - \lambda_n Ax_n), n = 1, 2, 3, \dots \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T^n P_C(y_n - \lambda_n Ay_n), n = 1, 2, 3, \dots \end{cases} \tag{3.1}$$

We now state and prove our strong convergence theorem as follows:

Theorem 3.1. *Let C be a non empty closed and convex subset of a real Hilbert space H. Let $f : C \to C$ be ρ-contraction mapping, $A : C \to H$ be an α-inverse strongly monotone mapping and let $T : C \to C$ be asymptotically non-expansive mapping. Assume that T is asymptotically regular on C such that $F(T) \cap VI(C, A) \neq \emptyset$. Let $\{\alpha_n\} \subset [0, 1]$ and $\{\lambda_n\} \subset [0, 2\alpha]$ satisfy*

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty,$

(ii) $0 < a \le \lambda_n \le b < 2\alpha,$

(iii) $\lim_{n\to\infty} (\lambda_n - \lambda_{n+1}) = 0, \lim_{n\to\infty} \frac{k_n - 1}{\alpha_n} = 0.$

For $x_1 \in C$, if $\{x_n\}$ is the sequence generated by the iterative scheme (3.1), then $\{x_n\}$ converges strongly to $z = P_{F(T)\cap VI(C,A)} f(z)$.

Proof. For all $x, y \in C$ and as $\lambda_n \in (0, 2\alpha)$, we have

$$\begin{aligned} &\|(I - \lambda_n A)x - (I - \lambda_n A)y\|^2 \\ &= \|(x - y) - \lambda_n(Ax - Ay)\|^2 \\ &= \|x - y\|^2 - 2\lambda_n\langle x - y, Ax - Ay\rangle + \lambda_n^2\|Ax - Ay\|^2 \end{aligned}$$

$$\leq \|x-y\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Ax - Ay\|^2$$

$$\leq \|x-y\|^2, \tag{3.2}$$

which implies that $I - \lambda_n A : C \to H$ is nonexpansive.
Let $z \in F(T) \cap VI(C, A)$ and $z_n = P_C(y_n - \lambda_n Ay_n)$. Then $z = P_C(z - \lambda_n Az)$. From (3.2), we have

$$\begin{aligned}
\|z_n - z\| &= \|P_C(y_n - \lambda_n Ay_n) - P_C(z - \lambda_n Az)\| \\
&\leq \|(y_n - \lambda_n Ay_n) - (z - \lambda_n Az)\| \leq \|y_n - z\| && (3.3)\\
&= \|P_C(x_n - \lambda_n Ax_n) - P_C(z - \lambda_n Az)\| \\
&\leq \|(x_n - \lambda_n Ax_n) - (z - \lambda_n Az)\| \\
&\leq \|x_n - z\|. && (3.4)
\end{aligned}$$

Take $\epsilon \in (0, 1-\rho)$. Since $\dfrac{(k_n - 1)}{\alpha_n} \to 0$ as $n \to \infty$, there exists $N \in \mathbb{N}$ such that
$(k_n - 1) < \epsilon\alpha_n$ for all $n \in \mathbb{N}$.

From (3.1) and (3.4) it follows that

$$\begin{aligned}
&\|x_{n+1} - z\| \\
&= \|\alpha_n f(x_n) + (1-\alpha_n)T^n z_n - z\| \\
&= \|\alpha_n(f(x_n) - f(z)) + \alpha_n(f(z) - z) + (1-\alpha_n)(T^n z_n - z)\| \\
&\leq \alpha_n\rho\|x_n - z\| + \alpha_n\|f(z) - z\| + (1-\alpha_n)\|T^n z_n - z\| \\
&\leq \alpha_n\rho\|x_n - z\| + \alpha_n\|f(z) - z\| + (1-\alpha_n)k_n\|z_n - z\| \\
&\leq \alpha_n\rho\|x_n - z\| + \alpha_n\|f(z) - z\| + (1-\alpha_n)k_n\|x_n - z\| \\
&= (1-\alpha_n(1-\rho))\|x_n - z\| + \alpha_n\|f(z) - z\| + (1-\alpha_n)(k_n - 1)\|x_n - z\| \\
&\leq (1-\alpha_n(1-\rho))\|x_n - z\| + \alpha_n\|f(z) - z\| + \epsilon\alpha_n\|x_n - z\| \\
&\leq (1-\alpha_n(1-\rho-\epsilon))\|x_n - z\| + \alpha_n\|f(z) - z\| \\
&\leq \max\{\|x_n - z\|, \frac{1}{1-\rho-\epsilon}\|f(z) - z\|\}.
\end{aligned}$$

By induction, we see that, for all $n \geq 1$
$\|x_n - z\| \leq \max\{\|x_1 - z\|, \frac{1}{1-\rho-\epsilon}\|f(z) - z\|\}$.

Thus, $\{x_n\}$, $\{y_n\}$, $\{Ax_n\}$, $\{f(x_n)\}$, $\{z_n\}$ and $\{T^n z_n\}$ are bounded. Next, we prove that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

We observe that:

$$\begin{aligned}
&\|z_{n+1} - z_n\| \\
&= \|P_C(y_{n+1} - \lambda_{n+1}Ay_{n+1}) - P_C(y_n - \lambda_n Ay_n)\| \\
&\leq \|(y_{n+1} - \lambda_{n+1}Ay_{n+1}) - (y_n - \lambda_n Ay_n)\| \\
&= \|(y_{n+1} - \lambda_{n+1}Ay_{n+1}) - (y_n - \lambda_{n+1}Ay_n) + (\lambda_n - \lambda_{n+1})Ay_n\| \\
&\leq \|(y_{n+1} - \lambda_{n+1}Ay_{n+1}) - (y_n - \lambda_{n+1}Ay_n)\| + |\lambda_n - \lambda_{n+1}|\|Ay_n\|
\end{aligned}$$

$$
\begin{aligned}
(3.5)\leq\quad & \|y_{n+1}-y_n\|+|\lambda_n-\lambda_{n+1}|\|Ay_n\| \\
= \quad & \|P_C(x_{n+1}-\lambda_{n+1}Ax_{n+1})-P_C(x_n-\lambda_n Ax_n)\|+|\lambda_n-\lambda_{n+1}|\|Ay_n\| \\
(3.6)\leq\quad & \|x_{n+1}-x_n\|+|\lambda_n-\lambda_{n+1}|(\|Ax_n\|+\|Ay_n\|).
\end{aligned}
$$

Since $y_n = P_C(x_n - \lambda_n Ax_n)$ and $y_{n+1} = P_C(x_{n+1} - \lambda_{n+1}Ax_{n+1})$, we have

$$
\langle Ax_n, y-y_n\rangle + \frac{1}{\lambda_n}\langle y-y_n, y_n-x_n\rangle \geq 0, \forall y\in C. \tag{3.7}
$$

and

$$
\langle Ax_{n+1}, y-y_{n+1}\rangle + \frac{1}{\lambda_{n+1}}\langle y-y_{n+1}, y_{n+1}-x_{n+1}\rangle \geq 0, \forall y\in C. \tag{3.8}
$$

Putting $y = y_{n+1}$ in (3.7) and $y = y_n$ in (3.8), we have

$$
\langle Ax_n, y_{n+1}-y_n\rangle + \frac{1}{\lambda_n}\langle y_{n+1}-y_n, y_n-x_n\rangle \geq 0. \tag{3.9}
$$

and

$$
\langle Ax_{n+1}, y_n-y_{n+1}\rangle + \frac{1}{\lambda_{n+1}}\langle y_n-y_{n+1}, y_{n+1}-x_{n+1}\rangle \geq 0. \tag{3.10}
$$

Combining (3.9) and (3.10) we have,

$$
\langle Ax_{n+1}-Ax_n, y_n-y_{n+1}\rangle + \left\langle y_{n+1}-y_n, \frac{y_n-x_n}{\lambda_n} - \frac{y_{n+1}-x_{n+1}}{\lambda_{n+1}}\right\rangle \geq 0.
$$

And hence,

$$
\begin{aligned}
0\leq\ & \left\langle y_n-y_{n+1}, \lambda_n(Ax_{n+1}-Ax_n)+\frac{\lambda_n}{\lambda_{n+1}}(y_{n+1}-x_{n+1})-(y_n-x_n)\right\rangle \\
=& \left\langle y_{n+1}-y_n, y_n-y_{n+1}+\left(1-\frac{\lambda_n}{\lambda_{n+1}}\right)y_{n+1}+(x_{n+1}-\lambda_n Ax_{n+1})\right\rangle \\
+& \left\langle y_{n+1}-y_n, (\lambda_n Ax_n - x_n) - x_{n+1} + \frac{\lambda_n}{\lambda_{n+1}}x_{n+1}\right\rangle \\
=& \left\langle y_{n+1}-y_n, y_n-y_{n+1}+\left(1-\frac{\lambda_n}{\lambda_{n+1}}\right)(y_{n+1}-x_{n+1})\right\rangle \\
+& \left\langle y_{n+1}-y_n, (x_{n+1}-\lambda_n Ax_{n+1})-(x_n-\lambda_n Ax_n)\right\rangle .
\end{aligned}
$$

It then follows that

$$
\|y_{n+1}-y_n\|^2 \leq \|y_{n+1}-y_n\|\left\{\left|1-\frac{\lambda_n}{\lambda_{n+1}}\right|\|y_{n+1}-x_{n+1}\|+\|x_{n+1}-x_n\|\right\},
$$

and so we have

$$
\|y_{n+1}-y_n\| \leq \left|1-\frac{\lambda_n}{\lambda_{n+1}}\right|\|y_{n+1}-x_{n+1}\|+\|x_{n+1}-x_n\|. \tag{3.11}
$$

Using condition (ii), we have

$$\begin{aligned}\|y_{n+1}-y_n\| &\le \|x_{n+1}-x_n\| + \frac{1}{\lambda_{n+1}}|\lambda_{n+1}-\lambda_n|\|y_{n+1}-x_{n+1}\| \\ &\le \|x_{n+1}-x_n\| + \frac{1}{a}|\lambda_{n+1}-\lambda_n|M, \end{aligned} \tag{3.12}$$

Hence, we have

$$\|y_n-y_{n-1}\| \le \|x_n-x_{n-1}\| + \frac{1}{a}|\lambda_n-\lambda_{n-1}|M. \tag{3.13}$$

Consider

$$\begin{aligned}\|T^n z_n - T^{n-1}z_{n-1}\| &\le \|T^n z_n - T^n z_{n-1}\| + \|T^n z_{n-1} - T^{n-1}z_{n-1}\| \\ &\le k_n\|z_n - z_{n-1}\| + \|T^n z_{n-1} - T^{n-1}z_{n-1}\|. \end{aligned} \tag{3.14}$$

From (3.1),(3.5), (3.13) and (3.14), we have that

$$\begin{aligned}
&\|x_{n+1}-x_n\| \\
&= \|\alpha_n f(x_n) + (1-\alpha_n)T^n z_n - \alpha_{n-1}f(x_{n-1}) - (1-\alpha_{n-1})T^{n-1}z_{n-1}\| \\
&\le \alpha_n\rho\|x_n-x_{n-1}\| + |\alpha_n-\alpha_{n-1}|\big(\|f(x_{n-1}\| + \|T^{n-1}z_{n-1}\|\big) \\
&\quad +(1-\alpha_n)\|T^n z_n - T^{n-1}z_{n-1}\| \\
&\le \alpha_n\rho\|x_n-x_{n-1}\| + |\alpha_n-\alpha_{n-1}|K + (1-\alpha_n)\|T^n z_n - T^{n-1}z_{n-1}\| \\
&\le \alpha_n\rho\|x_n-x_{n-1}\| + |\alpha_n-\alpha_{n-1}|K + (1-\alpha_n)k_n\|z_n - z_{n-1}\| \\
&\quad +(1-\alpha_n)\|T^n z_{n-1} - T^{n-1}z_{n-1}\| \\
&\le \alpha_n\rho\|x_n-x_{n-1}\| + |\alpha_n-\alpha_{n-1}|K + (1-\alpha_n)(k_n-1)\|z_n - z_{n-1}\| \\
&\quad +(1-\alpha_n)\|T^n z_{n-1} - T^{n-1}z_{n-1}\| + (1-\alpha_n)\|z_n-z_{n-1}\| \\
&\le \alpha_n\rho\|x_n-x_{n-1}\| + |\alpha_n-\alpha_{n-1}|K \\
&\quad +(1-\alpha_n)(k_n-1)\left[\|y_n-y_{n-1}\| + |\lambda_{n-1}-\lambda_n|M\right] \qquad (3.15) \\
&\quad +(1-\alpha_n)\|T^n z_{n-1} - T^{n-1}z_{n-1}\| \\
&\quad +(1-\alpha_n)\left[\|y_n-y_{n-1}\| + |\lambda_{n-1}-\lambda_n|M\right] \\
&\le \alpha_n\rho\|x_n-x_{n-1}\| + |\alpha_n-\alpha_{n-1}|K + (k_n-1)\|x_n-x_{n-1}\| \\
&\quad +(k_n-1)\left[\frac{1}{a}|\lambda_n-\lambda_{n-1}|M + |\lambda_{n-1}-\lambda_n|M\right] \\
&\quad +\|T^n z_{n-1} - T^{n-1}z_{n-1}\| \\
&\quad +(1-\alpha_n)\left[\|x_n-x_{n-1}\| + \frac{1}{a}|\lambda_n-\lambda_{n-1}|M + |\lambda_{n-1}-\lambda_n|M\right] \\
&\le (1-\alpha_n(1-\rho-\epsilon))\|x_n-x_{n-1}\| + |\alpha_n-\alpha_{n-1}|K + \frac{\epsilon\alpha_n}{a}|\lambda_n-\lambda_{n-1}|M \\
&\quad +(\epsilon\alpha_n + 1 - \alpha_n)|\lambda_{n-1}-\lambda_n|M \\
&\quad +\|T^n z_{n-1} - T^{n-1}z_{n-1}\| + \frac{(1-\alpha_n)}{a}|\lambda_n-\lambda_{n-1}|M, \\
&\le (1-\alpha_n(1-\rho-\epsilon))\|x_n-x_{n-1}\| + \alpha_n(1-\rho-\epsilon)|\lambda_n-\lambda_{n-1}|M
\end{aligned}$$

$$+(1+a+\alpha_n(\epsilon+\rho a))\frac{|\lambda_n-\lambda_{n-1}|}{a}M+|\alpha_n-\alpha_{n-1}|K$$
$$+\|T^n z_{n-1}-T^{n-1}z_{n-1}\|,$$

where $K=\sup\{\|f(x_n)\|+\|T^n z_n\|\}$ and where $M=\sup_{n\geq 1}\{\|y_n-x_n\|,\|Ay_n\|\}$. Put $\varepsilon_n=\alpha_n(1-\rho-\epsilon)$, $\beta_n=|\lambda_n-\lambda_{n-1}|M$ and $\gamma_n=(1+a+\alpha_n(\epsilon+\rho a))\frac{|\lambda_n-\lambda_{n-1}|}{a}M+|\alpha_n-\alpha_{n-1}|K+\|T^n z_{n-1}-T^{n-1}z_{n-1}\|$. Then,

$$\|x_{n+1}-x_n\|\leq (1-\varepsilon_n)\|x_n-x_{n-1}\|+\varepsilon_n\beta_n+\gamma_n$$

Using Lemma 2.3, we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.16}$$

Further by (3.6) and (3.12) with the condition that $\lim_{n\to\infty}(\lambda_n-\lambda_{n+1})=0$, we get

$$\lim_{n\to\infty}\|z_{n+1}-z_n\|=0. \tag{3.17}$$

and

$$\lim_{n\to\infty}\|y_{n+1}-y_n\|=0.$$

Since $x_n=\alpha_{n-1}f(x_{n-1})+(1-\alpha_{n-1})T^{n-1}z_{n-1}$, we have

$$\begin{aligned}\|x_n-T^n z_n\| &\leq \|x_n-T^{n-1}z_{n-1}\|+\|T^{n-1}z_{n-1}-T^n z_n\|\\ &\leq \|x_n-T^{n-1}z_{n-1}\|+\|T^{n-1}z_{n-1}-T^n z_{n-1}\|\\ &\quad+\|T^n z_{n-1}-T^n z_n\|\\ &\leq \alpha_{n-1}\|f(x_{n-1})-T^{n-1}z_{n-1}\|\\ &\quad+\|T^{n-1}z_{n-1}-T^n z_{n-1}\|+k_n\|z_{n-1}-z_n\|.\end{aligned}$$

From (3.17) with $\alpha_n\to 0$ as $n\to\infty$ and T is asymptotically regular on C. It follows that

$$\lim_{n\to\infty}\|T^n z_n-x_n\|=0. \tag{3.18}$$

Since $z\in F(T)\cap VI(C,A)$, from the convexity of $\|.\|^2$ and (3.3), we have

$$\begin{aligned}&\|x_{n+1}-z\|^2\\ &= \|\alpha_n f(x_n)+(1-\alpha_n)T^n z_n-z\|^2\\ &= \|\alpha_n(f(x_n)-z)+(1-\alpha_n)(T^n z_n-z)\|^2\\ &\leq \alpha_n\|f(x_n)-z\|^2+(1-\alpha_n)\|T^n z_n-z\|^2\\ &\leq \alpha_n\|f(x_n)-z\|^2+(1-\alpha_n)k_n^2\|z_n-z\|^2\\ &\leq \alpha_n\|f(x_n)-z\|^2+(1-\alpha_n)k_n^2\|y_n-z\|^2 \qquad (3.19)\end{aligned}$$

$$
\begin{aligned}
&\le \alpha_n\|f(x_n)-z\|^2+(1-\alpha_n)k_n^2(\|(x_n-\lambda_nAx_n)-(z-\lambda_nAz)\|^2)\\
&\le \alpha_n\|f(x_n)-z\|^2\\
&\quad +(1-\alpha_n)k_n^2(\|x_n-z\|^2+\lambda_n(\lambda_n-2\alpha)\|Ax_n-Az)\|^2)\\
&\le \alpha_n\|f(x_n)-z\|^2\\
&\quad +(1-\alpha_n)k_n^2\|x_n-z\|^2\\
&\quad +(1-\alpha_n)k_n^2\lambda_n(\lambda_n-2\alpha)\|Ax_n-Az)\|^2\\
&\le \alpha_n\|f(x_n)-z\|^2\\
&\quad +(1-\alpha_n)(k_n^2-1)\|x_n-z\|^2+(1-\alpha_n)\|x_n-z\|^2\\
&\quad +(1-\alpha_n)k_n^2\lambda_n(\lambda_n-2\alpha)\|Ax_n-Az)\|^2\\
&\le \alpha_n\|f(x_n)-z\|^2\\
&\quad +(1-\alpha_n)(k_n^2-1)\|x_n-z\|^2+\|x_n-z\|^2\\
&\quad +(1-\alpha_n)k_n^2\lambda_n(\lambda_n-2\alpha)\|Ax_n-Az)\|^2, \qquad (3.20)
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\lambda_n(2\alpha-\lambda_n)(1-\alpha_n)k_n^2\|Ax_n-Az\|^2\\
&\quad\le \alpha_n\|f(x_n)-z\|^2+(1-\alpha_n)(k_n^2-1)\|x_n-z\|^2\\
&\quad\quad +\|x_n-z\|^2-\|x_{n+1}-z\|^2. \qquad (3.21)
\end{aligned}
$$

Since $\alpha_n\to 0$, $k_n\to 1$, $\|x_{n+1}-x_n\|\to 0$ as $n\to\infty$ and both $\{f(x_n)\}$ and $\{x_n\}$ are bounded, from (3.21), we obtain $\|Ax_n-Az\|\to 0$ as $n\to\infty$.
From (1.1) and the fact that $I-\lambda_nA$ is nonexpansive, we have

$$
\begin{aligned}
&\|y_n-z\|^2\\
&\quad= \|P_C(x_n-\lambda_nAx_n)-P_C(z-\lambda_nAz)\|^2\\
&\quad\le \langle y_n-z,(x_n-\lambda_nAx_n)-(z-\lambda_nAz)\rangle\\
&\quad= \frac{1}{2}[\|(x_n-\lambda_nAx_n)-(z-\lambda_nAz)\|^2+\|y_n-z\|^2\\
&\qquad -\|(x_n-\lambda_nAx_n)-(z-\lambda_nAz)-(y_n-z)\|^2]\\
&\quad\le \frac{1}{2}[\|x_n-z\|^2+\|y_n-z\|^2-\|(x_n-y_n)-\lambda_n(Ax_n-Az)\|^2]\\
&\quad= \frac{1}{2}[\|x_n-z\|^2+\|y_n-z\|^2-\|x_n-y_n\|^2+2\lambda_n\langle x_n-y_n,Ax_n-Az\rangle\\
&\qquad -\lambda_n^2\|Ax_n-Az\|^2],
\end{aligned}
$$

and so, we obtain
(3.22)

$$
\|y_n-z\|^2\le\|x_n-z\|^2-\|x_n-y_n\|^2+2\lambda_n\langle x_n-y_n,Ax_n-Az\rangle-\lambda_n^2\|Ax_n-Az\|^2.
$$

Hence from (3.19) and (3.22), we have

$$
\begin{aligned}
&\|x_{n+1}-z\|^2\\
&\quad\le \alpha_n\|f(x_n)-z\|^2+(1-\alpha_n)k_n^2[\|x_n-z\|^2-\|x_n-y_n\|^2
\end{aligned}
$$

$$
\begin{aligned}
&+2\lambda_n\langle x_n-y_n, Ax_n-Az\rangle-\lambda_n^2\|Ax_n-Az\|^2] \\
=\ & \alpha_n\|f(x_n)-z\|^2+(1-\alpha_n)k_n^2\|x_n-z\|^2-(1-\alpha_n)k_n^2\|x_n-y_n\|^2 \\
&+2\lambda_n(1-\alpha_n)k_n^2\langle x_n-y_n, Ax_n-Az\rangle-(1-\alpha_n)k_n^2\lambda_n^2\|Ax_n-Az\|^2 \\
\leq\ & \alpha_n\|f(x_n)-z\|^2+(1-\alpha_n)(k_n^2-1)\|x_n-z\|^2+(1-\alpha_n)\|x_n-z\|^2 \\
&-(1-\alpha_n)k_n^2\|x_n-y_n\|^2+2\lambda_n(1-\alpha_n)k_n^2\|x_n-y_n\|\|Ax_n-Az\| \\
\leq\ & \alpha_n\|f(x_n)-z\|^2+(1-\alpha_n)(k_n^2-1)\|x_n-z\|^2+\|x_n-z\|^2 \\
&-(1-\alpha_n)k_n^2\|x_n-y_n\|^2+2\lambda_n(1-\alpha_n)k_n^2\|x_n-y_n\|\|Ax_n-Az\|,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&(1-\alpha_n)k_n^2\|x_n-y_n\| \\
&\quad\leq\ \alpha_n\|f(x_n)-z\|^2+(1-\alpha_n)(k_n^2-1)\|x_n-z\|^2+\|x_n-z\|^2 \\
&\qquad -\|x_{n+1}-z\|^2+2\lambda_n(1-\alpha_n)k_n^2\|x_n-y_n\|\|Ax_n-Az\|. \qquad (3.23)
\end{aligned}
$$

Since $\alpha_n\to 0$, $k_n\to 1$, $\|x_n-x_{n+1}\|\to 0$ and $\|Ax_n-Az\|\to 0$ as $n\to\infty$, from (3.23), we have

$$\lim_{n\to\infty}\|x_n-y_n\|=0. \qquad (3.24)$$

Since

$$
\begin{aligned}
&\|T^nz_n-z_n\| \\
&\leq\ \|T^nz_n-x_n\|+\|x_n-y_n\|+\|y_n-z_n\| \\
&=\ \|T^nz_n-x_n\|+\|x_n-y_n\|+\|P_C(x_n-\lambda_nAx_n)-P_C(y_n-\lambda_nAy_n)\| \\
&\leq\ \|T^nz_n-x_n\|+\|x_n-y_n\|+\|(x_n-\lambda_nAx_n)-(y_n-\lambda_nAy_n)\| \\
&\leq\ \|T^nz_n-x_n\|+2\|x_n-y_n\|+\lambda_n\|Ax_n-Ay_n\| \\
&\leq\ \|T^nz_n-x_n\|+(2+\frac{1}{\alpha}\lambda_n)\|x_n-y_n\|.
\end{aligned}
$$

From (3.18) and (3.24), we have $\|T^nz_n-z_n\|\to 0$ as $n\to\infty$. Hence as $n\to\infty$, we have

$$\|z_n-y_n\|\leq\|z_n-T^nz_n\|+\|T^nz_n-x_n\|+\|x_n-y_n\|\to 0. \qquad (3.25)$$

Since A is Lipschitz continuous, we have $Ay_n-Az_n\to 0$ as $n\to\infty$. By combining (3.24) and (3.25), we have

$$\|z_n-x_n\|\leq\|z_n-y_n\|+\|y_n-x_n\|\to 0.$$

We have

$$\|Tx_n-x_n\|\leq\|x_n-T^nz_n\|+\|T^nz_n-z_n\|+\|z_n-x_n\|,$$

which implies $\|Tx_n-x_n\|\to 0$ as $n\to\infty$. Further, we have

$$\|Ty_n-y_n\|\leq\ \|Ty_n-Tx_n\|+\|Tx_n-x_n\|+\|x_n-y_n\|$$

$$\leq k_1\|y_n - x_n\| + \|Tx_n - x_n\| + \|x_n - y_n\|$$
$$\leq (k_1+1)\|y_n - x_n\| + \|Tx_n - x_n\|,$$

which implies that $\|Ty_n - y_n\| \to 0$ as $n \to \infty$, since $P_{F(T)\cap VI(C,A)}f : C \to C$ is a ρ-contraction mapping. Therefore by the Banach contraction principle there exists a unique $z_0 \in F(T)\cap VI(C,A)$ such that $z_0 = P_{F(T)\cap VI(C,A)}f(z_0)$. We shall show that

$$\limsup_{n\to\infty}\langle f(z_0) - z_0, x_n - z_0\rangle \leq 0. \tag{3.26}$$

Since $\{y_n\}$ is bounded, we have that there exists a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that

$$\begin{aligned}\lim_{n\to\infty}\langle f(z_0) - z_0, x_n - z_0\rangle &= \limsup_{n\to\infty}\langle f(z_0) - z_0, y_n - z_0\rangle \\ &= \lim_{i\to\infty}\langle f(z_0) - z_0, y_{n_i} - z_0\rangle.\end{aligned} \tag{3.27}$$

Without loss of generality, we may assume that $y_{n_i} \rightharpoonup x^*$. Since C is closed and convex, C is weakly closed. So we have $x^* \in C$. Now we show that $x^* \in F(T)$. In fact, since $y_{n_i} \rightharpoonup x^*$ and $Ty_n - y_n \to 0$, by Lemma 2.2, we have $x^* \in F(T)$.
Next, we show that, $x^* \in VI(C,A)$. Since $y_n - z_n \to 0$, we have $z_{n_i} \rightharpoonup x^*$. Let

$$Tv = \begin{cases} Av + N_C v & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then T is maximal monotone. Let $(v,w) \in G(T)$ and $z_n \in C$ this implies that $w \in Tv = Av + N_C v$ and hence $w - Av \in N_C v$.
Now we get

$$\langle v - z_n, w - Av\rangle \geq 0. \tag{3.28}$$

On the other hand, from $z_n = P_C(y_n - \lambda_n Ay_n)$ and $v \in C$, we have that

$$\langle y_n - \lambda_n Ay_n - z_n, z_n - v\rangle \geq 0,$$

and hence

$$\langle v - z_n, \frac{(z_n - y_n)}{\lambda_n} + Ay_n\rangle \geq 0.$$

Replacing n by n_i in (3.28), we have

$$\begin{aligned}&\langle v - z_{n_i}, w\rangle \\ &\quad\geq \langle v - z_{n_i}, Av\rangle \\ &\quad\geq \langle v - z_{n_i}, Av\rangle - \langle v - z_{n_i}, Ay_{n_i} + \frac{(z_{n_i} - y_{n_i})}{\lambda_{n_i}}\rangle \\ &\quad= \langle v - z_{n_i}, Av - Ay_{n_i} - \frac{(z_{n_i} - y_{n_i})}{\lambda_{n_i}}\rangle\end{aligned}$$

$$\begin{aligned} &= \langle v - z_{n_i}, Av - Az_{n_i}\rangle + \langle v - z_{n_i}, Az_{n_i} - Ay_{n_i}\rangle - \langle v - z_{n_i}, \frac{(z_{n_i} - y_{n_i})}{\lambda_{n_i}}\rangle \\ &\geq \langle v - z_{n_i}, Az_{n_i} - Ay_{n_i}\rangle - \langle v - z_{n_i}, \frac{(z_{n_i} - y_{n_i})}{\lambda_{n_i}}\rangle. \end{aligned}$$

hence, we get
$\langle v - x^*, w\rangle \geq 0$ as $i \to \infty$. Since T is maximal monotone (see, Rockafellar,[13]), we have $x^* \in T^{-1}0$ and hence $x^* \in VI(C, A)$.
Since $x^* \in F(T) \cap VI(C, A)$, from (3.27) and the property of metric projection, we have

$$\begin{aligned} \limsup_{n\to\infty}\langle f(z_0) - z_0, x_n - z_0\rangle &= \limsup_{n\to\infty}\langle f(z_0) - z_0, y_n - z_0\rangle \\ &= \lim_{i\to\infty}\langle f(z_0) - z_0, y_{n_i} - z_0\rangle \\ &= \langle f(z_0) - z_0, x^* - z_0\rangle \leq 0. \end{aligned} \tag{3.29}$$

Finally, we prove that $\lim_{n\to\infty}\|x_n - z_0\| = 0$. From (3.1) and Lemma 2.4 we obtain

$$\begin{aligned} &\|x_{n+1} - z_0\|^2 \\ &= \|\alpha_n(f(x_n) - z_0) + (1 - \alpha_n)(T^n z_n - z_0)\|^2 \\ &\leq (1 - \alpha_n)^2\|T^n z_n - z_0\|^2 + 2\langle\alpha_n(f(x_n) - z_0), x_{n+1} - z_0\rangle \\ &= (1 - \alpha_n)^2\|T^n z_n - z_0\|^2 + 2\alpha_n\langle f(x_n) - z_0, x_{n+1} - z_0\rangle \\ &\leq [(1 - \alpha_n)k_n]^2\|z_n - z_0\|^2 + 2\alpha_n\langle f(x_n) - z_0, x_{n+1} - z_0\rangle \\ &= [(1 - \alpha_n)k_n]^2\|z_n - z_0\|^2 + 2\alpha_n\langle f(x_n) - f(z_0), x_{n+1} - z_0\rangle \\ &\quad +2\alpha_n\langle f(z_0) - z_0, x_{n+1} - z_0\rangle \\ &\leq [(1 - \alpha_n)k_n]^2\|x_n - z_0\|^2 + 2\alpha_n\langle f(x_n) - f(z_0), x_{n+1} - z_0\rangle \\ &\quad + 2\alpha_n\langle f(z_0) - z_0, x_{n+1} - z_0\rangle \\ &\leq [(1 - \alpha_n)k_n]^2\|x_n - z_0\|^2 + 2\alpha_n\rho\|x_n - z_0\|\|x_{n+1} - z_0\| \\ &\quad +2\alpha_n\langle f(z_0) - z_0, x_{n+1} - z_0\rangle \\ &\leq [(1 - \alpha_n)k_n]^2\|x_n - z_0\|^2 + \alpha_n\rho(\|x_n - z_0\|^2 + \|x_{n+1} - z_0\|^2) \\ &\quad +2\alpha_n\langle f(z_0) - z_0, x_{n+1} - z_0\rangle \\ &= [(1 - \alpha_n)(k_n - 1) + (1 - \alpha_n)]^2\|x_n - z_0\|^2 + \alpha_n\rho\|x_n - z_0\|^2 \\ &\quad +\alpha_n\rho\|x_{n+1} - z_0\|^2 + 2\alpha_n\langle f(z_0) - z_0, x_{n+1} - z_0\rangle \\ &= [1 - (2 - \rho)\alpha_n + \alpha_n^2 + (1 - \alpha_n)^2(1 - k_n)^2 + 2(1 - \alpha_n)^2(k_n - 1)]\|x_n - z_0\|^2 \\ &\quad +\alpha_n\rho\|x_{n+1} - z_0\|^2 + 2\alpha_n\langle f(z_0) - z_0, x_{n+1} - z_0\rangle \\ &\leq [1 - (2 - \rho)\alpha_n + \alpha_n^2 + (1 - k_n)^2 + 2(k_n - 1)]\|x_n - z_0\|^2 \\ &\quad +\alpha_n\rho\|x_{n+1} - z_0\|^2 + 2\alpha_n\langle f(z_0) - z_0, x_{n+1} - z_0\rangle. \end{aligned}$$

Let $P_n = \sup_{n\in\mathbb{N}}\|x_n - z_0\|^2$, so now we have

$$\|x_{n+1} - z_0\|^2 \leq \frac{1 - (2 - \rho)\alpha_n}{1 - \rho\alpha_n}\|x_n - z_0\|^2 + \frac{\alpha_n^2 + (k_n - 1)^2 + 2(k_n - 1)}{1 - \rho\alpha_n}P_n$$

$$
\begin{aligned}
&+ \frac{2\alpha_n}{1-\rho\alpha_n}\langle f(z_0)-z_0, x_{n+1}-z_0\rangle \\
&= [1-\frac{(2-\rho)\alpha_n}{1-\rho\alpha_n}]\|x_n-z_0\|^2 + \frac{\alpha_n^2+(k_n-1)^2+2(k_n-1)}{1-\rho\alpha_n}P_n \\
&+ \frac{2\alpha_n}{1-\rho\alpha_n}\langle f(z_0)-z_0, x_{n+1}-z_0\rangle \\
&= (1-a_n)\|x_n-z_0\|^2 + a_n\beta_n,
\end{aligned}
$$

where $\beta_n = \frac{\alpha_n^2+(k_n-1)^2+2(k_n-1)}{2(1-\rho)\alpha_n}P_n + \frac{1}{1-\rho}\langle f(z_0)-z_0, x_{n+1}-z_0\rangle$ and $a_n = \frac{2(1-\rho)\alpha_n}{1-\rho\alpha_n}$. Since $\lim_{n\to\infty} a_n = 0$, $\sum_{n=0}^{\infty} a_n = \infty$ and $\limsup_{n\to\infty}\beta_n \leq 0$ by (3.29). Then by Lemma 2.5, we conclude that $\lim_{n\to\infty}\|x_n-z_0\| = 0$. □

4. Applications

Using our Theorem 3.1, we prove the following theorems:

Browder and Patryshyn [4] introduced $k-$ strictly pseudocontractive mapping which is as follows:
A mapping $S : C \to C$ is called k- strictly pseudocontractive if there exists $k \in [0,1)$ such that,

$$\|Sx-Sy\|^2 \leq \|x-y\|^2 + k\|(I-S)x-(I-S)y\|^2, \forall x,y \in C.$$

Putting $A = I - S$, we know that

$$\langle x-y, Ax-Ay\rangle \geq \frac{1-k}{2}\|Ax-Ay\|^2, \forall x,y \in C.$$

Theorem 4.1. *Let C be a non empty closed and convex subset of a real Hilbert space H. Let $f : C \to C$ be ρ-contraction mapping, S be a k-strictly pseudo contractive mapping of C into itself and let $T : C \to C$ be asymptotically non-expansive mapping. Assume that T is asymptotically regular on C such that $F(T) \cap F(S) \neq \emptyset$, where $A = I - S$. Let $\{x_n\}$ be a sequence generated by*

$$
(4.1) \qquad \begin{cases} x_1 \in C, \\ y_n = x_n - \lambda_n A x_n, n = 1,2,3,\ldots \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T^n(y_n - \lambda_n A y_n), n = 1,2,3,\ldots \end{cases}
$$

where $\{\alpha_n\} \subset [0,1]$ and $\{\lambda_n\} \subset [0,1-k]$ satisfy

(i) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$,

(ii) $0 < a \leq \lambda_n \leq b < 1-k$,

(iii) $\lim_{n\to\infty}(\lambda_n - \lambda_{n+1}) = 0$,

(iv) $\lim_{n\to\infty} \frac{k_n - 1}{\alpha_n} = 0$.

Then $\{x_n\}$ *converges strongly to* $z = P_{F(T)\cap F(S)} f(z)$.

Proof. In Theorem 3.1, put $A = I - S$ and $P_C = I$. Then A is $\frac{1-k}{2}$-inverse strongly monotone mapping. We have that $F(S) = VI(C, A)$ and $P_C(x_n - \lambda_n A x_n) = (I - \lambda_n)x_n + \lambda_n S x_n$. So by Theorem 3.1, we obtain the desired result. □

Theorem 4.2. *Let H be a real Hilbert space. Let $f : H \to H$ be ρ-contraction mapping and $T : H \to H$ be asymptotically non-expansive mapping. Assume that T is asymptotically regular on H such that $F(T) \cap A^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 \in C, \\ y_n = x_n - \lambda_n A x_n, n = 1, 2, 3, \dots \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T^n (y_n - \lambda_n A y_n), n = 1, 2, 3, \dots \end{cases} \tag{4.2}$$

where $\{\alpha_n\} \subset [0, 1]$ *and* $\{\lambda_n\} \subset [0, 2\alpha]$ *satisfy*

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < a \leq \lambda_n \leq b < 2\alpha$,

(iii) $\lim_{n\to\infty} (\lambda_n - \lambda_{n+1}) = 0$,

(iv) $\lim_{n\to\infty} \frac{k_n - 1}{\alpha_n} = 0$.

Then $\{x_n\}$ *converges strongly to* $z = P_{F(T)\cap A^{-1}0} f(z)$.

Proof. Since $A^{-1}0 = VI(H, A)$ and $P_C = I$. So, by Theorem 3.1, we obtain the desired result. □

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